ON WILLEMS' CONJECTURE ON BRAUER CHARACTER DEGREES

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Abstract. In 2005 Wolfgang Willems put forward a conjecture proposing a lower bound for the sum of squares of the degrees of the irreducible p-Brauer characters of a finite group G. We prove this conjecture for the prime $p = 2$. For this we rely on the recent reduction of Willems' conjecture to a question on quasi-simple groups by Tong-Viet. We also verify the conditions of Tong-Viet for certain families of finite quasi-simple groups and odd primes. On the way we obtain lower bounds for the number of regular semisimple conjugacy classes in finite groups of Lie type.

1. Introduction

Let G be a finite group and $\mathrm{Irr}(G)$ the set of its ordinary irreducible characters. Frobenius showed the classical formula

$$\sum_{\chi\in\mathrm{Irr}(G)} \chi(1)^2 = |G|$$

for the dimensions of the irreducible complex representations of G. No analogue of this equation is known in the modular setting, that is, for the set $\mathrm{IBr}(G)$ of irreducible p-Brauer characters of G, when p is a prime dividing $|G|$. In 2005, Willems [13] put forward a conjecture giving a lower bound in terms of the prime-to-p part $|G|_{p'}$ of the group order:

Conjecture 1 (Willems (2005))**.** *Let G be a finite group and p be a prime. Then*

$$|G|_{p'} \leq \sum_{\varphi\in\mathrm{IBr}(G)} \varphi(1)^2.$$

Willems [13] points out that his conjecture holds for groups with cyclic Sylow p-subgroups as well as for p-solvable groups, and he proves it for groups of Lie type in defining characteristic p. Here we show:

Theorem 2. *Willems' Conjecture 1 holds for all groups for the prime $p = 2$.*

Our proof relies on a reduction of (a strengthening of) the conjecture by Tong-Viet [11, Prop. 1.1] to the case of quasi-simple groups. He shows that Conjecture 1 holds for all finite groups at the prime p if the following conjecture on p-Brauer characters is true for all quasi-simple groups:

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Conjecture 3 (Tong-Viet (2019)). *Let G be a finite quasi-simple group with centre $Z(G)$ of p'-order. Then*

$$|G/Z(G)|_{p'} \leq \sum_{\varphi \in \mathrm{IBr}(G|\theta)} \varphi(1)^2$$

for all faithful characters $\theta \in \mathrm{Irr}(Z(G))$.

Here, for $N \trianglelefteq G$ and $\theta \in \mathrm{IBr}(N)$ we denote by $\mathrm{IBr}(G|\theta)$ the set of irreducible Brauer characters of G above θ. Also, recall that a finite group G is *quasi-simple* if G is perfect and $G/Z(G)$ is simple.

Tong-Viet [11, Prop. 2.1] has verified his conjecture for the finitely many quasi-simple groups of Lie type with exceptional Schur multiplier, as well as for all covering groups of sporadic simple groups and for alternating groups of small degree. We invoke the classification of finite simple groups to deal with the general case, obtaining a complete answer at least when $p = 2$. It will turn out that in many cases the inequality in Conjecture 3 is already satisfied with one single suitable Brauer character, like for symmetric groups in characteristic 2 or many groups of Lie type in defining characteristic, but on the other hand there are groups for which the degrees of a large number of characters have to be taken into consideration; there is no absolute upper bound on the number of necessary characters even in the class of quasi-simple groups.

To show that there are sufficiently many such characters, we are led to derive lower bounds for the number of conjugacy classes of finite groups of Lie type containing regular semisimple elements (Proposition 4.1) which may be of independent interest. As an application we obtain that any simple group of Lie type in characteristic p has at least two conjugacy classes of length divisible by the p-part of the group order (Corollary 4.2); this was recently used by Sambale [10].

After collecting some basic observations, we study the finite simple groups according to their classification, starting with the groups of Lie type. In Section 3 we suppose that p is the defining prime; here our results are only partial for certain types. In Section 4 we show Conjecture 3 in the non-defining characteristic situation for all cases. Finally, in Section 5 we verify Conjecture 3 for alternating groups for the prime $p = 2$.

2. Preliminary results

Proposition 2.1. *Let G be a finite group and $N \trianglelefteq G$ with G/N solvable. Assume Conjecture 1 holds for G at the prime p. Then it also holds for N at p.*

Proof. Let $N \leq M \leq G$ be a maximal (normal) subgroup. Then G/M is cyclic of prime order. Using Clifford theory and an easy counting argument we conclude that Conjecture 1 holds for M. The general statement thus follows by induction over a composition series of G/N. □

We will use the following consequence of a result of Kiyota and Wada:

Lemma 2.2. *Conjecture 3 holds for quasi-simple groups with cyclic Sylow p-subgroups.*

Proof. It was shown by Kiyota and Wada [4, Prop. 4.7] (see also [3, Prop. 3.1]) that for every p-block B of a finite group G with a cyclic defect group D we have

$$\dim B \leq |D| \sum_{\varphi\in\mathrm{IBr}(B)} \varphi(1)^2.$$

Now assume that G is quasi-simple with a cyclic Sylow p-subgroup and centre $Z(G)$ of p'-order. Let $\theta \in \mathrm{Irr}(Z(G))$ be faithful. By [8, Thm. (9.2)], for example, there is a union $\mathcal{B}$ of p-blocks of G with $\mathrm{Irr}(G|\theta) = \bigcup_{B\in\mathcal{B}} \mathrm{Irr}(B)$ and $\mathrm{IBr}(G|\theta) = \bigcup_{B\in\mathcal{B}} \mathrm{IBr}(B)$. Let d be the maximal order of a defect group of any $B \in \mathcal{B}$. Then, using the above inequality we have

$$|G/Z(G)|_{p'} \leq \frac{1}{d}|G/Z(G)| = \frac{1}{d}\sum_{\chi\in\mathrm{Irr}(G|\theta)} \chi(1)^2 = \frac{1}{d}\sum_{B\in\mathcal{B}} \dim B \leq \sum_{\varphi\in\mathrm{IBr}(G|\theta)} \varphi(1)^2,$$

as claimed by Conjecture 3. □

In many cases, we will make use of characters of *p-defect zero*, that is, irreducible characters χ of a finite group G such that $\chi(1)$ contains the full p-part of the group order $|G|$. It is a basic result of Brauer that these remain irreducible under p-modular reduction and thus furnish irreducible p-Brauer characters of the same degree.

3. Groups of Lie type in defining characteristic

Throughout this section $\mathbf{G}$ denotes a simple, simply connected linear algebraic group over the algebraic closure of a finite field and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism with finite group of fixed point $\mathbf{G}^F$. Any simple group of Lie type can then be obtained as $\mathbf{G}^F/Z(\mathbf{G}^F)$ for a suitable such $\mathbf{G}$, except for ${}^2F_4(2)'$, for which Conjecture 3 already shown in [11, Prop. 2.1].

In this section we consider the case where p is the defining characteristic of $\mathbf{G}$. Here, Willems observed that the Steinberg character (of p-defect zero) already has large enough degree for Conjecture 1 to hold. Note, however, that this result does not imply Conjecture 3 for these groups, since generally there will be more than one block of positive defect. Here, substantially more work is needed. In fact, it seems that not enough is currently known about large degree irreducible modular characters in this situation to derive a complete answer unless $p = 2$. We will use the theory of highest weight representations; a basic introduction can be found for example in [6, §16].

Remark 3.1. Let $\mathbf{G}$ be simply connected and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism with respect to an $\mathbb{F}_p$-structure. Recall that any irreducible representation of $\mathbf{G}^F$ over $\overline{\mathbb{F}_p}$ is the restriction of a highest weight representation of $\mathbf{G}$ with p-restricted highest weight. Let $\{\psi_i \mid i \in I\}$ be a set of irreducible representations of $\mathbf{G}$ (and hence of $\mathbf{G}^F$) with restricted highest weights, all lying over the same central character θ of $\mathbf{G}$ (and hence over the same central character of $\mathbf{G}^F$). Further assume that $\sum_{i\in I}\psi_i(1)^2 \geq |\mathbf{G}^F : (\ker\theta)^F|_{p'}$. Let χ be the Steinberg representation of $\mathbf{G}$ and let $r \geq 1$. Then by Steinberg's tensor product theorem (see, e.g., [6, Thm. 16.12]), all $\psi_{i,r} := \psi_i \otimes \chi^{(1)} \otimes \cdots \otimes \chi^{(r-1)}$ are irreducible representations of $\mathbf{G}^{F^r}$, where $\chi^{(j)}$ denotes the jth Frobenius twist of χ. Since we have

$\chi(1)^2 > |\mathbf{G}^F|_{p'}$,
$$\sum_{i\in I} \psi_{i,r}(1)^2 \geq |\mathbf{G}^{F^r} : (\ker\theta)^{F^r}|_{p'}.$$
Thus, if Conjecture 3 holds for $\mathbf{G}^F$, it holds for all $\mathbf{G}^{F^r}$, $r \geq 1$. In proving Conjecture 3 for $\mathbf{G}^{F^r}$ in the defining characteristic, we may therefore restrict attention to the groups defined over the prime field whenever convenient.

Let us first deal with some small rank cases.

Proposition 3.2. *Conjecture 3 holds for all central quotients of the groups* $\mathrm{SL}_2(q)$, $\mathrm{SL}_3(q)$, $\mathrm{SU}_3(q)$, $\mathrm{SL}_4(q)$, $\mathrm{SU}_4(q)$ *and* $\mathrm{Sp}_4(q)$ *in defining characteristic.*

Proof. Let G be as in the statement. By [13, Exmp. 2.3(a)] we may restrict our investigations to the case that $Z(G) \neq 1$ and by Remark 3.1 we may assume that $q = p$. For $G = \mathrm{SL}_2(q)$, we may hence assume that $q = p$ is odd. Here G has a p-restricted faithful irreducible Brauer character of degree $p-1$, and $(p-1)^2 \geq (p^2-1)/2 = |G/Z(G)|_{p'}$.

Next assume $G = \mathrm{SL}_3(p)$ or $\mathrm{SU}_3(p)$ with $Z(G)$ non-trivial of order 3. For $1 \leq i \leq p-1$ let χ_i be the irreducible p-Brauer character of G with (restricted) highest weight $(i, i-1)$. We obtain a lower bound on $\chi_i(1)$ by adding up the dimensions of certain weight spaces in the corresponding highest weight module $L(i, i-1)$. According to the result of Premet [9, Thm. 1], all weights in the corresponding characteristic 0 irreducible highest weight module also occur in $L(i, i-1)$. But these are exactly the weights subdominant to $(i, i-1)$. It is now an easy exercise to determine all these weights and to find the lengths of their orbits under the Weyl group. This shows that $\chi_i(1) = \dim L(i, i-1) \geq 3i^2$. Hence $\sum_{i=1}^{p-1} \chi_i(1)^2 \geq (p^2-1)(p^3+1)/3$ by a straightforward calculation. Thus we conclude, since all characters χ_i lie above a fixed faithful character of $Z(\mathrm{SL}_3)$ (see e.g. [5, App. A.2]).

Next, assume that $G = \mathrm{Sp}_4(p)$. Since $|Z(G)| = (2, p-1)$ we may assume that p is odd. Note that a module with restricted highest weight (i, j) is faithful on $Z(G)$ if and only if j is odd. Arguing as in the previous case, one sees that a lower bound for the dimension of the restricted module with highest weight $(i, 2j+1)$ is $2i^2 + (8j+6)i + 4(j+1)^2$, and the sum of the squares of this over all $i = 0, \ldots, p-1$ and $j = 0, \ldots, (p-3)/2$ is larger than p^6 and thus larger than $|G/Z(G)|_{p'} = (p^2-1)(p^4-1)/2$.

Finally, for $G = \mathrm{SL}_4(p)$ or $\mathrm{SU}_4(p)$, lower bounds for the weight space dimensions in restricted representations with highest weight (m_1, m_2, m_3) with $m_i \leq p-1$ and $m_1+2m_2+3m_3 \equiv 1 \pmod 4$ (for faithful representations of G), respectively $m_1+2m_2+3m_3 \equiv 2 \pmod 4$ (for those with a central subgroup of order 2 in the kernel), yield a sum of squares at least $(p^2-1)(p^3 \pm 1)(p^4-1)/(4, p \pm 1)$. □

Proposition 3.3. *Let G be quasi-simple of Lie type in characteristic p, but not a spin, half spin or symplectic group, nor of type ${}^{(2)}A_{n-1}$ with $n \leq 5$. Then Conjecture 3 holds for G at the prime p.*

Proof. We may assume that G is not an exceptional covering group of its simple quotient, since these have centre of order divisible by the characteristic p (see e.g. [6, Tab. 24.3]). Thus, G is a central factor group of a simply connected group of Lie type. For those, as pointed out by Willems [13, Exmp. 2.3(a)], Conjecture 3 holds for groups with trivial centre. It remains to discuss the groups with non-trivial centre. In particular we need not worry about Suzuki- and Ree groups. Let $G = \mathbf{G}^F$, with $\mathbf{G}$ simple of simply connected type and F a Frobenius endomorphism of $\mathbf{G}$ with respect to an $\mathbb{F}_q$-structure.

First assume that $G = E_6(q)$ or ${}^2E_6(q)$ with $|Z(G)| = 3$. Then G has a subgroup $H = F_4(q) \times Z(G)$, where the first factor is the centraliser of a graph automorphism of G. Let $\psi \in \mathrm{IBr}(H)$ be the Steinberg character of $F_4(q)$ times a faithful character of $Z(G)$. Then $\psi(1) = q^{24}$. Clearly, G has to have a faithful irreducible p-Brauer character φ of at least that degree. But then $\varphi(1)^2 \geq q^{48} > |G|_{p'}$. Next, assume $G = E_7(q)$ with $|Z(G)| = 2$. Here, consider the subgroup $H = E_6(q) \times Z(G)$, and let $\psi \in \mathrm{IBr}(H)$ be the Steinberg character of $E_6(q)$ times the faithful linear character of $Z(G)$. This shows that G has an irreducible p-Brauer character φ with $\varphi(1)^2 \geq \psi(1)^2 = q^{72} > |G|_{p'}$.

Next, let $G = \mathrm{SL}_n(q)$. Then G has a subgroup $H = \mathrm{SL}_{n-1}(q) \times Z(G)$. The Steinberg character of $\mathrm{SL}_{n-1}(q)$ times a linear character of $Z(G)$ has degree $q^{(n-1)(n-2)/2}$, so G has an irreducible p-Brauer character of degree at least that large, while $|G|_{p'} \leq q^{(n-1)(n+2)/2}$. Thus, we are done when $n \geq 6$.

For $G = \mathrm{SU}_n(q)$ with $n \geq 6$, we again obtain an irreducible p-Brauer character from a Steinberg character ψ of a subgroup $\mathrm{SU}_{n-1}(q) \times Z(G)$, with $\psi(1)^2 = q^{(n-1)(n-2)}$. Now using that $(q^k - 1)(q^{k+1} + 1) \leq q^{2k+1}$ for all $k \geq 2$ we find

$$|G|_{p'} = \prod_{k=2}^{n} (q^k - (-1)^k) \leq q^{(n-1)(n+2)/2},$$

which again is smaller than $\psi(1)^2$ when $n \geq 6$.

Since we exclude the spin, half spin and symplectic groups by assumption, it only remains to discuss the groups $G = \mathrm{SO}_{2n}^{\pm}(q)'$, q odd, $n \geq 4$. These contain a subgroup $H = \mathrm{SO}_{2n-1}(q)'Z(G)$, and the product of the Steinberg character of $\mathrm{SO}_{2n-1}(q)'$ with the faithful character of $Z(G)$ yields an irreducible Brauer character of degree $q^{(n-1)^2}$. Thus G itself also has a faithful character of at least that degree, and its square is larger than $|G|_{p'}$ when $n \geq 4$. □

Proposition 3.4. *Let G be quasi-simple of classical Lie type in characteristic $p = 2$. Then Conjecture 3 holds for G at the prime 2.*

Proof. By the results of Propositions 3.2 and 3.3 we only need to consider the groups $\mathrm{SL}_n(q)$, $\mathrm{SU}_n(q)$ for $n = 5$ and $q = 2^f$, as the spin, half spin and symplectic groups have trivial centre in characteristic 2. For $\mathrm{SL}_5(q)$ the tensor product of the natural module with $f-1$ twists of the 1024-dimensional restricted Steinberg module yields an irreducible representation of sufficiently large degree when $f \geq 3$. For $f \leq 2$ the centre of $\mathrm{SL}_5(2^f)$ is trivial. For the unitary groups, we can argue similarly for $f \geq 3$. For $f = 2$ we take the tensor product of the Steinberg character with the exterior square of the natural representation, and for $\mathrm{SU}_5(2)$ the centre is trivial. □

We now state some further partial results.

Lemma 3.5. *Let $G = \mathrm{Sp}_{2n}(q)$ with $q = p^f$. Then Conjecture 3 holds for G at p if either $n \geq 5$ or $f \geq 2$.*

Proof. Recall that we only need to consider the faithful characters of G. The symplectic group $\mathrm{Sp}_{2n+2}(q)$ has a subsystem subgroup $\mathrm{Sp}_{2n}(q) \times \mathrm{Sp}_2(q)$, and its image in

$H := \mathrm{PSp}_{2n+2}(q)$ is a central product $G_1 := \mathrm{Sp}_{2n}(q) \circ \mathrm{Sp}_2(q)$ containing a subgroup $\mathrm{Sp}_{2n}(q)$. Now the Steinberg character of H of degree $q^{(n+1)^2}$ has p-defect zero, so yields an irreducible p-Brauer character of H of that degree. Hence the subgroup G_1 has to have a faithful irreducible Brauer character of degree at least $q^{(n+1)^2}/|H : G_1|$. Since the faithful irreducible Brauer characters of $\mathrm{Sp}_2(q)$ have degree at most $q-1$, G must have a faithful irreducible Brauer character of degree at least $q^{(n+1)^2}/((q-1)|H : G_1|)$. But the square of this is at least $|G|_{p'}$ whenever $n \geq 5$. When $n \leq 4$ but $f \geq 2$, we may take the tensor product of that character at $q = p$ with twists of the Steinberg character. □

Thus, with Proposition 3.2 among symplectic groups $\mathrm{Sp}_{2n}(q)$ only the faithful block for $n = 3, 4$ and $q = p$ odd remains open.

Proposition 3.6. *Let G be a covering group of a simple orthogonal group of rank $n \geq 3$ over $\mathbb{F}_q$, and assume that $q = p^f$ with $f \geq 2$. Then Conjecture 3 holds for all covering groups of G at p unless possibly when $G = \mathrm{Spin}_8^-(q)$, where it holds when $f \geq 3$.*

Proof. By Proposition 3.4 it remains to discuss the spin groups and half spin groups for odd q. Also, $\mathrm{Spin}_{4n}^+(q)$ has non-cyclic centre, so we need not consider it. First assume that $G = \mathrm{Spin}_{2n}^+(p)$ with $n \geq 3$ odd. Let H be the preimage in G of the stabiliser $\mathrm{GL}_n(p)$ of a totally isotropic subspace for $\mathrm{SO}_{2n}^+(p)$. As n is odd, $\mathrm{PSL}_n(p)$ has odd Schur multiplier, so the Steinberg character of $\mathrm{SL}_n(p)$ extends to a faithful p-Brauer character ψ of H, of degree $p^{n(n-1)/2}$. Thus, G has a faithful irreducible character of at least that degree. Then, by Steinberg's tensor product theorem, $\mathrm{Spin}_{2n}^+(q)$ has a faithful irreducible Brauer character of degree at least $(q/p)^{n^2}\psi(1)$, which is larger than $|G|_{p'}$ when $f \geq 2$. Now $\mathrm{Spin}_{2n+1}(p)$ contains a subgroup $\mathrm{Spin}_{2n}^+(p)$ whose centre lies in the centre of $\mathrm{Spin}_{2n+1}(p)$. As the Steinberg character of $\mathrm{Spin}_{2n+1}(p)$ has degree p^{n^2}, this shows that $\mathrm{Spin}_{2n+1}(q)$ has a character as claimed (where still n is odd). Unless $n = 3$ the same argument also applies to $\mathrm{HSpin}_{2n+2}^+(q)$, $\mathrm{Spin}_{2n+2}^-(q)$, $\mathrm{Spin}_{2n+3}(q)$ and $\mathrm{Spin}_{2n+4}^-(q)$.

The group $\mathrm{HSpin}_8^+(q)$ is isomorphic to $\mathrm{SO}_8^+(q)$ by the triality automorphism and thus by Proposition 3.3 has a faithful Brauer character of degree q^9. Using this, we also obtain our claim for $\mathrm{Spin}_9(q)$ and $\mathrm{Spin}_{10}^-(q)$. □

4. The non-defining characteristic case

We now turn to groups of Lie type G in cross characteristic. That is, we assume that p is not the defining characteristic of G. Here, Willems [13, Thm. 3.1] obtained certain asymptotic results on Conjecture 1, which were improved upon for several families of groups in the thesis of Maslowski [7]. Nonetheless, both of these fall short of proving Conjecture 3 in the case at hand.

We will again argue using suitable characters of p-defect zero, as considered by Willems [12], but in order to obtain complete results we need to show that there exist sufficiently many of these.

Let $\mathbf{G}$ be a simple algebraic group and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism. The regular semisimple elements are known to be dense in $\mathbf{G}$, so we certainly expect a large proportion of regular semisimple classes in the finite group $\mathbf{G}^F$. Here, motivated by our application in character theory, we quantify this expectation by giving a (rather weak) lower bound for their number.

Proposition 4.1. *Let* $(\mathbf{G}, F)$ *be as above, with*

$$\mathbf{G}^F \neq \mathrm{SU}_4(2),\ \mathrm{Sp}_4(2),\ \mathrm{Sp}_6(2),\ \mathrm{O}_8^+(2),\ \mathrm{O}_8^-(2),\ \mathrm{SU}_3(3).$$

Then a lower bound for the number $n_{\mathrm{reg}}(T)$ *of conjugacy classes of regular semisimple conjugacy classes meeting certain maximal tori* T *of* $\mathbf{G}^F$ *is bounded below as given in Table* 1 *for classical groups, and in Table* 2 *for exceptional type groups.*

Proof. The proof is independent of the isogeny type of $\mathbf{G}$. Let q be the absolute value of the eigenvalues of F on the character group of an F-stable maximal torus of $\mathbf{G}$. We will use the following well-known result of Zsigmondy: for every integer $e \geq 3$ and prime power q there exists a prime $z_e(q)$ dividing $q^e - 1$ but no $q^m - 1$ for $1 \leq m < e$, unless $(e, q) = (6, 2)$. Since q has multiplicative order e modulo $z_e(q)$, we see that $z_e(q) \geq e + 1$.

For each group $G = \mathbf{G}^F$ of classical type we have given in Table 1 (the order of) two maximal tori $T \leq G$. These two tori have been chosen such that the greatest common divisor of their orders is exactly the order of the centre of the simply connected group G_{sc} of that type, which is also the order of the commutator factor group of the adjoint group of that type. Furthermore, the image of T in $G_{\mathrm{sc}}/Z(G_{\mathrm{sc}})$ is cyclic, except for the first two tori listed for type D_n. For each torus the table also gives an integer e such that $|T|$ is divisible by a Zsigmondy prime $r := z_e(q)$, whenever that exists, that is, if $e \geq 3$ and $(e, q) \neq (6, 2)$. In all cases listed in the table, the description of Sylow p-subgroups in [6, Thm. 25.14] together with the order formula for G (see [6, Tab. 24.1]) shows that the Sylow r-subgroups of G are cyclic. Moreover, by the parametrisation of maximal tori via conjugacy classes of the Weyl group (see [6, Prop. 25.3]), the conjugates of T are the only maximal tori of G containing elements of order r, except for the torus of order $(q^{n-1} - 1)(q + 1)$ in types B_n, C_n. In particular, apart from that latter case, all elements of T of order divisible by r are necessarily regular. Now T certainly contains at least $|T|(r - 1)/r$ such elements. Since $r \geq e + 1$, we thus find at least $|T|e/(e + 1)$ regular elements in T. In the case of the torus T of order $(q^{n-1} - 1)(q + 1)$, all products of elements of order r by an element of order at least 3 in the factor of order $q + 1$ are regular, and thus there exist at least $(q^{n-1} - 1)(q - 1)e/(e + 1)$ regular elements in T.

The fusion of elements in any maximal torus is controlled by its normaliser, so the number $n_{\mathrm{reg}}(T)$ of regular conjugacy classes in G with representatives in T is at least $|N_G(T) : T|^{-1}$ times the number of regular elements in T. These are exactly the entries in the last column of Table 1.

We now discuss the cases in Table 1 when there do not always exist Zsigmondy primes. For $G = A_1(q)$ a maximal torus of order $q \pm 1$ contains at least $q \pm 1 - \gcd(2, q-1)$ regular elements and hence representatives from $(q \pm 1 - \gcd(2, q-1))/2$ regular conjugacy classes. For $G = A_2(q)$, it can be seen by direct calculation that a maximal torus of order $q^2 - 1$ intersects $(q^2 - q)/2 \geq |T|/3$ regular classes, as stated in the table. For $G = {}^2A_2(q)$, a maximal torus of order $q^2 - 1$ meets $(q + 1)(q - 2)/2$ regular classes, which is smaller than $|T|/3$ only when $q = 2, 3$. In the first case, G is solvable, the second was excluded. For $G = B_2(q)$ a maximal torus of order $q^2 - 1$ meets $(q - 1)(q - 2)/4$ regular semisimple classes, as stated in Table 1.

Table 1. Lower bounds in classical groups

G	$\lvert T\rvert$	e	$\lvert N_G(T):T\rvert$	$n_{\rm reg}(T)\geq$
$A_1(q)$	$q+1$	2	2	$(q-1)/2$
	$q-1$	1	2	$(q-3)/2$
$A_{n-1}(q),\ n\geq 3$	$(q^n-1)/(q-1)$	n	n	$\lvert T\rvert/(n+1)$
	$q^{n-1}-1$	$n-1$	$n-1$	$\lvert T\rvert/n$
${}^2A_{n-1}(q)$,	$(q^n+1)/(q+1)$	$2n$	n	$2\lvert T\rvert/(2n+1)$
$3\leq n\equiv 1\,(2)$	$q^{n-1}-1$	$n-1$	$n-1$	$\lvert T\rvert/n$
${}^2A_{n-1}(q)$,	$q^{n-1}+1$	$2n-2$	$n-1$	$2\lvert T\rvert/(2n-1)$
$4\leq n\equiv 0\,(2)$	$(q^n-1)/(q+1)$	n	n	$\lvert T\rvert/(n+1)$
$B_n(q), C_n(q),\ n\geq 2$	q^n+1	$2n$	$2n$	$\lvert T\rvert/(2n+1)$
$n=2:$	q^2-1	2	4	$(q-1)(q-2)/4$
$4\leq n\equiv 0\,(2):$	$(q^{n-1}-1)(q+1)$	$n-1$	$4n-4$	$(q^{n-1}-1)(q-1)/(4n)$
$n\equiv 1\,(2):$	q^n-1	n	$2n$	$\lvert T\rvert/(2n+2)$
$D_n(q),\ n\geq 4$	$(q^{n-1}+1)(q+1)$	$2n-2$	$2n-2$	$\lvert T\rvert/(2n-1)$
$n\equiv 0\,(2):$	$(q^{n-1}-1)(q-1)$	$n-1$	$2n-2$	$\lvert T\rvert/(2n)$
$n\equiv 1\,(2):$	q^n-1	n	n	$\lvert T\rvert/(n+1)$
${}^2D_n(q),\ n\geq 4$	q^n+1	$2n$	n	$2\lvert T\rvert/(2n+1)$
	$(q^{n-1}+1)(q-1)$	$2n-2$	$2n-2$	$\lvert T\rvert/(2n-1)$

To complete the discussion of Zsigmondy exceptions, we finally consider the cases when $(e,q)=(6,2)$ in Table 1. This concerns the following groups:

G	$\mathrm{SL}_6(2)$	$\mathrm{SL}_7(2)$	$\mathrm{SU}_4(2)$	$\mathrm{SU}_6(2)$	$\mathrm{SU}_7(2)$	$\mathrm{Sp}_6(2)$	$\mathrm{O}_8^+(2)$	$\mathrm{O}_8^-(2)$
$\lvert T\rvert$	63	63	9	21	63	9	27	9
$n_{\rm reg}(T)$	9	9	2	3	9	1	3	1

The numbers $n_{\rm reg}(T)$ can be read off from the known character tables. For $\mathrm{SL}_6(2)$, $\mathrm{SL}_7(2)$, $\mathrm{SU}_6(2)$ and $\mathrm{SU}_7(2)$ this agrees with the numbers given in Table 1, while the other groups are listed as exceptions.

We now turn to the groups of exceptional type, for which the arguments are very similar. In Table 2 for each type we give a maximal torus T, respectively two tori for G of type E_7. In all cases, T is cyclic by [6, Thm. 25.14] and for all groups different from $E_7(q)$, all elements of T of order not dividing $\lvert Z(G_{\rm sc})\rvert$ are regular by the order formula for G (see [6, Tab. 24.1]). For $E_7(q)$, the maximal tori are Coxeter tori of maximal rank subgroups of type ${}^2A_7(q)$, $A_7(q)$ respectively, and all of their elements not lying in the subgroup of order $q\pm 1$ are regular.

Each regular element in T is conjugate to $\lvert N_G(T):T\rvert$ elements of T. Since torus normalisers control fusion of their elements, we obtain the stated lower bounds for the number $n_{\rm reg}(T)$ of G-conjugacy classes of regular semisimple elements meeting T. □

Note that the total number of semisimple classes for G of simply connected type was shown by Steinberg to be equal to q^l, where l is the rank of $\mathbf{G}$. With a lot more effort

Table 2. Lower bounds in exceptional groups

G	$\lvert T\rvert$	$\lvert N_G(T):T\rvert$	$n_{\mathrm{reg}}(T)$
${}^2B_2(q^2),\ q^2 \geq 8$	Φ_8''	4	$(\lvert T\rvert-1)/4$
${}^2G_2(q^2),\ q^2 \geq 27$	Φ_{12}''	6	$(\lvert T\rvert-1)/6$
$G_2(q),\ q \equiv 1\,(3)$	Φ_6	6	$(\lvert T\rvert-1)/6$
$q \not\equiv 1\,(3)$	Φ_3	6	$(\lvert T\rvert-1)/6$
${}^3D_4(q)$	Φ_{12}	4	$(\lvert T\rvert-1)/4$
${}^2F_4(q^2),\ q^2 \geq 8$	Φ_{24}''	12	$(\lvert T\rvert-1)/12$
$F_4(q)$	Φ_{12}	12	$(\lvert T\rvert-1)/12$
$E_6(q)$	Φ_9	9	$(\lvert T\rvert-(3,q-1))/9$
${}^2E_6(q)$	Φ_{18}	9	$(\lvert T\rvert-(3,q+1))/9$
$E_7(q)$	$\Phi_2\Phi_{14}$	14	$(q^7-q)/14$
	$\Phi_1\Phi_7$	14	$(q^7-q)/14$
$E_8(q)$	Φ_{24}	24	$(\lvert T\rvert-1)/24$

Here $\Phi_8''=q^2+\sqrt{2}q+1$, $\Phi_{12}''=q^2+\sqrt{3}q+1$, $\Phi_{24}''=q^4+\sqrt{2}q^3+q^2+\sqrt{2}q+1$.

it would be possible to derive estimates for the number of regular semisimple classes that are asymptotically much closer to q^l.

We note the following easy consequence, which has been used in the proof of a recent result by Sambale [10, Thm. 10]:

Corollary 4.2. *Let G be a simple group of Lie type in characteristic p. Then there exist at least two conjugacy classes of elements of G with centraliser order prime to p.*

Proof. Let $\mathbf{G}$ be simple of simply connected type with a Steinberg endomorphism F such that $G=\mathbf{G}^F/Z(\mathbf{G}^F)$. This is possible unless $G={}^2F_4(2)'$, for which the claim is easily verified. Let $x\in\mathbf{G}^F$ be regular semisimple. Then its centraliser is a maximal torus, of order prime to p. Thus it suffices to show that $\mathbf{G}^F$ has at least two conjugacy classes of regular semisimple elements which do not have the same image in G. For exceptional type groups G it is immediate that the numbers $n_{\mathrm{reg}}(T)$ in Table 2 are all strictly bigger than $|Z(\mathbf{G}^F)|$, and so we obtain at least two regular semisimple G-classes. For the groups of classical type, we take regular semisimple classes coming from the two different types of maximal tori given in Table 1. For the exceptions in Proposition 4.1 the claim is again readily verified. □

Theorem 4.3. *Conjecture 3 holds for quasi-simple groups of Lie type for non-defining primes p.*

Proof. By [11, Prop. 2.1] the claim holds for the exceptional covering groups, thus we may assume that G is a central quotient $G=\mathbf{G}^F/Z$ of a quasi-simple group $\mathbf{G}^F$ of simply connected Lie type. We will construct sufficiently many irreducible Deligne–Lusztig characters of G of p-defect zero using Proposition 4.1. First, the assertion is readily checked from the known Brauer character tables for the six groups listed as exceptions in that result. So we may assume that G is not a covering group of one of those. As discussed in the proof of Proposition 4.1, for G of classical type, the two maximal tori of $\mathbf{G}^F$ in

Table 1 are such that their images in $\mathbf{G}^F/Z(\mathbf{G}^F)$ have coprime orders. Thus, we may choose at least one of them, say T, with $|T|_p = |Z(G)|_p$ for our given prime p. The same applies to groups of type E_7. For the remaining groups of exceptional type, all elements of $T \setminus Z(\mathbf{G}^F)$ are regular and the corresponding Sylow subgroups of G are cyclic. Since by Lemma 2.2 we need not consider primes p for which Sylow subgroups are cyclic, we again have that $|T|_p = |Z(G)|_p$ for the tori T in Table 2.

Let $\mathbf{T} \le \mathbf{G}$ be an F-stable maximal torus with $\mathbf{T}^F = T$, and let $(\mathbf{T}^*, \mathbf{G}^*, F)$ be dual to $\mathbf{T}, \mathbf{G}, F)$ (see [2, Def. 1.5.17]), so $T^* := \mathbf{T}^{*F}$ has the same order as T. To each regular element $s \in T^*$, up to G^*-conjugation, there exists an irreducible Deligne–Lusztig character $\pm R_s$ of $\mathbf{G}^F$ of degree $|\mathbf{G}^F : T|_{q'}$ (see [2, Def. 2.5.17 and Thm. 2.2.12]). Hence, by the choice of T, R_s is of central p-defect, that is, the defect of χ equals the p-part of the centre of $\mathbf{G}^F$. So the p-modular reduction of R_s is irreducible (see e.g. [8, Thm. (9.13)]). Now by the character formula [2, Prop. 2.2.18], R_s is a faithful character of $G = \mathbf{G}^F/Z$ with $Z = \ker(\theta) \cap Z(\mathbf{G}^F)$, where s is in duality with $(\mathbf{T}, \theta)$, for $\theta \in \mathrm{Irr}(T)$. By construction the Zsigmondy prime r for T occurring in the proof of Proposition 4.1 does not divide $|\mathbf{G}^{*F} : [\mathbf{G}^{*F}, \mathbf{G}^{*F}]|$. So we may choose a system of representatives R for the cosets of $T^* \cap [\mathbf{G}^{*F}, \mathbf{G}^{*F}]$ in T^* consisting of elements of order prime to r. Thus, if $s \in T^*$ has order divisible by r then so has st, for $t \in R$. It follows that the regular elements in T^* of order divisible by r are distributed equally across the various cosets of $T^* \cap [\mathbf{G}^{*F}, \mathbf{G}^{*F}]$ in T^*. Thus, the irreducible Deligne–Lusztig characters of central p-defect constructed above contribute

$$n_{\mathrm{reg}}(T)/|Z| \cdot |\mathbf{G}^F : T|_{q'}^2$$

to the sum in Conjecture 3. So, we need to see that

$$n_{\mathrm{reg}} \ge \frac{|G|_q \cdot |T|^2}{|G|_{q'} \cdot |P|}, \qquad (*)$$

where P denotes a Sylow p-subgroup of G.

For this, we estimate $|P|$. Since, as pointed out before, we only need to consider primes p for which Sylow p-subgroups of G are non-cyclic, p divides at least two (not necessarily distinct) cyclotomic factors occurring in the order formula for G (see [6, Thm. 25.14]). Note that if $p > 2$ and q has order d modulo p, then $p \ge d+1$, while for $p = 2$, at least one of $q-1$, $q+1$ is divisible at least by 4. Again first assume that $\mathbf{G}$ is of classical type, say of rank n. A Sylow p-subgroup P of G has order at least p^2. As the Weyl group W of $\mathbf{G}$ contains a symmetric group $\mathfrak{S}_n$, this shows $|P| \ge (n+1)^2$ if p does not divide the order of W. If $p \le n$, then $|P| \ge p^{\lfloor n/p \rfloor + p}$, which is still at least $(n+1)^2$ unless $p = 2$. But for $p = 2$ we have $|P| \ge 2^{\lfloor n/2 \rfloor + 4}$, and it follows that $|P| \ge (n+1)^2$ in all cases.

For groups of exceptional type, from the order formula for G (see [6, Tab. 24.1]) it is easy to check that the order of the smallest non-cyclic Sylow p-subgroups is at least 25, and even at least 121 when $G = E_8(q)$.

Now comparing $(*)$ with the right hand side of the formula in Conjecture 3 we see that our conclusion follows except when G is one of $\mathrm{SL}_3(2), \mathrm{Sp}_4(2)$, for which the claim is easily checked directly, or $G = \mathrm{Sp}_{2n}(q)$ or $\mathrm{Spin}_{2n+1}(q)$ with $n = 4$ and $q \le 5$, $n = 6$ and $q \le 4$, $n = 8, 10$ with $q \le 3$ or $(n, q) = (12, 2)$, and in all cases, p divides the order $q^n + 1$ of the first torus from Table 1. Among these, the only cases in which Sylow p-subgroups of G

are non-cyclic are for

$$(n,q,p)\in\{(6,2,5),(6,3,5),(6,4,17),(10,2,5),(10,3,5),(12,2,17)\}.$$

In all these cases except for $(n,p,q)=(6,2,5)$, the desired inequality holds when we use the exact order of P (which is at least p^3 here). Finally, for $\mathrm{Sp}_{12}(2)$, the ordinary character table allows us to see that the sum of squares of 5-defect zero characters exceeds the bound. □

5. Alternating groups

It remains to discuss the covering groups of simple alternating groups. At present we only see how to treat the prime $p=2$.

Theorem 5.1. *Let G be a covering group of the alternating group $\mathfrak{A}_n$, $n\geq 5$. Then Conjecture 3 holds for G at the prime 2.*

Proof. By Proposition 2.1 it is sufficient to consider the covering groups of the symmetric groups. The exceptional covering groups of $\mathfrak{A}_6$ and $\mathfrak{A}_7$ are easily checked, so we may assume that $n\geq 8$ and so $Z(G)=1$, that is $G=\mathfrak{S}_n$. We will show that for large enough n there exists a single irreducible 2-Brauer character whose degree satisfies the desired inequality. In fact, as for the groups of Lie type, it is the 2-modular reduction of an ordinary character, namely, a faithful character of the 2-fold cover $2.\mathfrak{S}_n$, but not of 2-defect zero.

For $l\geq 1$ consider the two partitions

$$p_1=(4l-3,4l-7,\dots,1)\vdash n_{l,1}:=l(2l-1),$$
$$p_2=(4l-1,4l-5,\dots,3)\vdash n_{l,2}:=l(2l+1).$$

These label irreducible characters χ_l^i of the 2-fold cover $2.\mathfrak{S}_{n_{l,i}}$ of $\mathfrak{S}_{n_{l,i}}$. Their degree is given by the analogue of the hook length formula (see [1, Thm. 2.8]): for a partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_m)\vdash n$ with distinct parts λ_i, the degree of the corresponding spin character of $2.\mathfrak{S}_n$ equals

$$2^{\lfloor (n-m)/2\rfloor}\frac{n!}{\prod_i\lambda_i!}\cdot\prod_{i<j}\frac{\lambda_i-\lambda_j}{\lambda_i+\lambda_j}.$$

From this one easily computes that

$$\frac{\chi_{l+1}^1(1)}{\chi_l^1(1)}=2^{4l-1}\binom{n_{l+1,1}}{4l+1}\binom{4l-1}{2l}^{-1}\quad\text{and}\quad\frac{\chi_{l+1}^2(1)}{\chi_l^2(1)}=2^{4l+1}\binom{n_{l+1,2}}{4l+3}\binom{4l+1}{2l+1}^{-1}.$$

We claim that for $l\geq 8$ we have

$$\chi_l^1(1)^2\geq (n_{l,2}-1)!_{2'}\qquad\text{and}\qquad\chi_l^2(1)^2\geq(n_{l+1,1}-1)!_{2'}\,. \tag{$*$}$$

Assume that the inequalities have already been shown up to l. Now

$$\binom{4l-1}{2l}=\frac{1}{2}\binom{4l}{2l}\leq 2^{4l-1}/\sqrt{4l}\quad\text{and}\quad\binom{4l+1}{2l+1}=\frac12\binom{4l+2}{2l+1}\leq 2^{4l+1}/\sqrt{4l+2},$$

by the standard estimate for the middle binomial coefficient, and so

$$\chi_{l+1}^1(1)^2\geq 4l\binom{n_{l+1,1}}{4l+1}^2(n_{l,2}-1)!_{2'},\qquad\chi_{l+1}^2(1)^2\geq(4l+2)\binom{n_{l+1,2}}{4l+3}^2(n_{l+1,1}-1)!_{2'}$$

by our inductive assumption. Further, with $n := n_{l+1,1}$ we have

$$\binom{n}{4l+1} \geq c_1\Big(\frac{n}{4l+1}\Big)^{4l+1}\Big(\frac{n}{n-4l-1}\Big)^{n-4l-1}\sqrt{\frac{n}{2\pi(4l+1)(n-4l-1)}}$$
$$\geq c_1\sqrt{\frac{1}{2\pi(4l+1)}}\Big(1+\frac{4l+1}{n-4l-1}\Big)^{n-4l-1}\Big(\frac{n}{4l+1}\Big)^{4l+1}$$
$$\geq c_2\sqrt{\frac{1}{2\pi(4l+1)}}\ e^{4l+1}\Big(\frac{n}{4l+1}\Big)^{4l+1}$$

for some constants $c_1, c_2 \geq 0.5$ independent of l by a well-known estimate for binomial coefficients. On the other hand by Stirling's formula

$$\frac{(n+2l+1)!}{(n-2l-2)!} \leq c_3\frac{\sqrt{2\pi(n+2l+1)}}{\sqrt{2\pi(n-2l-2)}}\cdot\frac{(n+2l+1)^{n+2l+1}\,e^{n-2l-2}}{(n-2l-2)^{n-2l-2}\,e^{n+2l-1}}$$
$$= c_4e^{-4l-3}\Big(1+\frac{4l+3}{n-2l-2}\Big)^{n-2l-2}(n+2l+1)^{4l+3}$$
$$\leq c_4(n+2l+1)^{4l+3},$$

where again $c_3, c_4 \leq 2$ are independent of l. Putting things together, this shows that

$$\frac{\chi_{l+1}^1(1)^2}{(n_{l+1,2}-1)!_{2'}} \geq \frac{c_2^2}{c_4}4l\frac{e^{8l+2}}{2\pi(4l+1)}\Big(\frac{n}{4l+1}\Big)^{8l+2}\Big(\frac{1}{n+2l+1}\Big)^{4l+3}\frac{(n_{l+1,2}-1)!_2}{(n_{l,2}-1)!_2}.$$

Now, using that $n = n_{l+1,1} = (l+1)(2l+1)$ we find

$$\frac{n^2}{(4l+1)^2(n+2l+1)} \geq \frac{1}{8},$$

and since $(n_{l+1,2}-1)!_2/(n_{l,2}-1)!_2 \geq 2^{4l+3}/(4l+3)$,

$$\frac{\chi_{l+1}^1(1)^2}{(n_{l+1,2}-1)!_{2'}} \geq c_5\frac{4l}{(4l+1)(4l+3)}\Big(\frac{1}{n+2l+1}\Big)^2\Big(\frac{e^2}{4}\Big)^{4l-1}.$$

The last term on the right hand side clearly dominates when $l \to \infty$ and so the left hand side is eventually bigger than 1. A more precise estimate and checking the first 50 values by computer then completes the proof of our claim for χ_l^1. A very similar computation shows (*) for χ_l^2.

The desired assertion then follows: by a result of Fayers [1, Thm. 3.3], both χ_l^1 and χ_l^2 remain irreducible modulo 2. So for all $n \geq n_{l+1,i}$, with $l \geq 8$, there exists an irreducible 2-Brauer character of $\mathfrak{S}_n$ of degree at least $\chi_{l+1}^i(1)$. By (*), $\chi_{l+1}^1(1)$ satisfies the inequality from Conjecture 3 for $\mathfrak{S}_n$ for all $n_{l+1,1} \leq n \leq n_{l+1,1}+2l+1 = n_{l+1,2}-1$, and similarly, $\chi_{l+1}^2(1)$ satisfies our inequality for $\mathfrak{S}_n$ whenever $n_{l+1,2} \leq n \leq n_{l+2,1}-1$. Finally, for $l < 8$, that is, for $n < 120$ it is easily checked using the criterion in [1, Thm. 3.3] that there always is an irreducible 2-Brauer character with the desired property, labelled by a sum of one of our partitions $\lambda_{l,1}$ or $\lambda_{l,2}$ and a suitable Carter partition. □

It should be noted that we need to use both characters χ_l^1 and χ_l^2 for our approach to work.

For odd primes, the combinatorics seem more daunting, and furthermore, there are two quite different cases to consider, corresponding to faithful and non-faithful characters of $2.\mathfrak{S}_n$. Note that by Lemma 2.2, we may assume that $p \leq n/2$ for $\mathfrak{S}_n$ and $2.\mathfrak{S}_n$.

Our Main Theorem 2 now follows by combining Proposition 3.4, Theorem 4.3 and Theorem 5.1 with the result of Tong-Viet [11, Prop. 2.1] for sporadic groups.

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