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SPLIT SPETSES FOR PRIMITIVE REFLECTION GROUPS

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Michel Broué, Gunter Malle, Jean Michel

Abstract. — Let W be an exceptional spetsial irreducible reflection group acting on a complex vector space V, *i.e.*, a group G_n for

$$n \in \{4, 6, 8, 14, 23, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37\}$$

in the Shephard-Todd notation. We describe how to determine some data associated to the corresponding (split) "spets" $\mathbb{G} = (V, W)$, given complete knowledge of the same data for all proper subspetses (the method is thus inductive).

The data determined here are the set $\mathrm{Uch}(\mathbb{G})$ of "unipotent characters" of $\mathbb{G}$ and its repartition into families, as well as the associated set of Frobenius eigenvalues. The determination of the Fourier matrices linking unipotent characters and "unipotent character sheaves" will be given in another paper.

The approach works for all split reflection cosets for primitive irreducible reflection groups. The result is that all the above data exist and are unique (note that the cuspidal unipotent degrees are only determined up to sign).

We keep track of the complete list of axioms used. In order to do that, we explain in detail some general axioms of "spetses", generalizing (and sometimes correcting) [**BMM99**] along the way.

Note that to make the induction work, we must consider a class of reflection cosets slightly more general than the split irreducibles ones: the reflection cosets with split semi-simple part, *i.e.*, cosets $(V, W\varphi)$ such that $V = V_1 \oplus V_2$ with $W \subset \mathrm{GL}(V_1)$ and $\varphi|_{V_1} = \mathrm{Id}$. We need also to consider some non-exceptional cosets, those associated to imprimitive complex reflection groups which appear as parabolic subgroups of the exceptional ones.

***Résumé* (Spetses déployés pour les groupes de réflexion primitifs)**

Soit W un groupe de réflexions spetsial exceptionnel agissant sur un espace vectoriel complexe V, *i.e.*, un groupe G_n (dans la notation de Shephard–Todd) pour

$$n \in \{4, 6, 8, 14, 23, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37\}\,.$$

Nous décrivons comment calculer des données attachées au "spets" $\mathbb{G} = (V, W)$ déployé correspondant, si nous connaissons les mêmes données pour tous les sous-spetses propres (la méthode est donc récursive).

Les donnes déterminées ici sont l'ensemble $\mathrm{Uch}(\mathbb{G})$ des "caractères unipotents" de $\mathbb{G}$ et sa répartition en familles, ainsi que l'ensemble des valeurs propres de Frobenius associées. La détermination des matrices de Fourier reliant les caractères unipotents aux "faisceaux caractères unipotents" sera donnée dans un prochain article.

Cette approche s'applique aussi bien à toutes les données de réflexions primitives irréductibles "presque tordues". Notre principal résultat est que les données mentionnées ci-dessus existent et sont uniques (noter que les degrés unipotents cuspidaux ne sont déterminés qu'au signe près).

Nous précisons la liste complète des axiomes utilisés. Dans ce but, nous exposons en détail quelques-uns des axiomes généraux des "spetses", généralisant (et parfois corrigeant) ainsi [**BMM99**].

Il est à noter que, pour appliquer la méthode inductive, nous devons considérer une classe de données de réflexions plus générale que les données déployées irréductibles : celles dont la partie semi-simple est déployée, *i.e.*, les données $(V, W\varphi)$ telles que $V = V_1 \oplus V_2$ avec $W \subset \mathrm{GL}(V_1)$ et $\varphi|_{V_2} = \mathrm{Id}$. Nous devons également considérer quelques données de réflexions non-exceptionnelles qui apparaissent comme sous-données paraboliques de données exceptionnelles.

CONTENTS

FROM WEYL GROUPS TO COMPLEX REFLECTION GROUPS

Let $\mathbf{G}$ be a connected reductive algebraic group over an algebraic closure of a finite field $\mathbb{F}_q$ and $F : \mathbf{G} \to \mathbf{G}$ an isogeny such that F^δ (where δ is a natural integer) defines an $\mathbb{F}_{q^\delta}$-rational structure on $\mathbf{G}$. The group of fixed points $G := \mathbf{G}^F$ is a finite group of Lie type, also called finite reductive group. Lusztig has given a classification of the irreducible complex characters of such groups. In particular he has constructed the important subset $\mathrm{Un}(G)$ of unipotent characters of G. In a certain sense, which is made precise by Lusztig's Jordan decomposition of characters, the unipotent characters of G and of various Levi subgroups of G determine all irreducible characters of G.

The unipotent characters are constructed as constituents of representations of G on certain ℓ-adic cohomology groups, on which F^δ also acts. Lusztig shows that for a given unipotent character $\gamma \in \mathrm{Un}(G)$, there exists a root of unity or a root of unity times the square root of q^δ, that we denote $\mathrm{Fr}(\gamma)$, such that the eigenvalue of F^δ on any γ-isotypic part of such ℓ-adic cohomology groups is given by $\mathrm{Fr}(\gamma)$ times an integral power of q^δ.

The unipotent characters are naturally partitioned into so-called Harish-Chandra series, as follows. If $\mathbf{L}$ is an F-stable Levi subgroup of some F-stable parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, then *Harish-Chandra induction*

$$R_L^G := \mathrm{Ind}_P^G \circ \mathrm{Infl}_L^P : \mathbb{Z}\mathrm{Irr}(L) \longrightarrow \mathbb{Z}\mathrm{Irr}(G)$$

where $L := \mathbf{L}^F$ and $P := \mathbf{P}^F$ defines a homomorphism of character groups independent of the choice of $\mathbf{P}$. A unipotent character of G is called *cuspidal* if it does not occur in $R_L^G(\lambda)$ for any proper Levi subgroup $L < G$ and any $\lambda \in \mathrm{Un}(L)$. The set of constituents

$$\mathrm{Un}(G, (L, \lambda)) := \{\gamma \in \mathrm{Un}(G) \mid \langle \gamma, R_L^G(\lambda) \rangle \neq 0\}$$

where $\lambda \in \mathrm{Un}(L)$ is cuspidal, is called the *Harish-Chandra series* above (L, λ). It can be shown that the Harish-Chandra series form a partition of $\mathrm{Un}(G)$, if (L, λ) runs over a system of representatives of the G-conjugacy classes of such pairs. Thus, given $\gamma \in \mathrm{Un}(G)$ there is a unique pair (L, λ) up to conjugation such that L is a Levi subgroup of G, $\lambda \in \mathrm{Un}(L)$ is cuspidal and γ occurs as a constituent in $R_L^G(\lambda)$. Furthermore, if $\gamma \in \mathrm{Un}(G, (L, \lambda))$ then $\mathrm{Fr}(\gamma) = \mathrm{Fr}(\lambda)$.

Now let $W_G(L, \lambda) := N_G(\mathbf{L}, \lambda)/L$, the *relative Weyl group* of (L, λ). This is always a finite Coxeter group. Then $\mathrm{End}_{\mathbb{C}G}(R_L^G(\lambda))$ is an Iwahori-Hecke algebra $\mathcal{H}(W_G(L, \lambda))$ for $W_G(L, \lambda)$ for a suitable choice of parameters. This gives a natural parametrization of $\mathrm{Un}(G, (L, \lambda))$ by characters of $\mathcal{H}(W_G(L, \lambda))$, and thus, after a choice of a suitable specialization for the corresponding generic Hecke algebra, a parametrization

$$\mathrm{Irr}(W_G(L, \lambda)) \longrightarrow \mathrm{Un}(G, (L, \lambda)), \quad \chi \mapsto \gamma_\chi,$$

of the Harish-Chandra series above (L, λ) by $\mathrm{Irr}(W_G(L, \lambda))$. In particular, the characters in the *principal series* $\mathrm{Un}(G, (T, 1))$, where T denotes a maximally split torus, are indexed by $\mathrm{Irr}(W^F)$, the irreducible characters of the F-fixed points of the Weyl group W.

More generally, if $d \geq 1$ is an integer and if $\mathbf{T}$ is an F-stable subtorus of $\mathbf{G}$ such that

- $\mathbf{T}$ splits completely over $\mathbb{F}_{q^d}$
- but no subtorus of $\mathbf{T}$ splits over any smaller field,

then its centralizer $\mathbf{L} := C_{\mathbf{G}}(\mathbf{T})$ is an F-stable *d-split* Levi subgroup (not necessarily lying in an F-stable parabolic subgroup). We assume here and in the rest of the introduction that F is a Frobenius endomorphism to simplify the exposition; for the "very twisted" Ree and Suzuki groups one has to replace d by a cyclotomic polynomial over an extension of the rationals as is done in 1.48.

Here, again using ℓ-adic cohomology of suitable varieties Lusztig induction defines a linear map

$$R_L^G : \mathbb{Z}\mathrm{Irr}(L) \longrightarrow \mathbb{Z}\mathrm{Irr}(G),$$

where again $L := \mathbf{L}^F$. As before we say that $\gamma \in \mathrm{Un}(G)$ is *d-cuspidal* if it does not occur in $R_L^G(\lambda)$ for any proper d-split Levi subgroup $L < G$ and any $\lambda \in \mathrm{Un}(L)$, and we write $\mathrm{Un}(G, (L, \lambda))$ for the set of constituents of $R_L^G(\lambda)$, when $\lambda \in \mathrm{Un}(L)$ is d-cuspidal. By [**BMM93**, 3.2(1)] these *d-Harish-Chandra series*, for any fixed d, again form a partition of $\mathrm{Un}(G)$. The relative Weyl groups $W_G(L, \lambda) := N_G(\mathbf{L}, \lambda)/L$ are now in general complex reflection groups. It is shown (see [**BMM93**, 3.2(2)]) that again there exists a parametrization of $\mathrm{Un}(G, (L, \lambda))$ by the irreducible characters of some cyclotomic Hecke algebra $\mathcal{H}(W_G(L, \lambda))$ of $W_G(L, \lambda)$ and hence, after a choice of a suitable specialization for the corresponding generic Hecke algebra, a parametrization

$$\mathrm{Irr}(W_G(L, \lambda)) \longrightarrow \mathrm{Un}(G, (L, \lambda)), \quad \chi \mapsto \gamma_\chi,$$

of the d-Harish-Chandra series above (L, λ) by $\mathrm{Irr}(W_G(L, \lambda))$. Furthermore, there exist signs ϵ_χ such that the degrees of characters belonging to $\mathrm{Un}(G, (L, \lambda))$ are given by

$$\gamma_\chi(1) = \epsilon_\chi\,\lambda(1)/S_\chi\,,$$

where S_χ denotes the Schur element of χ with respect to the canonical trace form on $\mathcal{H}(W_G(L,\lambda))$ (see [**Mal00**, §7] for references).

Attached to $(\mathbf{G}, F)$ is the set $\mathrm{Ucsh}(G)$ of characteristic functions of F-stable unipotent character sheaves of $\mathbf{G}$. Lusztig showed that these are linearly independent and span the same subspace of $\mathbb{C}\mathrm{Irr}(G)$ as $\mathrm{Un}(G)$. The base change matrix S from $\mathrm{Un}(G)$ to $\mathrm{Ucsh}(G)$ is called the Fourier matrix of G. Define an equivalence relation on $\mathrm{Un}(G)$ as the transitive closure of the following relation:

$$\gamma \sim \gamma' \iff \text{ there exists } A \in \mathrm{Ucsh}(G) \text{ with } \langle\gamma, A\rangle \neq 0 \neq \langle\gamma', A\rangle\,.$$

The equivalence classes of this relation partition $\mathrm{Un}(G)$ (and also $\mathrm{Ucsh}(G)$) into so-called *families*. Lusztig shows that the intersection of any family with the principal series $\mathrm{Un}(G,(T,1))$, is a two-sided cell in $\mathrm{Irr}(W^F)$ (after identification of $\mathrm{Irr}(W^F)$ with the principal series $\mathrm{Un}(G,(T,1))$ as above).

All of the above data are *generic* in the following sense. Let $\mathbb{G}$ denote the complete root datum of $(\mathbf{G}, F)$, that is, the root datum of $\mathbf{G}$ together with the action of $q^{-1}F$ on it. Then there is a set $\mathrm{Uch}(\mathbb{G})$, together with maps

$$\mathrm{Deg} : \mathrm{Uch}(\mathbb{G}) \longrightarrow \mathbb{Q}[x]\,,\ \gamma \mapsto \mathrm{Deg}(\gamma)\,,$$
$$\lambda : \mathrm{Uch}(\mathbb{G}) \longrightarrow \mathbb{C}^\times[x^{1/2}]\,,\ \gamma \mapsto \mathrm{Fr}(\gamma)\,,$$

such that for all groups $(\mathbf{G}', F')$ with the same complete root datum $\mathbb{G}$ (where ${F'}^\delta$ defines a $\mathbb{F}_{q'^\delta}$-rational structure) there are bijections $\psi_{G'} : \mathrm{Uch}(\mathbb{G}) \longrightarrow \mathrm{Un}({\mathbf{G}'}^{F'})$ satisfying

$$\psi_{G'}(\gamma)(1) = \mathrm{Deg}(\gamma)(q') \quad \text{and} \quad \mathrm{Fr}(\psi_{G'}(\gamma)) = \mathrm{Fr}(\gamma)(q'^\delta)\,.$$

Furthermore, by results of Lusztig and Shoji, Lusztig induction R_L^G of unipotent characters is generic, that is, for any complete Levi root subdatum $\mathbb{L}$ of $\mathbb{G}$ with corresponding Levi subgroup L of G there is a linear map

$$R_{\mathbb{L}}^{\mathbb{G}} : \mathbb{Z}\mathrm{Uch}(\mathbb{L}) \longrightarrow \mathbb{Z}\mathrm{Uch}(\mathbb{G})$$

satisfying

$$R_L^G \circ \psi_L = \psi_G \circ R_{\mathbb{L}}^{\mathbb{G}}$$

(see [**BMM93**, 1.33]).

The following has been observed on the data: for W irreducible and any scalar $\xi \in Z(W)$ there is a permutation with signs E_ξ of $\mathrm{Uch}(\mathbb{G})$ such that

$$\mathrm{Deg}(E_\xi(\gamma))(x) = \mathrm{Deg}(\gamma)(\xi^{-1}x)\,.$$

We call this the *Ennola-transform*, by analogy with what Ennola first observed on the relation between characters of general linear and unitary groups. In the case considered here, $Z(W)$ has order at most 2. Such a permutation E_ξ turns out to be of order the square of the order of ξ if W is of type E_7 or E_8, and of the same order as ξ otherwise.

Thus, to any pair consisting of a finite Weyl group W and the automorphism induced by F on its reflection representation, is associated a complete root datum $\mathbb{G}$, and to this is associated a set $\mathrm{Uch}(\mathbb{G})$ with maps Deg, Fr, E_ξ $(\xi \in Z(W))$ and linear maps $R_{\mathbb{L}}^{\mathbb{G}}$ for any Levi subdatum $\mathbb{L}$ satisfying a long list of properties.

Our aim is to try and treat a complex reflection group as a Weyl group of some yet unknown object. Given W a finite subgroup generated by (pseudo)-reflections of a finite dimensional complex vector space V, and a finite order automorphism φ of V which normalizes W, we first define the corresponding *reflection coset* by $\mathbb{G} := (V, W\varphi)$. Then we try to build "unipotent characters" of $\mathbb{G}$, or at least to build their degrees (polynomials in x), Frobenius eigenvalues (roots of unity times a power (modulo 1) of x); in a coming paper we shall build their Fourier matrices.

Lusztig (see [**Lus93**] and [**Lus94**]) knew already a solution for Coxeter groups which are not Weyl groups, except the Fourier matrix for H_4 which was determined by Malle in 1994 (see [**Mal94**]).

Malle gave a solution for imprimitive spetsial complex reflection groups in 1995 (see [**Mal95**]) and proposed (unpublished) data for many primitive spetsial groups.

Stating now a long series of precise axioms — many of a technical nature — we can now show that *there is a unique solution for all primitive spetsial complex reflection groups, i.e.*, groups G_n for $n \in \{4, 6, 8, 14, 23, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37\}$ in the Shephard–Todd notation, and the symmetric groups.

Let us introduce our basic objects and some notation.

- A complex vector space V of dimension r, a finite reflection subgroup W of $\mathrm{GL}(V)$, a finite order element $\varphi \in N_{\mathrm{GL}(V)}(W)$.
- $\mathcal{A}(W) :=$ the reflecting hyperplanes arrangement of W, and for $H \in \mathcal{A}(W)$,
    - $W_H :=$ the fixator of H in W, a cyclic group of order e_H,
    - $j_H :=$ an eigenvector for reflections fixing H.
- $N_W^{\mathrm{hyp}} := |\mathcal{A}(W)|$ the number of reflecting hyperplanes.

The action of $N_{\mathrm{GL}(V)}(W)$ on the monomial of degree N_W^{hyp}

$$\prod_{H \in \mathcal{A}(W)} j_H \in SV$$

defines a linear character of $N_{\mathrm{GL}(V)}(W)$, which coincides with $\det_V$ on restriction to W, hence (by quotient with $\det_V$) defines a character

$$\theta : N_{\mathrm{GL}(V)}(W) \longrightarrow N_{\mathrm{GL}(V)}(W)/W \longrightarrow \mathbb{C}^\times .$$

We set

$$\mathbb{G} = (V, W\varphi)$$

and we define the "polynomial order" of $\mathbb{G}$ by the formula

$$|\mathbb{G}| := (-1)^r \theta(\varphi)\, x^{N_W^{\mathrm{hyp}}} \frac{1}{\frac{1}{|W|}\sum_{w\in W} \frac{1}{\det_V(1-w\varphi x)^*}} \in \mathbb{C}[x]$$

(where z^* denotes the complex conjugate of the complex number z).

Notice that when W is a true Weyl group and φ is a graph automorphism, then the polynomial $|\mathbb{G}|$ is the order polynomial discovered by Steinberg for the corresponding family of finite reductive groups.

A particular case.—

Let us quickly state our results for the cyclic group of order 3, the smallest complex reflection group which is not a Coxeter group.

For the purposes of that short exposition, we give some ad hoc definitions of the main notions (Hecke algebras, Schur elements, unipotent characters, ξ-series, etc.) which will be given in a more general and more systematic context in the paper below.

Let $\zeta := \exp(\frac{2\pi i}{3})$. We have

$$V := \mathbb{C}\,,\ W := \langle\zeta\rangle\,,\ \varphi := 1\,,\ \mathbb{G} := (\mathbb{C}, W)\,, \quad N^{\mathrm{hyp}} = 1\,,$$

$$|\mathbb{G}| = x(x^3-1)\,.$$

Generic Hecke algebra $\mathcal{H}(W,(a,b,c))$. —

For indeterminates a, b, c, we define an algebra over $\mathbb{Z}[a^{\pm1}, b^{\pm1}, c^{\pm1}]$ by

$$\mathcal{H}(W,(a,b,c)) := \langle\, \mathbf{s} \mid (\mathbf{s}-a)(\mathbf{s}-b)(\mathbf{s}-c) = 0\,\rangle\,.$$

The algebra $\mathcal{H}(W,(a,b,c))$ has three linear characters χ_a, χ_b, χ_c defined by $\chi_t(\mathbf{s}) = t$ for $t \in \{a,b,c\}$.

Canonical trace. —

The algebra $\mathcal{H}(W,(a,b,c))$ is endowed with the symmetrizing form defined by

$$\tau(\mathbf{s}^n) := \begin{cases} \displaystyle\sum_{\substack{\alpha,\beta,\gamma>0\\ \alpha+\beta+\gamma=n}} a^\alpha b^\beta c^\gamma & \text{for } n>0\,,\\ \displaystyle\sum_{\substack{\alpha,\beta,\gamma\le 0\\ \alpha+\beta+\gamma=n}} a^\alpha b^\beta c^\gamma & \text{for } n\le 0\,.\end{cases}$$

Schur elements of $\mathcal{H}(W,(a,b,c))$. —

We define three elements of $\mathbb{Z}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$ which we call Schur elements by

$$S_a = \frac{(b-a)(c-a)}{bc}\ ,\ S_b = \frac{(c-b)(a-b)}{ca}\ ,\ S_c = \frac{(a-c)(b-c)}{ab}\ ,$$

so that

$$\tau = \frac{\chi_a}{S_a} + \frac{\chi_b}{S_b} + \frac{\chi_c}{S_c}\,.$$

Spetsial Hecke algebra for the principal series. —
This is the specialization of the generic Hecke algebra defined by

$$\mathcal{H}(W,(x,\zeta,\zeta^2)) := \langle\, \mathbf{s} \mid (\mathbf{s}-x)(1+\mathbf{s}+\mathbf{s}^2) = 0\,\rangle\,.$$

Note that the specialization to the group algebra factorizes through this.

Unipotent characters. —

There are 4 unipotent characters of $\mathbb{G}$, denoted $\rho_0, \rho_\zeta, \rho_\zeta^*, \rho$. Their degrees and Frobenius eigenvalues, are given by the following table:

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
ρ_0	1	1
ρ_ζ	$\dfrac{1}{1-\zeta^2}x(x-\zeta^2)$	1
ρ_ζ^*	$\dfrac{1}{1-\zeta}x(x-\zeta)$	1
ρ	$\dfrac{-\zeta}{1-\zeta^2}x(x-1)$	ζ^2

We set $\mathrm{Uch}(\mathbb{G}) := \{\rho_0, \rho_\zeta, \rho_\zeta^*, \rho\}\,.$

Families. —

$\mathrm{Uch}(\mathbb{G})$ splits into two families: $\{\rho_0\}\,,\ \{\rho_\zeta, \rho_\zeta^*, \rho\}\,.$

Principal ξ*-series for* ξ *taking values* $1, \zeta, \zeta^2$. —

1. We define the principal ξ-series by
$$\mathrm{Uch}(\mathbb{G},\xi) := \{\gamma \mid \mathrm{Deg}(\gamma)(\xi) \neq 0\}\,,$$
and we say that a character γ is ξ-cuspidal if
$$\left(\frac{|\mathbb{G}|(x)}{\mathrm{Deg}(\gamma)(x)}\right)|_{x=\xi} \neq 0\,.$$
2. $\mathrm{Uch}(\mathbb{G}) = \mathrm{Uch}(\mathbb{G},\xi) \sqcup \{\gamma_\xi\}$ where γ_ξ is ξ-cuspidal.

3. Let $\mathcal{H}(W,\xi) := \mathcal{H}(W,(\xi^{-1}x,\zeta,\zeta^2))$ be the specialization of the generic Hecke algebra at $a=\xi^{-1}x\,, b=\zeta\,, c=\zeta^2$. There is a natural bijection

$$\mathrm{Irr}\,(\mathcal{H}(W,\xi)) \xrightarrow{\sim} \mathrm{Uch}(\mathbb{G},\xi) \quad , \quad \chi_t \mapsto \gamma_t \quad \text{for } t=a,b,c$$

such that:

(a) We have

$$\mathrm{Deg}(\gamma_t)(x) = \pm\left(\frac{x^3-1}{1-\xi x}\right)\frac{1}{S_t(x)}\,,$$

where $S_t(x)$ denotes the corresponding specialized Schur element.

(b) The intersections of the families with the set $\mathrm{Uch}(\mathbb{G},\xi)$ correspond to the Rouquier blocks of $\mathcal{H}(W,\xi)$.

Fourier matrix. —

The Fourier matrix for the 3-element family is

$$\frac{\zeta}{1-\zeta^2}\begin{pmatrix}\zeta^2 & -\zeta & -1\\ -\zeta & \zeta^2 & 1\\ -1 & 1 & 1\end{pmatrix}.$$

CHAPTER 1

REFLECTION GROUPS, BRAID GROUPS, HECKE ALGEBRAS

The following notation will be in force throughout the paper.

We denote by $\mathbb{N}$ the set of nonnegative integers.

We denote by $\boldsymbol{\mu}$ the group of all roots of unity in $\mathbb{C}^\times$. For $n \geq 1$, we denote by $\boldsymbol{\mu}_n$ the subgroup of n-th roots of unity, and we set $\zeta_n := \exp(2\pi i/n) \in \boldsymbol{\mu}_n$.

If K is a number field, a subfield of $\mathbb{C}$, we denote by $\mathbb{Z}_K$ the ring of algebraic integers of K. We denote by $\boldsymbol{\mu}(K)$ the group of roots of unity in K, and we set $m_K := |\boldsymbol{\mu}(K)|$. We denote by $\overline{K}$ the algebraic closure of K in $\mathbb{C}$.

We denote by $z \mapsto z^*$ the complex conjugation on $\mathbb{C}$. For a Laurent polynomial $P(x) \in \mathbb{C}[x, x^{-1}]$, we set $P(x)^\vee := P(1/x)^*$.

1.1. Complex reflection groups and reflection cosets

1.1.1. Some notation. —

Let V be a finite dimensional complex vector space, and let W be a finite subgroup of $\mathrm{GL}(V)$ generated by reflections (a *finite complex reflection group*).

We denote by $\mathcal{A}(W)$ (or simply by $\mathcal{A}$ when there is no ambiguity) the set of reflecting hyperplanes of reflections in W. If $H \in \mathcal{A}(W)$, we denote by e_H the order of the fixator W_H of H in W, a cyclic group consisting of 1 and all reflections around H. Finally, we call *distinguished reflection* around H the reflection with reflecting hyperplane H and non trivial eigenvalue $\exp(2\pi i/e_H)$.

An element of V is called *regular* if it belongs to none of the reflecting hyperplanes. We denote by V^{reg} the set of regular elements, that is

$$V^{\mathrm{reg}} = V - \bigcup_{H \in \mathcal{A}(W)} H \,.$$

We set

$$N_W^{\mathrm{ref}} := |\{w \in W \mid w \text{ is a reflection}\}| \quad\text{and}\quad N_W^{\mathrm{hyp}} := |\mathcal{A}(W)|$$

so that $N_W^{\mathrm{ref}} = \sum_{H\in\mathcal{A}(W)}(e_H - 1)$ and $N_W^{\mathrm{hyp}} = \sum_{H\in\mathcal{A}(W)} 1$. We set

$$e_W := \sum_{H\in\mathcal{A}(W)} e_H = N_W^{\mathrm{ref}} + N_W^{\mathrm{hyp}}\,,\ \text{so that } e_H = e_{W_H}\,. \tag{1.1}$$

The *parabolic subgroups* of W are by definition the fixators of subspaces of V: for $I \subseteq V$, we denote by W_I the fixator of I in W. Then the map

$$I \mapsto W_I$$

is an order reversing bijection from the set of intersections of elements of $\mathcal{A}(W)$ to the set of parabolic subgroups of W.

1.1.2. Some linear characters. —

Let W be a reflection group on V. Let SV be the symmetric algebra of V, and let SV^W be the subalgebra of elements fixed by W.

For $H \in \mathcal{A}(W)$, let us denote by $j_H \in V$ an eigenvector for the group W_H which does not lie in H, and let us set

$$J_W := \prod_{H\in\mathcal{A}(W)} j_H \in SV\,,$$

an element of the symmetric algebra of V well defined up to multiplication by a nonzero scalar, homogeneous of degree N_W^{hyp}.

For $w \in W$, we have (see e.g. [**Bro10**, 4.3.2]) $w.J_W = \det_V(w)J_W$ and more generally, there is a linear character on $N_{\mathrm{GL}(V)}(W)$ extending $\det_V|_W$ and denoted by $\widetilde{\det}_V^{(W)}$, defined as follows:

$$\nu.J_W = \widetilde{\det}_V^{(W)}(\nu)J_W \quad \text{for all } \nu \in N_{\mathrm{GL}(V)}(W)\,.$$

Remark 1.2. — The character $\widetilde{\det}_V^{(W)}$ is in general different from $\det_V$, as can easily be seen by considering its values on the center of $\mathrm{GL}(V)$. But by what we said above it coincides with $\det_V$ on restriction to W.

It induces a linear character

$$\det{}'_V : N_{\mathrm{GL}(V)}(W)/W \to \mathbb{C}^\times$$

defined as follows: for $\overline{\varphi} \in N_{\mathrm{GL}(V)}(W)/W$ with preimage $\varphi \in N_{\mathrm{GL}(V)}(W)$, we set

$$\det{}'_V(\overline{\varphi}) := \widetilde{\det}_V^{(W)}(\varphi)\det_V(\varphi)^*\,.$$

Similarly, the element (of degree N_W^{ref}) of SV defined by

$$J_W^\vee := \prod_{H \in \mathcal{A}(W)} j_H^{e_H - 1}$$

defines a linear character $\widetilde{\det}_V^{(W)\vee}$ on $N_{\mathrm{GL}(V)}(W)$, which coincides with $\det_V^{-1}$ on restriction to W, hence a character

$$\det_V'^{\vee} : N_{\mathrm{GL}(V)}(W)/W \to \mathbb{C}^\times \quad , \quad \overline{\varphi} \mapsto \widetilde{\det}_V^{(W)\vee}(\varphi)\det_V(\varphi)\,.$$

The *discriminant*, element of degree $N_W^{\mathrm{hyp}} + N_W^{\mathrm{ref}}$ of SV^W defined by

$$\mathrm{Disc}_W := J_W J_W^\vee = \prod_{H \in \mathcal{A}(W)} j_H^{e_H}\,,$$

defines a character

$$\Delta_W := \det_V' \det_V'^{\vee} : N_{\mathrm{GL}(V)}(W)/W \to \mathbb{C}^\times\,.$$

Let $\varphi \in N_{\mathrm{GL}(V)}(W)$ be an element of finite order. Let ζ be a root of unity. We recall (see [**Spr74**] or [**BM96**]) that an element $w\varphi \in W\varphi$ is called ζ*-regular* if there exists an eigenvector for $w\varphi$ in V^{reg} with eigenvalue ζ.

Lemma 1.3. — *Assume that $w\varphi$ is ζ-regular. Then*

1. $\widetilde{\det}_V^{(W)}(w\varphi) = \zeta^{N_W^{\mathrm{hyp}}}$ *and* $\widetilde{\det}_V^{(W)\vee}(w\varphi) = \zeta^{N_W^{\mathrm{ref}}}$.
2. $\det_V'(\overline{\varphi}) = \zeta^{N_W^{\mathrm{hyp}}}\det_V(w\varphi)^{-1}$ *and* $\det_V'^{\vee}(\overline{\varphi}) = \zeta^{N_W^{\mathrm{ref}}}\det_V(w\varphi)$.
3. $\Delta_W(\overline{\varphi}) = \zeta^{e_W}$.

Proof. —
Let V^* be the dual of V. We denote by $\langle \cdot , \cdot \rangle$ the natural pairing $V^* \times V \to K$, which extends naturally to a pairing $V^* \times SV \to K$ "evaluation of functions on V^*".

We denote by $V^{*\mathrm{reg}}$ the set of elements of V^* fixed by none of the reflections of W (acting on the right by transposition): this is the set of regular elements of V^* for the complex reflection group W acting through the contragredient representation.

Let $\alpha \in V^{*\mathrm{reg}}$ be such that $\alpha w\varphi = \zeta\alpha$. Since α is regular, we have $\langle \alpha, J_W \rangle \neq 0$. But

$$\begin{cases} \langle \alpha w\varphi, J_W \rangle = \zeta^{N_W^{\mathrm{hyp}}} \langle \alpha, J_W \rangle \\ \langle \alpha, w\varphi(J_W) \rangle = \widetilde{\det}_V^{(W)}(w\varphi)\langle \alpha, J_W \rangle \end{cases}$$

which shows that $\widetilde{\det}_V^{(W)}(w\varphi) = \zeta^{N_W^{\mathrm{hyp}}}$. A similar proof using $J_W^\vee$ shows that $\widetilde{\det}_V^{(W)\vee}(w\varphi) = \zeta^{N_W^{\mathrm{ref}}}$. Assertions (2) and (3) are then immediate. □

Remark 1.4. — As a consequence of the preceding lemma, we see that if $w \in W$ is a ζ-regular element, then

$$\det_V(w) = \zeta^{N_W^{\mathrm{hyp}}}\ ,\ \det_V(w)^{-1} = \zeta^{N_W^{\mathrm{ref}}} \text{ and thus } \zeta^{e_W} = 1\,.$$

1.1.3. Field of definition. —

The following theorem has been proved through a case by case analysis [**Ben76**] (see also [**Bes97**]).

Theorem 1.5. —

Let W be a finite reflection group on V. Then the field

$$\mathbb{Q}_W := \mathbb{Q}(\mathrm{tr}_V(w) \mid w \in W)$$

is a splitting field for all complex representations of W.

The ring of integers of $\mathbb{Q}_W$ will be denoted by $\mathbb{Z}_W$. If L is any number field, we set

$$L_W := L((\mathrm{tr}_V(w))_{w\in W})\,,$$

the composite of L with $\mathbb{Q}_W$.

1.1.4. Reflection cosets. —

Following [**BMM99**], we set the following definition.

Definition 1.6. — *A* reflection coset *on a characteristic zero field K is a pair $\mathbb{G} = (V, W\varphi)$ where*

- *V is a finite dimensional K-vector space,*
- *W is a finite subgroup of $\mathrm{GL}(V)$ generated by reflections,*
- *φ is an element of finite order of $N_{\mathrm{GL}(V)}(W)$.*

We then denote by

- $\overline{\varphi}$ the image of φ in $N_{\mathrm{GL}(V)}(W)/W$, so that the reflection coset may also be written $\mathbb{G} = (V, W, \overline{\varphi})$, and we denote by $\delta_{\mathbb{G}}$ the order of $\overline{\varphi}$,
- $\mathrm{Ad}(\varphi)$ the automorphism of W defined by φ ; it is the image of φ in $N_{\mathrm{GL}(V)}(W)/C_{\mathrm{GL}(V)}(W)$,
- $\mathrm{Out}(\varphi)$ (or $\mathrm{Out}(\overline{\varphi})$) the image of φ in the outer automorphism group of W, *i.e.*, the image of φ in $N_{\mathrm{GL}(V)}(W)/WC_{\mathrm{GL}(V)}(W)$ (note that $\mathrm{Out}(\varphi)$ is an image of both $\overline{\varphi}$ and $\mathrm{Ad}(\varphi)$).

The reflection coset $\mathbb{G} = (V, W, \overline{\varphi})$ is said to be *split* if $\overline{\varphi} = 1$ (*i.e.*, if $\delta_{\mathbb{G}} = 1$).

Definition 1.7. —

1. *If $K = \mathbb{Q}$ (so that W is a Weyl group), we say that $\mathbb{G}$ is* rational.

   A "generic finite reductive group" $(X, R, Y, R^\vee, W\varphi)$ as defined in [**BMM93**] *defines a rational reflection coset $\mathbb{G} = (\mathbb{Q} \otimes_{\mathbb{Z}} Y, W\varphi)$. We then say that* $(X, R, Y, R^\vee, W\varphi)$ is associated with $\mathbb{G}$.

2. *There are also* very twisted rational *reflection cosets* $\mathbb{G}$ *defined over* $K = \mathbb{Q}(\sqrt{2})$ *(resp.* $K = \mathbb{Q}(\sqrt{3})$*) by very twisted generic finite reductive groups associated with systems* ${}^2B_2$ *and* ${}^2F_4$ *(resp.* ${}^2G_2$*). Again, such very twisted generic finite reductive groups are said to be* associated with $\mathbb{G}$. *Note that, despite of the notation, very twisted rational reflection cosets are not defined over* $\mathbb{Q}$: W *is rational on* $\mathbb{Q}$ *but not* $W\varphi$.
3. *If* $K \subset \mathbb{R}$ *(so that* W *is a Coxeter group), we say that* $\mathbb{G}$ *is* real.

For details about what follows, the reader may refer to [**BM92**] and [**BMM93**].

– In the case where $\mathbb{G}$ is rational, given a prime power q, any choice of an associated generic finite reductive group determines a connected reductive algebraic group $\mathbf{G}$ defined over $\overline{\mathbb{F}}_q$ and endowed with a Frobenius endomorphism F defined by φ (*i.e.*, F acts as $q\varphi$ on $X(\mathbf{T})$ where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$). Such groups are called the *reductive groups associated with* $\mathbb{G}$.
– In the case where $K = \mathbb{Q}(\sqrt{2})$ (resp. $K = \mathbb{Q}(\sqrt{3})$), given $\mathbb{G}$ very twisted rational and q an odd power of $\sqrt{2}$ (resp. an odd power of $\sqrt{3}$), any choice of an associated very twisted generic finite reductive group determines a connected reductive algebraic group $\mathbf{G}$ defined over $\overline{\mathbb{F}}_{q^2}$ and endowed with an isogeny acting as $q\varphi$ on $X(\mathbf{T})$. Again, this group is called a *reductive group associated with* $\mathbb{G}$.

Theorem 1.5 has been generalized in [**Mal06**, Thm 2.16] to the following result.

Theorem 1.8. —
Let $\mathbb{G} = (V, W\varphi)$ *be a reflection coset. Let*

$$\mathbb{Q}_{\mathbb{G}} := \mathbb{Q}(\mathrm{tr}_V(w\varphi) \mid w \in W)$$

be the character field of the subgroup $\langle W\varphi\rangle$ *of* $\mathrm{GL}(V)$ *generated by* $W\varphi$. *Then every* φ*-stable complex irreducible character of* W *has an extension to* $\langle W\varphi\rangle$ *afforded by a representation defined over* $\mathbb{Q}_{\mathbb{G}}$.

1.1.5. Generalized invariant degrees. —

In what follows, K denotes a number field which is stable under complex conjugation, and $\mathbb{G} = (V, W\varphi)$ is a reflection coset over K.

Let r denote the dimension of V.

One defines the family $((d_1, \zeta_1), (d_2, \zeta_2), \ldots, (d_r, \zeta_r))$ of *generalized invariant degrees* of $\mathbb{G}$ (see for example [**Bro10**, 4.2.2]): there exists a family $(f_1, f_2, \ldots, f_r)$ of r homogeneous algebraically independent elements of SV^W and a family $(\zeta_1, \zeta_2, \ldots, \zeta_r)$ of elements of $\boldsymbol{\mu}$ such that

– $SV^W = K[f_1, f_2, \ldots, f_r]$,
– for $i = 1, 2, \ldots, r$, we have $\deg(f_i) = d_i$ and $\varphi.f_i = \zeta_i f_i$.

Remark 1.9. — Let $\mathrm{Disc}_W = \sum_{\mathbf{m}} a_{\mathbf{m}} f^{\mathbf{m}}$ be the expression of Disc_W as a polynomial in the fundamental invariants $f_1, \dots, f_r$, where the sum runs over the monomials $f^{\mathbf{m}} = f_1^{m_1} \cdots f_r^{m_r}$. Then for every $\mathbf{m}$ with $a_{\mathbf{m}} \neq 0$ we have $\Delta_W(\overline{\varphi}) = \zeta_1^{m_1} \cdots \zeta_r^{m_r}$.

In the particular case where $w \in W$ is ζ-regular and the order of ζ is one of the invariant degrees d_i, we recover 1.3(2) by using the result of Bessis [**Bes01**, 1.6] that in that case $f_i^{e_W/d_i}$ is one of the monomials occurring in Disc_W.

The character $\det'_V : N_{\mathrm{GL}(V)}(W)/W \to K^\times$ (see 1.1.2 above) defines the root of unity

$$\det_{\mathbb{G}} := \det'_V(\overline{\varphi})\,.$$

Similarly, the character ${\det'_V}^\vee$ attached to $J_W^\vee$ (see 1.1.2) defines a root of unity $\det_{\mathbb{G}}^\vee := {\det'_V}^\vee(\overline{\varphi})$ attached to $\mathbb{G}$.

The character Δ_W defined by the discriminant of W defines in turn a root of unity by

$$\Delta_{\mathbb{G}} := \det_{\mathbb{G}}.\det_{\mathbb{G}}^\vee = \Delta_W(\overline{\varphi})\,.$$

The following lemma collects a number of conditions under which $\Delta_{\mathbb{G}} = 1$.

Lemma 1.10. —

We have $\Delta_{\mathbb{G}} = 1$ if (at least) one of the following conditions is satisfied.

1. *If the reflection coset $\mathbb{G}$ is split. Moreover in that case we have* $\det_{\mathbb{G}} = \det_{\mathbb{G}}^\vee = 1$.
2. *If $W\varphi$ contains a 1-regular element.*
3. *If $\mathbb{G}$ is real (i.e., if $K \subset \mathbb{R}$).*

Proof. —

(1) is trivial.

(2) We have $\Delta_{\mathbb{G}} = \Delta_W(\overline{\varphi}) = 1^{e_W} = 1$ by Lemma 1.3(2).

(3) Consider the element $J_W = \prod_{H \in \mathcal{A}(W)} j_H$ introduced above. Since $\varphi \in N_{\mathrm{GL}(V)}(W)$, φ acts on $\mathcal{A}(W)$, hence J_W is an eigenvector of φ. If φ has finite order, the corresponding eigenvalue is an element of $\boldsymbol{\mu}(K)$, hence is ± 1 if K is real.

Moreover, all reflections in W are "true reflections", that is $e_H = 2$ for all $H \in \mathcal{A}(W)$. It follows that $\mathrm{Disc}_W = J_W^2$ and so that Disc_W is fixed by φ. □

1.2. Uniform class functions on a reflection coset

The next paragraph is extracted from [**BMM99**]. It is reproduced for the convenience of the reader since it fixes conventions and notation.

1.2.1. Generalities, induction and restriction. —

Let $\mathbb{G} = (V, W\varphi)$ be a reflection coset over K.

We denote by $\mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$ the $\mathbb{Z}_K$-module of all W-invariant functions on the coset $W\varphi$ (for the natural action of W on $W\varphi$ by conjugation) with values in $\mathbb{Z}_K$, called *uniform class functions on* $\mathbb{G}$. For $\alpha \in \mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$, we denote by α^* its complex conjugate.

For $\alpha, \alpha' \in \mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$, we set $\langle \alpha, \alpha' \rangle_{\mathbb{G}} := \dfrac{1}{|W|} \sum_{w \in W} \alpha(w\varphi)\alpha'(w\varphi)^*$.

Notation

- If $\mathbb{Z}_K \to \mathcal{O}$ is a ring morphism, we denote by $\mathrm{CF}_{\mathrm{uf}}(\mathbb{G}, \mathcal{O})$ the $\mathcal{O}$-module of W-invariant functions on $W\varphi$ with values in $\mathcal{O}$, which we call the module of *uniform class functions on* $\mathbb{G}$ with values in $\mathcal{O}$. We have $\mathrm{CF}_{\mathrm{uf}}(\mathbb{G}, \mathcal{O}) = \mathcal{O} \otimes_{\mathbb{Z}_K} \mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$.
- For $w\varphi \in W\varphi$, we denote by $\mathrm{ch}^{\mathbb{G}}_{w\varphi}$ (or simply $\mathrm{ch}_{w\varphi}$) the characteristic function of the orbit of $w\varphi$ under W. The family $\left(\mathrm{ch}^{\mathbb{G}}_{w\varphi}\right)$ (where $w\varphi$ runs over a complete set of representatives of the orbits of W on $W\varphi$) is a basis of $\mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$.
- For $w\varphi \in W\varphi$, we set

$$R^{\mathbb{G}}_{w\varphi} := |C_W(w\varphi)| \mathrm{ch}^{\mathbb{G}}_{w\varphi}$$

(or simply $R_{w\varphi}$).

Remark 1.11. —

In the case of reductive groups, we may choose $K = \mathbb{Q}$. For $(\mathbf{G}, F)$ associated to $\mathbb{G}$, let $\mathrm{Uch}(\mathbf{G}^F)$ be the set of unipotent characters of $\mathbf{G}^F$: then the map which associates to $R^{\mathbb{G}}_{w\varphi}$ the Deligne-Lusztig character $R^{\mathbf{G}}_{\mathbf{T}_{wF}}(\mathrm{Id})$ defines an isometric embedding (for the scalar products $\langle \alpha, \alpha' \rangle_{\mathbb{G}}$ and $\langle \alpha, \alpha' \rangle_{\mathbf{G}^F}$) from $\mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$ onto the $\mathbb{Z}$-submodule of $\mathbb{Q}\mathrm{Uch}(\mathbf{G}^F)$ consisting of the $\mathbb{Q}$-linear combinations of Deligne-Lusztig characters (i.e., "unipotent uniform functions") having integral scalar product with all Deligne-Lusztig characters.

- Let $\langle W\varphi \rangle$ be the subgroup of $\mathrm{GL}(V)$ generated by $W\varphi$. We recall that we denote by $\overline{\varphi}$ the image of φ in $\langle W\varphi \rangle / W$ — thus $\langle W\varphi \rangle / W$ is cyclic and generated by $\overline{\varphi}$.

For $\psi \in \mathrm{Irr}(\langle W\varphi \rangle)$, we denote by $R^{\mathbb{G}}_{\psi}$ (or simply R_ψ) the restriction of ψ to the coset $W\varphi$. We have $R^{\mathbb{G}}_{\psi} = \dfrac{1}{|W|} \sum_{w \in W} \psi(w\varphi) R^{\mathbb{G}}_{w\varphi}$, and we call such a function an *almost character* of $\mathbb{G}$.

Let $\mathrm{Irr}(W)^{\overline{\varphi}}$ denote the set of $\overline{\varphi}$-stable irreducible characters of W. For $\theta \in \mathrm{Irr}(W)^{\overline{\varphi}}$, we denote by $E_{\mathbb{G}}(\theta)$ (or simply $E(\theta)$) the set of restrictions to $W\varphi$ of the extensions of θ to characters of $\langle W\varphi \rangle$.

The next result is well-known (see *e.g.* [**DM85**, §II.2.c]), and easy to prove.

Proposition 1.12. —

1. *Each element* α *of* $E_{\mathbb{G}}(\theta)$ *has norm* 1 *(i.e.,* $\langle \alpha, \alpha \rangle_{\mathbb{G}} = 1$*),*
2. *the sets* $E_{\mathbb{G}}(\theta)$ *for* $\theta \in \mathrm{Irr}(W)^{\overline{\varphi}}$ *are mutually orthogonal,*

3. $\mathrm{CF}_{\mathrm{uf}}(\mathbb{G},K)=\bigoplus^{\perp}_{\theta\in\mathrm{Irr}(W)^{\overline{\varphi}}} KE_{\mathbb{G}}(\theta)$, *where we set* $KE_{\mathbb{G}}(\theta):=KR^{\mathbb{G}}_{\psi}$ *for some (any)* $\psi\in E_{\mathbb{G}}(\theta)$.

Induction and restriction

Let $\mathbb{L}=(V,W_{\mathbb{L}}w\varphi)$ be a subcoset of maximal rank of $\mathbb{G}$ [**BMM99**, §3.A], and let $\alpha\in\mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$ and $\beta\in\mathrm{CF}_{\mathrm{uf}}(\mathbb{L})$. We denote

- by $\mathrm{Res}^{\mathbb{G}}_{\mathbb{L}}\alpha$ the restriction of α to the coset $W_{\mathbb{L}}w\varphi$,
- by $\mathrm{Ind}^{\mathbb{G}}_{\mathbb{L}}\beta$ the uniform class function on $\mathbb{G}$ defined by

$$\mathrm{Ind}^{\mathbb{G}}_{\mathbb{L}}\beta(u\varphi):=\frac{1}{|W_{\mathbb{L}}|}\sum_{v\in W}\tilde{\beta}(vu\varphi v^{-1})\quad\text{for } u\varphi\in W\varphi, \tag{1.13}$$

where $\tilde{\beta}(x\varphi)=\beta(x\varphi)$ if $x\in W_{\mathbb{L}}w$, and $\tilde{\beta}(x\varphi)=0$ if $x\notin W_{\mathbb{L}}w$. In other words, we have

$$\mathrm{Ind}^{\mathbb{G}}_{\mathbb{L}}\beta(u\varphi)=\sum_{v\in W/W_{\mathbb{L}},\,{}^{v}(u\varphi)\in W_{\mathbb{L}}w\varphi}\beta({}^{v}(u\varphi))\,. \tag{1.14}$$

We denote by $1^{\mathbb{G}}$ the constant function on $W\varphi$ with value 1. For $w\in W$, let us denote by $\mathbb{T}_{w\varphi}$ the maximal torus of $\mathbb{G}$ defined by $\mathbb{T}_{w\varphi}:=(V,w\varphi)$. It follows from the definitions that

$$R^{\mathbb{G}}_{w\varphi}=\mathrm{Ind}^{\mathbb{G}}_{\mathbb{T}_{w\varphi}}1^{\mathbb{T}_{w\varphi}}. \tag{1.15}$$

For $\alpha\in\mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$, $\beta\in\mathrm{CF}_{\mathrm{uf}}(\mathbb{L})$ we have the *Frobenius reciprocity*:

$$\langle\alpha,\mathrm{Ind}^{\mathbb{G}}_{\mathbb{L}}\beta\rangle_{\mathbb{G}}=\langle\mathrm{Res}^{\mathbb{G}}_{\mathbb{L}}\alpha,\beta\rangle_{\mathbb{L}}\,. \tag{1.16}$$

Remark 1.17. —

In the case of reductive groups, assume that $\mathbb{L}$ is a Levi subcoset of $\mathbb{G}$ attached to the Levi subgroup **L**. Then $\mathrm{Ind}^{\mathbb{G}}_{\mathbb{L}}$ corresponds to Lusztig induction from **L** to **G** (this results from definition 1.13 applied to a Deligne-Lusztig character which, using the transitivity of Lusztig induction, agrees with Lusztig induction). Similarly, the Lusztig restriction of a uniform function is uniform by [**DL76**, Thm.7], so by (1.16) $\mathrm{Res}^{\mathbb{G}}_{\mathbb{L}}$ corresponds to Lusztig restriction.

For further details, like a Mackey formula for induction and restriction, the reader may refer to [**BMM99**].

We shall now introduce notions which extend or sometimes differ from those introduced in [**BMM99**]: here we introduce two polynomial orders $|\mathbb{G}^{\mathrm{nc}}|$ and $|\mathbb{G}^{\mathrm{c}}|$ which both differ slightly (for certain twisted reflection cosets) from the definition of polynomial order given in [**BMM99**].

1.2.2. Order and Poincaré polynomial. —

Poincaré polynomial

We recall that we denote by SV the symmetric algebra of V and by SV^W the subalgebra of fixed points under W.

The group $N_{\mathrm{GL}(V)}(W)/W$ acts on the graded vector space $SV^W = \bigoplus_{n=0}^{\infty} SV_n^W$. For any $\overline{\varphi} \in N_{\mathrm{GL}(V)}(W)/W$, define its graded character by

$$\mathrm{grchar}(\overline{\varphi}; SV^W) := \sum_{n=0}^{\infty} \mathrm{tr}(\overline{\varphi}; SV_n^W)x^n \in \mathbb{Z}_K[[x]]\,.$$

Let $\mathbb{G} = (V, W, \overline{\varphi})$ with $\dim V = r$.

Let us denote by $((d_1, \zeta_1), \ldots, (d_r, \zeta_r))$ the family of generalized invariant degrees (see 1.1.5 above). We have (see *e.g.* [**BMM93**, 3.5])

$$\mathrm{grchar}(\overline{\varphi}; SV^W) = \frac{1}{|W|}\sum_{w\in W} \frac{1}{\det_V(1 - w\varphi x)} = \frac{1}{\prod_{i=1}^{i=r}(1-\zeta_i x^{d_i})}\,.$$

The *Poincaré polynomial* $P_{\mathbb{G}}(x) \in \mathbb{Z}_K[x]$ of $\mathbb{G}$ is defined by

$$\begin{aligned} P_{\mathbb{G}}(x) &= \frac{1}{\mathrm{grchar}(\overline{\varphi}; SV^W)} = \frac{1}{\frac{1}{|W|}\sum_{w\in W} \frac{1}{\det_V(1 - w\varphi x)}} \\ &= \prod_{i=1}^{i=r}(1-\zeta_i x^{d_i})\,. \end{aligned} \tag{1.18}$$

The Poincaré polynomial is semi-palindromic (see [**BMM99**, §6.B]), that is,

$$P_{\mathbb{G}}(1/x) = (-1)^r \zeta_1\zeta_2\cdots\zeta_r x^{-(N_W^{\mathrm{ref}}+r)} P_{\mathbb{G}}(x)^*\,. \tag{1.19}$$

Graded regular representation

Let us denote by SV_+^W the maximal graded ideal of SV^W (generated by $f_1, f_2, \ldots, f_r$). We call the finite dimensional graded vector space

$$KW^{\mathrm{gr}} := SV/SV_+^W SV$$

the *graded regular representation.*

This has the following properties (see *e.g.* [**Bou68**, chap. V, §5, th. 2]).

Proposition 1.20. —

1. *KW^{gr} has a natural $N_{\mathrm{GL}(V)}(W)$-action, and we have an isomorphism of graded $KN_{\mathrm{GL}(V)}(W)$-modules*

$$SV \simeq KW^{\mathrm{gr}} \otimes_K SV^W\,.$$

2. *As a KW-module, forgetting the graduation, KW^{gr} is isomorphic to the regular representation of W.*

3. *Denoting by* $KW^{(n)}$ *the subspace of* KW^{gr} *generated by the elements of degree* n*, we have*
   (a) $KW^{\mathrm{gr}} = \bigoplus_{n=0}^{N_W^{\mathrm{ref}}} KW^{(n)}$,
   (b) *the* $\det_V$*-isotypic component of* KW^{gr} *is the one-dimensional subspace of* $KW^{(N_W^{\mathrm{hyp}})}$ *generated by* J_W*,*
   (c) $KW^{(N_W^{\mathrm{ref}})}$ *is the one-dimensional subspace generated by* $J_W^\vee$ *and is the* $\det_V^*$*-isotypic component of* KW^{gr}*.*

Fake degrees of uniform functions

• We denote by $\mathrm{tr}_{KW^{\mathrm{gr}}} \in \mathrm{CF}_{\mathrm{uf}}(\mathbb{G}, \mathbb{Z}_K[x])$ the uniform class function on $\mathbb{G}$ (with values in the polynomial ring $\mathbb{Z}_K[x]$) defined by the character of the graded regular representation KW^{gr}. Thus the value of the function $\mathrm{tr}_{KW^{\mathrm{gr}}}$ on $w\varphi$ is

$$\mathrm{tr}_{KW^{\mathrm{gr}}}(w\varphi) := \sum_{n=0}^{N_W^{\mathrm{ref}}} \mathrm{tr}(w\varphi; KW^{(n)})x^n.$$

We call $\mathrm{tr}_{KW^{\mathrm{gr}}}$ *the graded regular character*.

• We define the *fake degree*, a linear function $\mathrm{Feg}_{\mathbb{G}} : \mathrm{CF}_{\mathrm{uf}}(\mathbb{G}) \to K[x]$, as follows: for $\alpha \in \mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$, we set

$$\mathrm{Feg}_{\mathbb{G}}(\alpha) := \langle \alpha, \mathrm{tr}_{KW^{\mathrm{gr}}} \rangle_{\mathbb{G}} = \sum_{n=0}^{N_W^{\mathrm{ref}}} \left(\frac{1}{|W|} \sum_{w\in W} \alpha(w\varphi) \mathrm{tr}(w\varphi; KW^{(n)})^* \right) x^n \,. \tag{1.21}$$

We shall often omit the subscript $\mathbb{G}$ (writing then $\mathrm{Feg}(\alpha)$) when the context allows it. Notice that

$$\mathrm{Feg}(R_{w\varphi}^{\mathbb{G}}) = \mathrm{tr}_{KW^{\mathrm{gr}}}(w\varphi)^*, \tag{1.22}$$

and so in particular that

$$\mathrm{Feg}(R_{w\varphi}^{\mathbb{G}}) \in \mathbb{Z}_K[x]. \tag{1.23}$$

Lemma 1.24. —
We have

$$\mathrm{tr}_{KW^{\mathrm{gr}}} = \frac{1}{|W|} \sum_{w\in W} \mathrm{Feg}_{\mathbb{G}}(R_{w\varphi}^{\mathbb{G}})^* R_{w\varphi}^{\mathbb{G}} \,.$$

Proof of 1.24. —
It is an immediate consequence of the definition of $R_{w\varphi}^{\mathbb{G}}$ and of (1.22). □

Fake degrees of almost characters

Let E be a $K\langle W\varphi\rangle$-module. Its character θ is a class function on $\langle W\varphi\rangle$. Its restriction R_θ to $W\varphi$ is a uniform class function on $\mathbb{G}$. Then the fake degree of R_θ is :

$$\mathrm{Feg}_{\mathbb{G}}(R_\theta) = \mathrm{tr}(\varphi\,; \mathrm{Hom}_{KW}(KW^{\mathrm{gr}}, E)). \tag{1.25}$$

Notice that

$$\mathrm{Feg}_{\mathbb{G}}(R_\theta) \in \mathbb{Z}[\exp 2i\pi/\delta_{\mathbb{G}}][x] \tag{1.26}$$

(we recall that $\delta_{\mathbb{G}}$ is the order of the twist $\overline{\varphi}$ of $\mathbb{G}$).

The polynomial $\mathrm{Feg}_{\mathbb{G}}(R_\theta)$ is called *fake degree of* θ.

Let $\theta \in \mathrm{Irr}(W)^{\overline{\varphi}}$. Whenever $\psi \in \mathrm{Irr}(\langle W\varphi\rangle)$ is an extension of θ to $\langle W\varphi\rangle$, then $\mathrm{Reg}_\theta^{\mathbb{G}} := \mathrm{Feg}_{\mathbb{G}}(R_\psi)^* \cdot R_\psi$ depends only on θ and is the orthogonal projection of $\mathrm{tr}_{KW^{\mathrm{gr}}}$ onto $K[x]E_{\mathbb{G}}(\theta)$, so that in other words, we have

$$\mathrm{tr}_{KW^{\mathrm{gr}}} = \sum_{\theta\in\mathrm{Irr}(W)^{\overline{\varphi}}} \mathrm{Reg}_\theta^{\mathbb{G}}\,. \tag{1.27}$$

Polynomial order and fake degrees

From the isomorphism $SV \cong KW^{\mathrm{gr}} \otimes_K (SV)^W$ of $\langle W\varphi\rangle$-modules (see 1.20) we deduce for $w \in W$ that

$$\mathrm{tr}(w\varphi; SV) = \mathrm{tr}(w\varphi; KW^{\mathrm{gr}})\mathrm{tr}(w\varphi; SV^W)\,,$$

hence

$$\mathrm{tr}(w\varphi; SV) = \mathrm{tr}(w\varphi; KW^{\mathrm{gr}})\frac{1}{|W|}\sum_{v\in W}\frac{1}{\det_V(1 - xv\varphi)}\,.$$

Computing the scalar product with a class function α on $W\varphi$ gives

$$\frac{1}{|W|}\sum_{w\in W}\frac{\alpha(w\varphi)}{\det_V(1 - xw\varphi)^*} = \mathrm{Feg}_{\mathbb{G}}(\alpha)\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det_V(1 - xw\varphi)^*}, \tag{1.28}$$

or, in other words

$$\langle\alpha, \mathrm{tr}_{SV}\rangle_{\mathbb{G}} = \langle\alpha, \mathrm{tr}_{KW^{\mathrm{gr}}}\rangle_{\mathbb{G}}\langle 1^{\mathbb{G}}, \mathrm{tr}_{SV}\rangle_{\mathbb{G}}. \tag{1.29}$$

Let us set

$$S_{\mathbb{G}}(\alpha) := \langle\alpha, \mathrm{tr}_{SV}\rangle_{\mathbb{G}}\,. \tag{1.30}$$

Then (1.29) becomes

$$S_{\mathbb{G}}(\alpha) = \mathrm{Feg}_{\mathbb{G}}(\alpha)S_{\mathbb{G}}(1^{\mathbb{G}})\,. \tag{1.31}$$

By definition of the Poincaré polynomial we have $S_{\mathbb{G}}(1^{\mathbb{G}}) = 1/P_{\mathbb{G}}(x)^*$, hence

$$S_{\mathbb{G}}(\alpha) := \frac{\mathrm{Feg}_{\mathbb{G}}(\alpha)}{P_{\mathbb{G}}(x)^*}\,. \tag{1.32}$$

For a subcoset $\mathbb{L} = (V, W_{\mathbb{L}}w\varphi)$ of maximal rank of $\mathbb{G}$, by the Frobenius reciprocity (1.16) we have

$$\mathrm{Feg}_{\mathbb{G}}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}1^{\mathbb{L}}) = \langle 1^{\mathbb{L}}, \mathrm{Res}_{\mathbb{L}}^{\mathbb{G}}\mathrm{tr}_{KW^{\mathrm{gr}}}\rangle_{\mathbb{L}} = \sum_{n=0}^{N_W^{\mathrm{ref}}}\mathrm{tr}(w\varphi; (KW^{(n)})^{W_{\mathbb{L}}})^* x^n\,, \tag{1.33}$$

where $(KW^{(n)})^{W_{\mathbb{L}}}$ are the $W_{\mathbb{L}}$-invariants in $KW^{(n)}$.

Let us recall that every element $w\varphi \in W\varphi$ defines a maximal torus (a minimal Levi subcoset) $\mathbb{T}_{w\varphi} := (V, w\varphi)$ of $\mathbb{G}$.

Lemma 1.34. —

1. *We have*
$$\frac{P_{\mathbb{G}}(x)^*}{P_{\mathbb{L}}(x)^*} = \mathrm{Feg}_{\mathbb{G}}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} 1^{\mathbb{L}}),$$
2. $P_{\mathbb{L}}(x)$ *divides* $P_{\mathbb{G}}(x)$ *(in* $\mathbb{Z}_K[x]$ *),*
3. *for* $w\varphi \in W\varphi$*, we have*
$$\frac{P_{\mathbb{G}}(x)}{P_{\mathbb{T}_{w\varphi}}(x)} = \mathrm{tr}_{KW^{\mathrm{gr}}}(w\varphi) = \mathrm{Feg}_{\mathbb{G}}(R_{w\varphi}^{\mathbb{G}})^* .$$

Proof of 1.34. —

(1) By (1.32), we have $\mathrm{Feg}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} 1^{\mathbb{L}}) = S_{\mathbb{G}}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} 1^{\mathbb{L}}) P_{\mathbb{G}}(x)^*$. By Frobenius reciprocity, for any class function α on $\mathbb{L}$ we have
$$S_{\mathbb{G}}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} \alpha) = \langle \mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} \alpha, \mathrm{tr}_{SV} \rangle_{\mathbb{G}} = \langle \alpha, \mathrm{Res}_{\mathbb{L}}^{\mathbb{G}} \mathrm{tr}_{SV} \rangle_{\mathbb{L}} = S_{\mathbb{L}}(\alpha) ,$$
and so $S_{\mathbb{G}}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} 1^{\mathbb{L}}) = S_{\mathbb{L}}(1^{\mathbb{L}}) = \dfrac{1}{P_{\mathbb{L}}(x)^*}$.

(2) is an immediate consequence of (1).

(3) follows from (1) and from formulae (1.15) and (1.22). □

Let us now consider a Levi subcoset $\mathbb{L} = (V, W_{\mathbb{L}} w\varphi)$. By 1.34, for $vw\varphi \in W_{\mathbb{L}} w\varphi$, we have
$$\mathrm{tr}_{KW^{\mathrm{gr}}}(vw\varphi) = \frac{P_{\mathbb{G}}(x)}{P_{\mathbb{L}}(x)} \frac{P_{\mathbb{L}}(x)}{P_{\mathbb{T}_{vw\varphi}}(x)} = \frac{P_{\mathbb{G}}(x)}{P_{\mathbb{L}}(x)} \mathrm{tr}_{KW_{\mathbb{L}}^{\mathrm{gr}}}(vw\varphi) ,$$
and by 1.34, (1)
$$\mathrm{Res}_{\mathbb{L}}^{\mathbb{G}} \mathrm{tr}_{KW^{\mathrm{gr}}} = \mathrm{Feg}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} 1^{\mathbb{L}})^* \mathrm{tr}_{KW_{\mathbb{L}}^{\mathrm{gr}}} . \tag{1.35}$$

Lemma 1.36. —

For $\beta \in \mathrm{CF}_{\mathrm{uf}}(\mathbb{L})$ *we have*
$$\mathrm{Feg}_{\mathbb{G}}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} \beta) = \frac{P_{\mathbb{G}}(x)^*}{P_{\mathbb{L}}(x)^*} \mathrm{Feg}_{\mathbb{L}}(\beta) .$$

Indeed, by Frobenius reciprocity, (1.35), and Lemma 1.34,
$$\begin{aligned}
\mathrm{Feg}_{\mathbb{G}}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} \beta) &= \langle \mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} \beta, \mathrm{tr}_{KW^{\mathrm{gr}}} \rangle_{\mathbb{G}} = \langle \beta, \mathrm{Res}_{\mathbb{L}}^{\mathbb{G}} \mathrm{tr}_{KW^{\mathrm{gr}}} \rangle_{\mathbb{L}} \\
&= \mathrm{Feg}_{\mathbb{G}}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} 1^{\mathbb{L}}) \langle \beta, \mathrm{tr}_{KW_{\mathbb{L}}^{\mathrm{gr}}} \rangle_{\mathbb{L}} = \mathrm{Feg}_{\mathbb{G}}(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} 1^{\mathbb{L}}) \mathrm{Feg}_{\mathbb{L}}(\beta) \\
&= \frac{P_{\mathbb{G}}(x)^*}{P_{\mathbb{L}}(x)^*} \mathrm{Feg}_{\mathbb{L}}(\beta) .
\end{aligned}$$

Remark 1.37. —

In the case of reductive groups, it follows from (1.22) and (1.34 (3)) that $\mathrm{Feg}(R_{w\varphi})(q)$ is the degree of the Deligne-Lusztig character $R^{\mathbf{G}}_{\mathbf{T}_{w\varphi}}$. Since the regular representation of $\mathbf{G}^F$ is uniform, it follows that $\mathrm{tr}_{KW^{\mathrm{reg}}}$ corresponds to a (graded by x) version of the unipotent part of the regular representation of $\mathbf{G}^F$, and that Feg corresponds indeed to the (generic) degree for unipotent uniform functions on $\mathbf{G}^F$.

Changing x to $1/x$

As a particular uniform class function on $\mathbb{G}$, we can consider the function $\det_V$ restricted to $W\varphi$, which we still denote by $\det_V$. Notice that this restriction might also be denoted by $R^{\mathbb{G}}_{\det_V}$, since it is the almost character associated to the character of $\langle W\varphi\rangle$ defined by $\det_V$.

Lemma 1.38. —

Let α be a uniform class function on $\mathbb{G}$. We have

$$S_{\mathbb{G}}(\alpha\det_V^*)(x) = (-1)^r x^{-r} S_{\mathbb{G}}(\alpha^*)(1/x)^* \,.$$

Proof. —

Since $S_{\mathbb{G}}(\alpha) = \frac{1}{|W|}\sum_{w\in W}\frac{\alpha(w\varphi)}{\det_V(1-xw\varphi)^*}$, we see that

$$\begin{aligned} S_{\mathbb{G}}(\alpha\det_V^*)(x)^* &= \frac{1}{|W|}\sum_{w\in W}\frac{\alpha(w\varphi)^*\det_V(w\varphi)}{\det_V(1-xw\varphi)} \\ &= \frac{1}{|W|}\sum_{w\in W}\frac{\alpha(w\varphi)^*}{\det_V((w\varphi)^{-1}-x)} \\ &= (-1)^r x^{-r}\frac{1}{|W|}\sum_{w\in W}\frac{\alpha(w\varphi)^*}{\det_V(1-w\varphi/x)^*} \\ &= (-1)^r x^{-r} S_{\mathbb{G}}(\alpha^*)(1/x)\,. \end{aligned}$$

□

Corollary 1.39. —

We have

$$\mathrm{Feg}_{\mathbb{G}}(\det_V^*) = \zeta_1^*\zeta_2^*\cdots\zeta_r^* x^{N_W^{\mathrm{ref}}}\,.$$

Proof of 1.39. —

Applying Lemma 1.38 for $\alpha = 1^{\mathbb{G}}$ gives

$$S_{\mathbb{G}}(\det_V^*)(x) = (-1)^r x^{-r} S_{\mathbb{G}}(1^{\mathbb{G}})(1/x)^* = (-1)^r x^{-r}\frac{1}{P_{\mathbb{G}}(1/x)}\,,$$

hence by (1.19)

$$S_{\mathbb{G}}(\det_V^*)(x) = \zeta_1^*\zeta_2^*\cdots\zeta_r^* x^{N_W^{\mathrm{ref}}}\frac{1}{P_{\mathbb{G}}(x)^*}\,,$$

and the desired formula follows from (1.32). □

Fake degree of $\det_V$ *and some computations*

Proposition 1.40. —
We have

$$\begin{cases} \mathrm{Feg}_{\mathbb{G}}(\det_V)(x) = \det'_V(\overline{\varphi})x^{N_W^{\mathrm{hyp}}} \\ \mathrm{Feg}_{\mathbb{G}}(\det_V^*)(x) = \det_V'^{\vee}(\overline{\varphi})^* x^{N_W^{\mathrm{ref}}} \end{cases}$$

Corollary 1.41. —
We have

$$\begin{cases} \det_V'^{\vee}(\overline{\varphi}) = \zeta_1\zeta_2\cdots\zeta_r \\ \det'_V(\overline{\varphi}) = \Delta_W(\overline{\varphi})\zeta_1^*\zeta_2^*\cdots\zeta_r^* \end{cases}$$

Proof of 1.40 and 1.41. —
By Propositions 1.12 and 1.20(3), we see that

$$\begin{aligned} \mathrm{Feg}_{\mathbb{G}}(\det_V)(x) &= \left(\frac{1}{|W|}\sum_{w\in W}\det_V(w\varphi)\mathrm{tr}(w\varphi; KW^{(N_W^{\mathrm{hyp}})})^*\right)x^{N_W^{\mathrm{hyp}}} \\ &= \frac{\det_V(\varphi)}{\mathrm{tr}(\varphi; KW^{(N_W^{\mathrm{hyp}})})}x^{N_W^{\mathrm{hyp}}} = \det'_V(\overline{\varphi})x^{N_W^{\mathrm{hyp}}}\,. \end{aligned}$$

A similar proof holds for $\mathrm{Feg}_{\mathbb{G}}(\det_V^*)(x)$.
The corollary follows then from 1.39 and from $\det'_V\det_V'^{\vee} = \Delta_W$. □

Corollary 1.42. —
Assume that $w\varphi$ *is a* ζ*-regular element of* $W\varphi$*. Then*

$$\begin{cases} \mathrm{Feg}_{\mathbb{G}}(\det_V)(x) = \det_V(w\varphi)(\zeta^{-1}x)^{N_W^{\mathrm{hyp}}} \\ \mathrm{Feg}_{\mathbb{G}}(\det_V^*)(x) = \det_V^*(w\varphi)(\zeta^{-1}x)^{N_W^{\mathrm{ref}}}\,. \end{cases}$$

In particular, we have

$$\begin{cases} \mathrm{Feg}_{\mathbb{G}}(\det_V)(\zeta) = \det_V(w\varphi) \\ \mathrm{Feg}_{\mathbb{G}}(\det_V^*)(\zeta) = \det_V^*(w\varphi)\,. \end{cases}$$

Note that the last assertion of the above lemma will be generalized in 1.53.

Proof. —
By Proposition 1.40, and since $\det'_V(\overline{\varphi}) = \widetilde{\det}_V(w\varphi)\det_V(w\varphi)^*$, we have

$$\mathrm{Feg}_{\mathbb{G}}(\det_V)(x) = \det'_V(\overline{\varphi})^* x^{N_W^{\mathrm{hyp}}} = \widetilde{\det}_V(w\varphi)^*\det_V(w\varphi)x^{N_W^{\mathrm{hyp}}}\,.$$

Now by Lemma 1.3, we know that $\widetilde{\det}_V(w\varphi) = \zeta^{N_W^{\mathrm{hyp}}}$, which implies that $\mathrm{Feg}_{\mathbb{G}}(\det_V)(x) = \det_V(w\varphi)(\zeta^{-1}x)^{N_W^{\mathrm{hyp}}}$.
The proof of the second equality goes the same. □

Lemma 1.43. —

1. *For all $w \in W$ we have*

$$\mathrm{Feg}_{\mathbb{G}}(R_{w\varphi})(1/x) = \det_V(w\varphi)(\prod_{i=1}^{i=r} \zeta_i^*) x^{-N_W^{\mathrm{ref}}} \mathrm{Feg}_{\mathbb{G}}(R_{w\varphi})(x)^* \,.$$

2. *If $w\varphi$ is a ζ-regular element, we have*

$$\mathrm{Feg}_{\mathbb{G}}(R_{w\varphi})(x)^\vee = (\zeta^{-1}x)^{-N_W^{\mathrm{ref}}} \mathrm{Feg}_{\mathbb{G}}(R_{w\varphi})(x) \,.$$

Proof. —

(1) By (1.34, (3)), we have $\mathrm{Feg}_{\mathbb{G}}(R_{w\varphi}^{\mathbb{G}}) = \dfrac{P_{\mathbb{G}}(x)^*}{P_{\mathbb{T}_{w\varphi}}(x)^*}$. By (1.18), we have $P_{\mathbb{G}}(x) = \prod_{i=1}^{i=r}(1-\zeta_i x^{d_i})$. Moreover, $P_{\mathbb{T}_{w\varphi}}(x) = \det_V(1-w\varphi x)$. It follows that

$$\mathrm{Feg}_{\mathbb{G}}(R_{w\varphi}^{\mathbb{G}}) = \frac{\prod_{i=1}^{i=r}(1-\zeta_i^* x^{d_i})}{\det_V(1-w\varphi x)^*} \,.$$

The stated formula follows from the equality $\sum_{i=1}^{i=r}(d_i - 1) = N_W^{\mathrm{ref}}$, which is well known (see *e.g.* [**Bro10**, Thm. 4.1.(2)(b)]).

(2) By Corollary 1.41, we know that $\prod_{i=1}^{i=r} \zeta_i^* = {\det_V'}^\vee(\overline{\varphi})^*$, hence (by Lemma 1.3(2))

$$\prod_{i=1}^{i=r} \zeta_i^* = \zeta^{-N_W^{\mathrm{ref}}} \det_V(w\varphi)^* \,.$$

It follows from (1) that

$$\mathrm{Feg}_{\mathbb{G}}(R_{w\varphi})(1/x)^* = (\zeta^{-1}x)^{-N_W^{\mathrm{ref}}} \mathrm{Feg}_{\mathbb{G}}(R_{w\varphi})(x) \,.$$

□

1.2.3. The polynomial orders of a reflection coset. —

We define two "order polynomials" of $\mathbb{G}$, which are both elements of $\mathbb{Z}_K[x]$, and which coincide when $\mathbb{G}$ is real. This differs from [**BMM99**].

Definition 1.44. —

(nc) *The noncompact order polynomial of $\mathbb{G}$ is the element of $\mathbb{Z}_K[x]$ defined by*

$$\begin{aligned} |\mathbb{G}|_{\mathrm{nc}} :&= (-1)^r \mathrm{Feg}_{\mathbb{G}}(\det_V^*)(x) P_{\mathbb{G}}(x)^* \\ &= (-1)^r {\det_V'}^\vee(\overline{\varphi})^* x^{N_W^{\mathrm{ref}}} \prod_{i=1}^{i=r}(1-\zeta_i^* x^{d_i}) \\ &= (\zeta_1^*\zeta_2^*\cdots\zeta_r^*)^2 x^{N_W^{\mathrm{ref}}} \prod_{i=1}^{i=r}(x^{d_i} - \zeta_i) \,. \end{aligned}$$

(c) *The compact order polynomial of* $\mathbb{G}$ *is the element of* $\mathbb{Z}_K[x]$ *defined by*

$$\begin{aligned}|\mathbb{G}|_{\mathrm{c}} :&= (-1)^r \mathrm{Feg}_{\mathbb{G}}(\det_V)(x) P_{\mathbb{G}}(x)^* \\ &= (-1)^r \det{}'_V(\overline{\varphi})^* x^{N_W^{\mathrm{hyp}}} \prod_{i=1}^{i=r}(1-\zeta_i^* x^{d_i}) \\ &= \Delta_W(\overline{\varphi})^* x^{N_W^{\mathrm{hyp}}} \prod_{i=1}^{i=r}(x^{d_i}-\zeta_i)\,.\end{aligned}$$

Remark 1.45. —

1. In the particular case of a maximal torus $\mathbb{T}_{w\varphi} := (V, w\varphi)$ of $\mathbb{G}$, it is readily seen that

$$\begin{cases}|\mathbb{T}_{w\varphi}|_{\mathrm{nc}} = (-1)^r \det_V(w\varphi)^* \det(1-w\varphi x)^* \\ |\mathbb{T}_{w\varphi}|_{\mathrm{c}} = (-1)^r \det_V(w\varphi) \det(1-w\varphi x)^* = \det_V(x-w\varphi)\,.\end{cases}$$

2. In general the order polynomials are not monic. Nevertheless, if $\mathbb{G}$ is real they are equal and monic (see Lemma 1.10 for the case of $|\mathbb{G}|_{\mathrm{c}}$). If $\mathbb{G}$ is real, we set $|\mathbb{G}| := |\mathbb{G}|_{\mathrm{nc}} = |\mathbb{G}|_{\mathrm{c}}$.

Remark 1.46. — If $\mathbb{G}$ is rational or very twisted rational, and if $\mathbf{G}$ (together with the endomorphism F) is any of the associated reductive groups attached to the choice of a suitable prime power q, then (see *e.g.* [**Ste68**, 11.16])

$$|\mathbb{G}|_{x=q} = |\mathbf{G}^F|\,.$$

Proposition 1.47. —

1. *We have*

$$\begin{cases}\dfrac{|\mathbb{G}|_{\mathrm{c}}}{|\mathbb{T}_{w\varphi}|_{\mathrm{c}}} = \widetilde{\det}_V(w\varphi)^* x^{N_W^{\mathrm{hyp}}} \mathrm{Feg}_{\mathbb{G}}(R_{w\varphi})(x)\,, \\ \dfrac{|\mathbb{G}|_{\mathrm{nc}}}{|\mathbb{T}_{w\varphi}|_{\mathrm{nc}}} = \widetilde{\det}{}_V^{\vee}(w\varphi)^* x^{N_W^{\mathrm{ref}}} \mathrm{Feg}_{\mathbb{G}}(R_{w\varphi})(x)\,.\end{cases}$$

2. *If moreover* $w\varphi$ *is a* ζ*-regular element of* $W\varphi$*, we have*

$$\begin{cases}\dfrac{|\mathbb{G}|_{\mathrm{c}}}{|\mathbb{T}_{w\varphi}|_{\mathrm{c}}} = (\zeta^{-1}x)^{N_W^{\mathrm{hyp}}} \mathrm{Feg}_{\mathbb{G}}(R_{w\varphi})(x)\,, \\ \dfrac{|\mathbb{G}|_{\mathrm{nc}}}{|\mathbb{T}_{w\varphi}|_{\mathrm{nc}}} = (\zeta^{-1}x)^{N_W^{\mathrm{ref}}} \mathrm{Feg}_{\mathbb{G}}(R_{w\varphi})(x)\,.\end{cases}$$

Proof. —

(1) By Definition 1.44 and the above remark, 1, and by Lemma 1.34, we have

$$\begin{aligned}\frac{|\mathbb{G}|_{\mathrm{c}}}{|\mathbb{T}_{w\varphi}|_{\mathrm{c}}} &= \det{}'_V(\overline{\varphi})^* \det_V(w\varphi)^* x^{N_W^{\mathrm{hyp}}} \frac{P_{\mathbb{G}}(x)^*}{P_{\mathbb{T}_{w\varphi}}(x)^*} \\ &= \widetilde{\det}_V(w\varphi)^* x^{N_W^{\mathrm{hyp}}} \mathrm{Feg}_{\mathbb{G}}(R_{w\varphi})(x)\,,\end{aligned}$$

proving (1) in the compact type. The proof for the noncompact type is similar.

(2) follows from Lemma 1.3. □

1.3. Φ-Sylow theory and Φ-split Levi subcosets

1.3.1. The Sylow theorems. —

Here we correct a proof given in [**BMM99**], *viz.* in Th. 1.50 (4) below.

Definition 1.48. —

- *We call* K-cyclotomic polynomial *(or* cyclotomic polynomial in $K[x]$*) a monic irreducible polynomial of degree at least* 1 *in* $K[x]$ *which divides* $x^n - 1$ *for some integer* $n \geq 1$.
- *Let* $\Phi \in K[x]$ *be a cyclotomic polynomial. A* Φ-reflection coset *is a torus whose polynomial order is a power of* Φ.

Remark 1.49. —

If G is an associated finite reductive group, then a Φ_d-reflection coset is the reflection datum of a torus which splits over $\mathbb{F}_{q^d}$ but no subtorus splits over any proper subfield.

Theorem 1.50. —

Let $\mathbb{G}$ *be a reflection coset over* K *and let* Φ *be a* K*-cyclotomic polynomial.*

1. *If* Φ *divides* $P_{\mathbb{G}}(x)$*, there exist nontrivial* Φ*-subcosets of* $\mathbb{G}$.
2. *Let* $\mathbb{S}$ *be a maximal* Φ*-subcoset of* $\mathbb{G}$*. Then*
   (a) *there is* $w \in W$ *such that* $\mathbb{S} = \left(\ker \Phi(w\varphi), (w\varphi)|_{\ker \Phi(w\varphi)}\right)$,
   (b) $|\mathbb{S}| = \Phi^{a(\Phi)}$*, the full contribution of* Φ *to* $P_{\mathbb{G}}(x)$.
3. *Any two maximal* Φ*-subcosets of* $\mathbb{G}$ *are conjugate under* W.
4. *Let* $\mathbb{S}$ *be a maximal* Φ*-subcoset of* $\mathbb{G}$*. We set* $\mathbb{L} := C_{\mathbb{G}}(\mathbb{S})$ *and* $W_{\mathbb{G}}(\mathbb{L}) := N_W(\mathbb{L})/W_{\mathbb{L}}$*. Then*
$$\frac{|\mathbb{G}|_{\mathrm{nc}}}{|W_{\mathbb{G}}(\mathbb{L})||\mathbb{L}|_{\mathrm{nc}}} \equiv \frac{|\mathbb{G}|_{\mathrm{c}}}{|W_{\mathbb{G}}(\mathbb{L})||\mathbb{L}|_{\mathrm{c}}} \equiv 1 \mod \Phi.$$
5. *With the above notation, we have*
$$\mathbb{L} = (V, W_{\mathbb{L}} w\varphi) \text{ with } W_{\mathbb{L}} = C_W(\ker \Phi(w\varphi)).$$
We set $V(\mathbb{L}, \Phi) := \ker \Phi(w\varphi)$ *viewed as a vector space over the field* $K[x]/(\Phi(x))$ *through its natural structure of* $K[w\phi]$*-module. Then the pair* $(V(\mathbb{L}, \Phi), W_{\mathbb{G}}(\mathbb{L}))$ *is a reflection group.*

The maximal Φ-subcosets of $\mathbb{G}$ are called the *Sylow* Φ*-subcosets*.

Proof of 1.50. —

As we shall see, assertions (1) to (3) are consequences of the main results of Springer in [**Spr74**] (see also Theorem 3.4 in [**BM92**]).

Assertion (5) is nothing but a reformulation of a result of Lehrer and Springer (see for example [**Bro10**, Thm.5.6])

For each K-cyclotomic polynomial Φ and $w \in W$, we denote by $V(w\varphi, \Phi)$ the kernel of the endomorphism $\Phi(w\varphi)$ of V (*i.e.*, $V(\mathbb{L}, \Phi)$ viewed as a K-vector space). Thus

$$\mathbb{S}(w\varphi, \Phi) := (V(w\varphi, \Phi), (w\varphi)|_{V(w\varphi,\Phi)})$$

is a torus of $\mathbb{G}$.

Let us denote by K' a Galois extension of K which splits Φ, and set $V' := K' \otimes_K V$. For every root ζ of Φ in K', we set $V'(w\varphi, \zeta) := V'(w\varphi, (x - \zeta))$. It is clear that $V'(w\varphi, \Phi) = \bigoplus_\zeta V'(w\varphi, \zeta)$ where ζ runs over the set of roots of Φ, and thus

$$\dim V'(w\varphi, \Phi) = \deg(\Phi) \cdot \dim V'(w\varphi, \zeta)\,.$$

It follows from [**Spr74**], 3.4 and 6.2, that for all such ζ

(S1) $\max_{(w\in W)} \dim V'(w\varphi, \zeta) = a(\Phi)$,

(S2) for all $w \in W$, there exists $w' \in W$ such that $\dim V'(w'\varphi, \zeta) = a(\Phi)$ and $V'(w\varphi, \zeta) \subset V'(w'\varphi, \zeta)$,

(S3) if $w_1, w_2 \in W$ are such that $\dim V'(w_1\varphi, \zeta) = \dim V'(w_2\varphi, \zeta) = a(\Phi)$, there exists $w \in W$ such that $w \cdot V'(w_1\varphi, \zeta) = V'(w_2\varphi, \zeta)$.

Now, (S1) shows that there exists $w \in W$ such that the rank of $\mathbb{S}(w\varphi, \Phi)$ is $a(\Phi)\deg(\Phi)$, which implies the first assertion of Theorem 1.50.

If $V'(w\varphi, \zeta) \subseteq V'(w'\varphi, \zeta)$ then we have $V'(w\varphi, \sigma(\zeta)) \subseteq V'(w'\varphi, \sigma(\zeta))$ for all $\sigma \in \mathrm{Gal}(K'/K)$, hence

- $V'(w\varphi, \Phi) \subset V'(w'\varphi, \Phi)$,
- $w'\varphi|_{V(w\varphi,\Phi)} = w\varphi|_{V(w\varphi,\Phi)}$.

So (S2) implies that for all $w \in W$, there exists $w' \in W$ such that the rank of $\mathbb{S}(w'\varphi, \Phi)$ is $a(\Phi)\deg(\Phi)$ and $\mathbb{S}(w\varphi, \Phi)$ is contained in $\mathbb{S}(w'\varphi, \Phi)$, which proves assertion (2) of Theorem 1.50.

For the same reason, (S3) shows that if w_1 and w_2 are two elements of W such that both $\mathbb{S}(w_1\varphi, \Phi)$ and $\mathbb{S}(w_2\varphi, \Phi)$ have rank $a(\Phi)\deg(\Phi)$, there exists $w \in W$ such that $\mathbb{S}(ww_1\varphi w^{-1}, \Phi) = \mathbb{S}(w_2\varphi, \Phi)$, which proves assertion (3) of Theorem 1.50.

The proof of the fourth assertion requires several steps.

Lemma 1.51. —

For $\mathbb{S}$ a Sylow Φ-subcoset of $\mathbb{G}$ let $\mathbb{L} := C_\mathbb{G}(\mathbb{S})$. Then for any class function α on $\mathbb{G}$, we have

$$\mathrm{Feg}_\mathbb{G}(\alpha)(x) \equiv \frac{1}{|W_\mathbb{G}(\mathbb{L})|}\frac{P_\mathbb{G}(x)^*}{P_\mathbb{L}(x)^*}\mathrm{Feg}_\mathbb{L}(\mathrm{Res}^\mathbb{G}_\mathbb{L}\alpha)(x) \mod \Phi(x)\,.$$

Proof of 1.51. —

We have

$$P_{\mathbb{G}}(x)^* S_{\mathbb{G}}(\alpha) = \mathrm{Feg}_{\mathbb{G}}(\alpha)(x) = \frac{1}{|W|}\sum_{w\in W}\alpha(w\varphi)\frac{P_{\mathbb{G}}(x)^*}{\det(1-xw\varphi)^*}\,,$$

and it follows from the first two assertions of Theorem 1.50 that

$$\mathrm{Feg}_{\mathbb{G}}(\alpha)(x) \equiv \frac{P_{\mathbb{G}}(x)^*}{|W|}\sum_{w}\frac{\alpha(w\varphi)}{\det(1-xw\varphi)^*} \mod \Phi(x),$$

where w runs over those elements of W such that $V(w\varphi,\Phi)$ is of maximal dimension. These subspaces are permuted transitively by W (see (S3) above). Let $V(w_0\varphi,\Phi)$ be one of them, and let $\mathbb{S}$ be the Sylow Φ-subcosets defined by

$$\mathbb{S} = (V(w_0\varphi,\Phi), (w_0\varphi)|_{V(w_0\varphi,\Phi)}).$$

Let us recall that we set $\mathbb{L} = C_{\mathbb{G}}(\mathbb{S}) = (V, W_{\mathbb{L}}w_0\varphi)$. The group $N_W(\mathbb{S})$ consists of all $w\in W$ such that

$$w.V(w_0\varphi,\Phi)=V(w_0\varphi,\Phi) \quad\text{and}\quad (ww_0\varphi w^{-1})|_{V(w_0\varphi,\Phi)} = (w_0\varphi)|_{V(w_0\varphi,\Phi)}\,.$$

Since in this case every $w\in W$ such that $w.V(w_0\varphi,\Phi) = V(w_0\varphi,\Phi)$ belongs to $N_W(\mathbb{S})$, we have

$$\mathrm{Feg}(\alpha)(x) \equiv |W : N_W(\mathbb{S})|\frac{P_{\mathbb{G}}(x)^*}{|W|}\sum_{w\sim w_0}\frac{\alpha(w\varphi)}{\det(1-xw\varphi)^*} \mod \Phi(x)\,,$$

where "$w\sim w_0$" means that $V(w\varphi,\Phi) = V(w_0\varphi,\Phi)$.

Following (S3) above, the elements $w\sim w_0$ are those such that ww_0^{-1} acts trivially on $V(w_0\varphi,\Phi)$, *i.e.*, are the elements of the coset $W_{\mathbb{L}}w_0$. Thus

$$\sum_{w\sim w_0}\frac{\alpha(w\varphi)}{\det(1-xw\varphi)^*} = \sum_{w\in W_{\mathbb{L}}}\frac{\alpha(ww_0\varphi)}{\det(1-xww_0\varphi)^*} = |W_{\mathbb{L}}|S_{\mathbb{L}}(\mathrm{Res}_{\mathbb{L}}^{\mathbb{G}}(\alpha)).$$

We know that $N_W(\mathbb{S})\subset N_W(\mathbb{L})$. Let us check the reverse inclusion. The group $N_W(\mathbb{L})/W_{\mathbb{L}}$ acts on $V^{W_{\mathbb{L}}}$ and centralizes $(w_0\varphi)|_{V^{W_{\mathbb{L}}}}$. Hence it stabilizes the characteristic subspaces of $(w_0\varphi)|_{V^{W_{\mathbb{L}}}}$, among them $V(w_0\varphi,\Phi)$. This shows that $N_W(\mathbb{L}) = N_W(\mathbb{S})$. In particular $|N_W(\mathbb{S})| = |W_{\mathbb{L}}||W_{\mathbb{G}}(\mathbb{L})|$. □

Applying Lemma 1.51 to the case where $\alpha = 1^{\mathbb{G}}$ gives

Proposition 1.52. —

Let $\mathbb{S}$ be a Sylow Φ-subcoset of $\mathbb{G}$, and set $\mathbb{L} = C_{\mathbb{G}}(\mathbb{S})$.

1. $\dfrac{1}{|W(\mathbb{L})|}\dfrac{P_{\mathbb{G}}(x)^*}{P_{\mathbb{L}}(x)^*} \equiv 1 \mod \Phi(x)\,.$
2. *Whenever α is a class function on $\mathbb{G}$, we have*

$$\mathrm{Feg}_{\mathbb{G}}(\alpha)(x) \equiv \mathrm{Feg}_{\mathbb{L}}(\mathrm{Res}_{\mathbb{L}}^{\mathbb{G}}(\alpha))(x) \mod \Phi(x)\,.$$

Now applying Lemma 1.51 and Proposition 1.52 to the cases where α is $\det_V$ and $\det_V^*$ gives the desired congruences in Theorem 1.50(4) (thanks to Definition 1.44)

$$|\mathbb{G}|_{\mathrm{nc}}/(|W_{\mathbb{G}}(\mathbb{L})||\mathbb{L}|_{\mathrm{nc}}) \equiv |\mathbb{G}|_{\mathrm{c}}/(|W_{\mathbb{G}}(\mathbb{L})||\mathbb{L}|_{\mathrm{c}}) \equiv 1 \mod \Phi(x)\,.$$

□

Fake degrees and regular elements

Let α be a class function on $\mathbb{G}$. Let $w\varphi$ be Φ-regular, so that the maximal torus $\mathbb{T}_{w\varphi}$ of $\mathbb{G}$ is a Sylow Φ-subcoset. We get from Proposition 1.52(2) that

$$\mathrm{Feg}_{\mathbb{G}}(\alpha)(x) \equiv \alpha(w\varphi) \mod \Phi(x),$$

which can be reformulated into the following proposition.

Proposition 1.53. —
Let $w\varphi$ be ζ-regular for some root of unity ζ.

1. *For any $\alpha \in \mathrm{CF}_{\mathrm{uf}}(\mathbb{G})$ we have* $\mathrm{Feg}_{\mathbb{G}}(\alpha)(\zeta) = \alpha(w\varphi)$.
2. *In particular, we have* $\mathrm{Feg}_{\mathbb{G}}(R_{w\varphi})(\zeta) = |C_W(w\varphi)|\,$.

1.4. The associated braid group

1.4.1. Definition. —

Here we let V be a complex vector space of finite dimension r, and $W \subset \mathrm{GL}(V)$ be a complex reflection group on V.

We recall some notions and results from [**BMR98**].

Choosing a base point $x_0 \in V^{\mathrm{reg}} = V - \bigcup_{H\in\mathcal{A}(W)} H$, we denote by $\mathbf{B}_W := \pi_1(V^{\mathrm{reg}}/W, x_0)$ the corresponding braid group, and we set $\mathbf{P}_W := \pi_1(V^{\mathrm{reg}}, x_0)$

Since the covering $V^{\mathrm{reg}} \twoheadrightarrow V^{\mathrm{reg}}/W$ is Galois by Steinberg's theorem (see *e.g.* [**Bro10**, 4.2.3]), we have the corresponding short exact sequence

$$1 \to \mathbf{P}_W \to \mathbf{B}_W \to W \to 1\,.$$

A *braid reflection* $\mathbf{s}$ in $\mathbf{B}_W$ is a generator of the monodromy around the image in V/W of a reflecting hyperplane $H \in \mathcal{A}(W)$. We then say that $\mathbf{s}$ is a *braid reflection around* H (or *around the orbit of* H *under* W).

1.4.2. Lengths. —

For each $H \in \mathcal{A}(W)$, there is a linear character

$$l_H : \mathbf{B}_W \to \mathbb{Z}$$

such that, whenever $\mathbf{s}$ is a braid reflection in $\mathbf{B}_W$,

$$l_H(\mathbf{s}) = \begin{cases} 1 & \text{if } \mathbf{s} \text{ is a braid reflection around the orbit of } H, \\ 0 & \text{if not}. \end{cases}$$

We have $l_H = l_{H'}$ if and only if H and H' are in the same W-orbit. We set

$$l := \sum_{H \in \mathcal{A}(W)/W} l_H .$$

1.4.3. The element $\boldsymbol{\pi}_W$. —

We denote by $\boldsymbol{\pi}_W$ (or simply by $\boldsymbol{\pi}$ if there is no ambiguity) the element of $\mathbf{P}_W$ defined by the loop

$$\boldsymbol{\pi}_W : [0,1] \to V^{\mathrm{reg}} \ , \ t \mapsto \exp(2\pi i t)x_0 .$$

We have $\boldsymbol{\pi}_W \in Z\mathbf{B}_W$, and if W acts irreducibly on V, we know by [**DMM11**, Thm. 1.2] (see also [**Bes06**, 12.4] and [**BMR98**, 2.24]) that $\boldsymbol{\pi}_W$ *is a generator of* $Z\mathbf{P}_W$ called *the positive generator of* $Z\mathbf{P}_W$.

We have

$$l_H(\boldsymbol{\pi}_W) = |\text{orbit of } H \text{ under } W| e_H$$

and in particular (by formula 1.1)

$$l(\boldsymbol{\pi}_W) = N_W^{\mathrm{ref}} + N_W^{\mathrm{hyp}} .$$

1.4.4. Lifting regular automorphisms. —

For this section one may refer to [**Bro10**, §18] (see also [**DM06**, §3]).

A. Lifting a ζ-regular element $w\varphi$. —

- We fix a root of unity ζ and a ζ-regular element $w\varphi$ in $W\varphi$, and we let δ denote the order of $w\varphi$ modulo W. Notice that since $(w\varphi)^\delta$ is a ζ^δ-regular element of W, then (by 1.4) $\zeta^{\delta e_W} = 1$.

  Let us also choose $a, d \in \mathbb{N}$ such that $\zeta = \zeta_d^a$ (a/d is well defined in $\mathbb{Q}/\mathbb{Z}$). By what precedes we know that $d \mid e_W a\delta$, or, in other words, $e_W \delta a/d \in \mathbb{Z}$.
- Let us denote by x_1 a ζ-eigenvector of $w\varphi$, and let us choose a path γ from x_0 to x_1 in V^{reg}.

– We denote by $\pi_{x_1,a/d}$ the path in V^{reg} from x_1 to ζx_1 defined by

$$\pi_{x_1,a/d} : t \mapsto \exp(2\pi i a t/d).x_1 .$$

Note that $\pi_{x_1,a/d}$ does depend on the choice of $a/d \in \mathbb{Q}$ and not only on ζ.

Following [**Bro10**, 5.3.2.], we have

– a path $[w\varphi]_{\gamma,a/d}$ (sometimes abbreviated $[w\varphi]$) in V^{reg} from x_0 to $(w\varphi)(x_0)$, defined as follows:

$$[w\varphi]_{\gamma,a/d} : \quad x_0 \overset{\gamma}{\rightsquigarrow} x_1 \overset{\pi_{x_1,a/d}}{\longrightarrow} \zeta x_1 \overset{(w\varphi)(\gamma^{-1})}{\rightsquigarrow} (w\varphi)(x_0)$$

– an automorphism $\mathbf{a}(w\varphi)_{\gamma,a/d}$ (sometimes abbreviated $\mathbf{a}(w\varphi)$) of $\mathbf{B}_W$, defined as follows: for $g \in W$ and $\mathbf{g}$ a path in V^{reg} from x_0 to gx_0, the path $\mathbf{a}(w\varphi)_{\gamma,a/d}(\mathbf{g})$ from x_0 to $\operatorname{Ad}(w\varphi)(g)x_0$ is

(1.54)

$$\mathbf{a}(w\varphi)_{\gamma,a/d}(\mathbf{g}) \ : \quad x_0 \overset{[w\varphi]}{\rightsquigarrow} (w\varphi)(x_0) \overset{(w\varphi)(\mathbf{g})}{\rightsquigarrow} (w\varphi g)(x_0) \overset{\operatorname{Ad}(w\varphi)(g)([w\varphi]^{-1})}{\rightsquigarrow} \operatorname{Ad}(w\varphi)(g)(x_0)$$

with the following properties.

Lemma 1.55. —

1. *The automorphism* $\mathbf{a}(w\varphi)_{\gamma,a/d}$ *has finite order, equal to the order of* $\operatorname{Ad}(w\varphi)$ *acting on* W.
2. *The path*

$$\boldsymbol{\rho}_{\gamma,a/d} := [w\varphi]_{\gamma,a/d} \cdot \mathbf{a}(w\varphi)([w\varphi]_{\gamma,a/d}) \cdots \mathbf{a}(w\varphi)^{\delta-1}([w\varphi]_{\gamma,a/d})$$

(often abbreviated $\boldsymbol{\rho}$*) defines an element of* $\mathbf{B}_W$ *which satisfies*

$$\boldsymbol{\rho}_{\gamma,a/d}^{d} = \boldsymbol{\pi}^{\delta a} .$$

Remark 1.56. —

1. Notice that $a/d \in \mathbb{Q}$ is unique up to addition of an integer, so that $\boldsymbol{\rho}$ is defined by $w\varphi$ up to multiplication by a power of $\boldsymbol{\pi}^{\delta}$.
2. Let us consider another path γ' from x_0 to x'_1, another eigenvector of $w\varphi$ with eigenvalue ζ. Then
   (a) the element $\boldsymbol{\rho}_{\gamma',a/d}$ is conjugate to $\boldsymbol{\rho}_{\gamma,a/d}$ by an element of $\mathbf{P}_W$, and
   (b) the element $\mathbf{a}(w\varphi)_{\gamma',a/d}$ is conjugate to $\mathbf{a}(w\varphi)_{\gamma,a/d}$ by an element of $\mathbf{P}_W$.

B. When φ is 1-regular. —

Now assume moreover that φ is 1-regular, and choose for base point x_0 an element fixed by φ. Let us write $1 = \exp(2\pi i n)$ for some $n \in \mathbb{Z}$ (which plays here the role played by a/d above).

Lemma 1.57. —

1. *The corresponding loop $[\varphi]$ defines $\boldsymbol{\pi}^n$.*
2. *The path $[w\varphi]_{\gamma,n}$ defines a lift $\mathbf{w}_{\gamma,n}$ (abbreviated to $\mathbf{w}$) of w in $\mathbf{B}_W$.*
3. *We have $\mathbf{a}(w\varphi)_{\gamma,n} = \mathrm{Ad}(\mathbf{w}_{\gamma,n}) \cdot \mathbf{a}(\varphi)$.*

Proof. —
(1) is obvious. (2) results from the fact that the path $[w\varphi]_{\gamma,a/d}$ starts at x_0 and ends at $w\varphi x_0 = wx_0$.

Since $\varphi(x_0) = x_0$, Definition 1.54 becomes simply $\mathbf{a}(\varphi)(\mathbf{g}) = \varphi(\mathbf{g})$, a lift of $\mathrm{Ad}(\varphi)(g)$ to $\mathbf{B}_W$. To prove (3), we notice that by (2) we have

$$\mathbf{a}(w\varphi)_{\gamma,a/d}(\mathbf{g}) =$$
$$x_0 \overset{\mathbf{w}}{\rightsquigarrow} wx_0 \overset{(w\varphi)(\mathbf{g})}{\rightsquigarrow} (w\varphi g)(x_0) \overset{\mathrm{Ad}(w\varphi)(g)(\mathbf{w}^{-1})}{\rightsquigarrow} \mathrm{Ad}(w\varphi)(g)(x_0).$$

Since

$$\mathrm{Ad}(\mathbf{w})(\mathbf{g}) =$$
$$x_0 \overset{\mathbf{w}}{\rightsquigarrow} wx_0 \overset{w(\mathbf{g})}{\rightsquigarrow} wgx_0 \overset{\mathrm{Ad}(w)(g)(\mathbf{w}^{-1})}{\rightsquigarrow} \mathrm{Ad}(w)(g)(x_0)$$

we see that

$$\mathrm{Ad}(\mathbf{w})(\varphi(\mathbf{g})) =$$
$$x_0 \overset{\mathbf{w}}{\rightsquigarrow} wx_0 \overset{(w\varphi)(\mathbf{g})}{\rightsquigarrow} wgx_0 \overset{\mathrm{Ad}(\varphi w)(g)(\mathbf{w}^{-1})}{\rightsquigarrow} \mathrm{Ad}(w)(g)(x_0)\,,$$

thus showing that $\mathrm{Ad}(\mathbf{w})(\varphi(\mathbf{g})) = \mathbf{a}(w\varphi)(\mathbf{g})$. □

In that case, we can consider the semidirect product $\mathbf{B}_W \rtimes \langle \mathbf{a}(\varphi) \rangle$, in which we set $\boldsymbol{\varphi} := \mathbf{a}(\varphi)$. Then assertion (2) of Lemma 1.55 becomes

$$\boldsymbol{\rho} = (\mathbf{w}\boldsymbol{\varphi})^\delta \quad \text{and} \quad (\mathbf{w}\boldsymbol{\varphi})^{\delta d} = \boldsymbol{\pi}^{\delta a}\,.$$

C. Centralizer in W and centralizer in $\mathbf{B}_W$. —

Now we return to the general situation (we are no more assuming that φ is 1-regular).

By (5) in Theorem 1.50, we know that the centralizer of $w\varphi$ in W, denoted by $W(w\varphi)$, is a complex reflection group on the ζ-eigenspace $V(w\varphi)$ of $w\varphi$.

Assume from now on that the base point x_0 is chosen in $V(w\varphi)^{\mathrm{reg}}$. Let us denote by $\mathbf{B}_W(w\varphi)$ the braid group (at x_0) of $W(w\varphi)$ on $V(w\varphi)$, and by $\mathbf{P}_W(w\varphi)$ its pure braid group.

Since the reflecting hyperplanes of $W(w\varphi)$ are the intersections with $V(w\varphi)$ of the reflecting hyperplanes of W (see for example [**Bro10**, 18.6]), the inclusion of $V(w\varphi)$ in V induces an inclusion

$$V(w\varphi)^{\mathrm{reg}} \hookrightarrow V^{\mathrm{reg}}$$

which in turn induces a natural morphism (see again [**Bro10**, 18.6])

$$\mathbf{B}_W(w\varphi) \to C_{\mathbf{B}_W}(\mathbf{a}(w\varphi))\,.$$

The next statement has been proved in all cases if $\overline{\varphi} = 1$ [**Bes06**, 12.5,(3)].

Theorem–Conjecture 1.58. —
The following assertion is true if $\overline{\varphi} = 1$, and it is a conjecture in the general case: The natural morphism

$$\mathbf{B}_W(w\varphi) \to C_{\mathbf{B}_W}(\mathbf{a}(w\varphi))$$

is an isomorphism.

Remark 1.59. — It results from the above Theorem–Conjecture that the positive generator $\boldsymbol{\pi}_W(w\varphi)$ of the center $Z\mathbf{P}_W(w\varphi)$ of $\mathbf{P}_W(w\varphi)$ is identified with the positive generator $\boldsymbol{\pi}$ of $\mathbf{P}_W$.

1.5. The generic Hecke algebra

1.5.1. Definition. —

The generic Hecke algebra $\mathcal{H}(W)$ of W is defined as follows. Let us choose a W-equivariant set of indeterminates

$$\mathbf{u} := (u_{H,i})_{(H\in\mathcal{A}(W))(i=0,\dots,e_H-1)}\,.$$

The algebra $\mathcal{H}(W)$ is the quotient of the group algebra of $\mathbf{B}_W$ over the ring of Laurent polynomials $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}] := \mathbb{Z}[(u_{H,i}^{\pm 1})_{H,i}]$ by the ideal generated by the elements $\prod_{i=0}^{e_H-1}(\mathbf{s}-u_{H,i})$ for $H \in \mathcal{A}(W)$ and $\mathbf{s}$ running over the set of braid reflections around H.

The linear characters of the generic Hecke algebra $\mathcal{H}(W)$ are described as follows.

Let $\chi : \mathcal{H}(W) \to \mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$ be an algebra morphism. Then there is a W-equivariant family of integers $(j_H^\chi)_{H\in\mathcal{A}(W)}, j_H^\chi \in \{0,\dots,e_H-1\}$, such that, whenever $\mathbf{s}$ is a braid reflection around H, we have $\chi(\mathbf{s}) = u_{H,j_H^\chi}$.

1.5.2. Parabolic subalgebras. —

Let I be an intersection of reflecting hyperplanes of W, and let $\mathbf{B}_{W_I}$ be the braid group of the parabolic subgroup W_I of W.

If $\mathbf{u} = (u_{H,i})_{(H\in\mathcal{A}(W))(i=0,\dots,e_H-1)}$ is a W-equivariant family of indeterminates as above, then the family

$$\mathbf{u}_I := (u_{H,i})_{(H\in\mathcal{A}(W_I))(i=0,\dots,e_H-1)}$$

is a W_I-equivariant family of indeterminates.

We denote by $\mathcal{H}(W_I, W)$ the quotient of the group algebra of $\mathbf{B}_{W_I}$ over $\mathbb{Z}[\mathbf{u}_I, \mathbf{u}_I^{-1}]$ by the ideal generated by the elements $\prod_{i=0}^{e_H-1}(\mathbf{s} - u_{H,i})$ for $H \in \mathcal{A}(W_I)$ and $\mathbf{s}$ a braid reflection of $\mathbf{B}_{W_I}$ around H.

The algebra $\mathcal{H}(W_I, W)$ is a specialization of the generic Hecke algebra of W_I, called the *parabolic subalgebra of* $\mathcal{H}(W)$ *associated with* I.

The natural embeddings of $\mathbf{B}_{W_I}$ into $\mathbf{B}_W$ (see *e.g.* [**BMR98**, §4]) are permuted transitively by $\mathbf{P}_W$. The choice of such an embedding defines a morphism of $\mathcal{H}(W_I, W)$ onto a subalgebra of $\mathcal{H}(W)$ ([**BMR98**, §4]).

1.5.3. The main Theorem–Conjecture. —

Notation.—

- An element $P(\mathbf{u}) \in \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ is called *multi-homogeneous* if, for each $H \in \mathcal{A}(W)$, it is homogeneous as a Laurent polynomial in the indeterminates $\{u_{H,i} \mid i = 0, \dots, e_H - 1\}$.
- The group $\mathfrak{S}_W := \prod_{H\in\mathcal{A}(W)/W} \mathfrak{S}_{e_H}$ acts naturally on the set of indeterminates $\mathbf{u}$.
- We denote by $\xi \mapsto \xi^\vee$ the involutive automorphism of $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ which sends $u_{H,i}$ to $u_{H,i}^{-1}$ (for all H and i).

The following assertion is conjectured to be true for all finite reflection groups. It has been proved for almost all irreducible complex reflection groups (see [**BMM99**] and [**MM10**] for more details).

Theorem–Conjecture 1.60. —

1. *The algebra* $\mathcal{H}(W)$ *is free of rank* $|W|$ *over* $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$*, and by extension of scalars to the field* $\mathbb{Q}(\mathbf{u})$ *it becomes semisimple.*
2. *There exists a unique symmetrizing form*

$$\tau_W : \mathcal{H}(W) \to \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$$

(usually denoted simply τ*) with the following properties.*

   (a) *Through the specialization* $u_{H,j} \mapsto \exp(2i\pi j/e_H)$, *the form* τ *becomes the canonical symmetrizing form on the group algebra of* W.

   (b) *For all* $b \in B$, *we have* $\tau(b^{-1})^\vee = \dfrac{\tau(b\boldsymbol{\pi})}{\tau(\boldsymbol{\pi})}$, *where* $\boldsymbol{\pi} := \boldsymbol{\pi}_W$ *(see 1.4.3).*

3. *If I is an intersection of reflecting hyperplanes of W, the restriction of τ_W to a naturally embedded parabolic subalgebra $\mathcal{H}(W_I, W)$ is the corresponding specialization of the form τ_{W_I}.*
4. *The form τ satisfies the following conditions.*
   (a) *For $b \in B$, $\tau(b)$ is invariant under the action of $\mathfrak{S}_W$.*
   (b) *As an element of $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$, $\tau(b)$ is multi-homogeneous of degree $l_H(b)$ in the indeterminates $\{u_{H,i} \mid i = 0, \ldots, e_H - 1\}$ for all $H \in \mathcal{A}(W)$. In particular, we have*
   $$\tau(1) = 1 \quad \textit{and} \quad \tau(\boldsymbol{\pi}_W) = (-1)^{N_W^{\mathrm{ref}}} \prod_{\substack{H \in \mathcal{A}(W) \\ 0 \leqslant i \leqslant e_H - 1}} u_{H,i}\,.$$

1.5.4. Splitting field. —

An irreducible complex reflection group in $\mathrm{GL}(V)$ is said to be *well-generated* if it can be generated by $\dim(V)$ reflections (see *e.g.* [**Bro10**, §4.4.2] for more details).

The following theorem has been proved in [**Mal99**].

Theorem 1.61. —
Assume assertion (1) of Theorem–Conjecture 1.60 holds.
Let W be an irreducible complex reflection group, and let
$$m_W := \begin{cases} |ZW| & \textit{if } W \textit{ is well-generated,} \\ |\boldsymbol{\mu}(\mathbb{Q}_W)| & \textit{else.} \end{cases}$$
Let us choose a W-equivariant set of indeterminates $\mathbf{v} := (v_{H,j})$ subject to the conditions
$$v_{H,j}^{m_W} = \zeta_{e_H}^{-j} u_{H,j}\,.$$
Then the field $\mathbb{Q}_W(\mathbf{v})$ is a splitting field for $\mathcal{H}(W)$.

We denote by $\mathrm{Irr}(\mathcal{H}(W))$ the set of irreducible characters of
$$\mathbb{Q}_W(\mathbf{v})\mathcal{H}(W) := \mathbb{Q}_W(\mathbf{v}) \otimes_{\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]} \mathcal{H}(W)\,,$$
which is also the set of irreducible characters of
$$\overline{\mathbb{Q}}(\mathbf{v})\mathcal{H}(W) := \overline{\mathbb{Q}}(\mathbf{v}) \otimes_{\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]} \mathcal{H}(W)\,.$$

Following [**Mal00**, §2D], we see that the action of the group $\mathfrak{S}_W$ on $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ by permutations of the indeterminates $u_{H,j}$ extends to an action on $\overline{\mathbb{Q}}[\mathbf{v}, \mathbf{v}^{-1}]$. Indeed, we let $\mathfrak{S}_W$ act trivially on $\overline{\mathbb{Q}}$ and for all $\sigma \in \mathfrak{S}_W$ we set
$$\sigma(v_{H,j}) := \exp\left(2\pi i(\sigma(j) - j)/e_H m_W\right) v_{H,j}\,.$$
That action of $\mathfrak{S}_W$ induces an action on $\mathcal{H}(W)$, and then an action on $\mathrm{Irr}(\mathcal{H}(W))$ by

$$\sigma(\chi)(h) := \sigma(\chi(\sigma^{-1}(h))) \quad \text{for } \sigma \in \mathfrak{S}_W\,,\ h \in \mathcal{H}(W)\,,\ \chi \in \mathrm{Irr}(\mathcal{H}(W))\,. \tag{1.62}$$

1.5.5. Schur elements. —

The next statement follows from Theorem 1.61 by a general argument which goes back to Geck [**Gec93**].

Proposition 1.63. —
Assume Theorem–Conjecture 1.60 holds.
For each $\chi \in \operatorname{Irr}(\mathcal{H}(W))$ *there is a non-zero* $S_\chi \in \mathbb{Z}_W[\mathbf{v},\mathbf{v}^{-1}]$ *such that*

$$\tau = \sum_{\chi\in\operatorname{Irr}(\mathcal{H}(W))} \frac{1}{S_\chi}\chi\,.$$

The Laurent polynomials S_χ are called the *generic Schur elements* of $\mathcal{H}(W)$ (or of W).

Let us denote by $S \mapsto S^\vee$ the involution of $\mathbb{Q}_W(\mathbf{v})$ consisting in
- $v_{H,j}^\vee := v_{H,j}^{-1}$ for all $H \in \mathcal{A}(W)$, $j = 0,\dots,e-1$,
- complex conjugating the scalar coefficients.

Note that this extends the previous operation $\lambda \mapsto \lambda^\vee$ on $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$ defined above in 1.5.3.

The following property of the Schur elements (see [**BMM99**, 2.8]) is an immediate consequence of the characteristic property (see Theorem 1.60(2)(b)) of the canonical trace τ.

Lemma 1.64. —
Assume Theorem–Conjecture 1.60 holds.
Whenever $\chi \in \operatorname{Irr}(\mathcal{H}(W))$*, we have*

$$S_\chi(\mathbf{v})^\vee = \frac{\tau(\boldsymbol{\pi})}{\omega_\chi(\boldsymbol{\pi})} S_\chi(\mathbf{v})\,,$$

where ω_χ *denotes the central character corresponding to* χ*.*

1.5.6. About Theorem–Conjecture 1.60. —

We make some remarks about assertions (4) and (3) of 1.60.

Note that the equality $\tau(1) = 1$ results from the formula $\tau(b^{-1})^\vee = \tau(b\boldsymbol{\pi})/\tau(\boldsymbol{\pi})$ (condition (2)(b)) applied with $b = 1$. The same formula applied with $b = \boldsymbol{\pi}^{-1}$ shows that $\tau(\boldsymbol{\pi})$ is an invertible element of the Laurent polynomial ring, thus a monomial. Multi-homogeneity and invariance by $\mathfrak{S}_W$ will then imply the claimed equality in (4)(b) up to a constant; that constant can be checked by specialization.

Thus in order to prove (4)(b) (assuming (2)), we just have to prove multi-homogeneity. It is stated in [**BMM99**, p.179] that (a) and (b) are implied by [**BMM99**, Ass. 4] (which is the same as conjecture 2.6 of [**MM10**]), assumption that we repeat below (Assumption 1.65).

In [**BM97**] and [**GIM00**], it is shown that 1.65 holds for all imprimitive irreducible complex reflection groups.

Assumption 1.65. —
There is a section

$$W \hookrightarrow \mathbf{B}_W\,,\ w \mapsto \mathbf{w}$$

with image $\mathbf{W}$*, such that* $1 \in \mathbf{W}$*, and for* $\mathbf{w} \in \mathbf{W} - \{1\}$ *we have* $\tau(\mathbf{w}) = 0$.

Let us spell out a proof of that implication.

Lemma 1.66. —
Under Assumption 1.65, properties (4)(a) and (4)(b) of Theorem 1.60 hold.

Proof. —
By the homogeneity property of the character values (see *e.g.* [**BMM99**, Prop. 7.1, (2)]), applied to the grading given by each function l_H, we see that for $\chi \in \mathrm{Irr}(\mathcal{H}(W))$ and $b \in \mathbf{B}_W$, the value $\chi(b)$ is multi-homogeneous of degree $l_H(b)$.

From the definition of the Schur elements S_χ, it follows that $\tau(b)$ is multi-homogeneous of degree $l(b)$ if and only if the Schur elements S_χ are multi-homogeneous of degree 0.

Let M be the matrix $\{\chi(\mathbf{w})\}_{\chi\in\mathrm{Irr}(\mathcal{H}(W)),\mathbf{w}\in\mathbf{C}}$ where $\mathbf{C}$ is a subset of $\mathbf{W}$ which consists of the lift of one representative of each conjugacy class of W. Then the equation for the Schur elements reads $S = X \cdot M^{-1}$ where S is the vector $(1/S_\chi)_{\chi\in\mathrm{Irr}(\mathcal{H}(W))}$ and X is the vector $(1, 0, \ldots, 0)$ (assuming that $\mathbf{C}$ starts with 1). From this equation, it results that the inverses of the Schur elements are the cofactors of the first column of M divided by the determinant of M, which are multi-homogeneous of degree 0. From the same equation, since X is invariant by $\mathfrak{S}_W$, it results that for $\sigma \in \mathfrak{S}_W$, we have (see 1.62) $\sigma(S_\chi) = S_{\sigma(\chi)}$, which implies that τ is invariant by $\mathfrak{S}_W$. □

Note that, for the above proof, we just need the existence of $\mathbf{C}$ and not of $\mathbf{W}$.

We now turn to assertion (3) of Theorem–Conjecture 1.60. The proof of unicity of τ given in [**BMM99**] using part (2) of Theorem-Assumption 1.56 applies to any parabolic subalgebra of the generic algebra. Hence assertion (3) would follow from the next assumption.

Assumption 1.67. —
Let $\mathbf{B}_I$ *be a parabolic subgroup of* $\mathbf{B}_W$ *corresponding to the intersection of hyperplanes* I*, and let* $\boldsymbol{\pi}_I$ *be the corresponding element of the center of* $\mathbf{B}_I$*. Then for any* $b \in \mathbf{B}_I$*, we have* $\tau(b^{-1})^\vee = \tau(b\boldsymbol{\pi}_I)/\tau(\boldsymbol{\pi}_I)$.

From now on we shall assume that Theorem–Conjecture 1.60 holds.

1.5.7. The cyclic case. —

For what follows we refer to [**BM93**, §2].

Assume that $W = \langle s \rangle \subset \mathrm{GL}_1(\mathbb{C})$ is cyclic of order e. Denote by $\mathbf{s}$ the corresponding braid reflection in $\mathbf{B}_W$. Let $\mathbf{u} = (u_i)_{i=0,\ldots,e-1}$ be a set of indeterminates.

Then clearly there exists a unique symmetrizing form τ on the generic Hecke algebra $\mathcal{H}(W)$ of W (an algebra over $\mathbb{Z}[(u_i^{\pm 1})_{i=0,\ldots,e-1}]$) such that

$$\tau(1) = 1 \quad \text{and } \tau(\mathbf{s}^i) = 0 \text{ for } i = 1, \ldots, e-1\,.$$

This is the form from 1.60.

For $0 \leq i \leq e-1$, let us denote by $\chi_i : \mathcal{H}(W) \to \mathbb{Q}(\mathbf{u})$ the character defined by $\chi_i(\mathbf{s}) = u_i$. We set $S_i(\mathbf{u}) := S_{\chi_i}(\mathbf{u})$.

Lemma 1.68. —

The Schur elements $S_i(\mathbf{u})$ of $W = \langle s \rangle$ have the form

$$S_i(\mathbf{u}) = \prod_{j \neq i} \frac{u_j - u_i}{u_j} = \frac{1}{P(0,\mathbf{u})} \left(t \frac{d}{dt} P(t,\mathbf{u}) \right) |_{t=u_i}\,,$$

where $P(t,\mathbf{u}) := (t-u_0)\cdots(t-u_{e-1})$.

Proof. —

The first formula is in [**BM93**, Bem. 2.4]. For the second, notice that we have

$$\frac{d}{dt} P(t,\mathbf{u}) = P(0,\mathbf{u}) \sum_i \frac{1}{u_i} \prod_{j \neq i} \frac{t-u_i}{u_i - u_j} S_i(\mathbf{u})\,.$$

□

The following Lemma will be used later. Its proof is a straightforward computation (see Lemma 1.64).

Lemma 1.69. —

With the above notation, we have

$$S_i(\mathbf{u})^\vee = (-1)^{e-1} u_i^{-e} (\prod_{j=0}^{e-1} u_j)\, S_i(\mathbf{u}) = -u_i^{-e} P(0,\mathbf{u}) S_i(\mathbf{u})\,.$$

1.6. Φ-cyclotomic Hecke algebras, Rouquier blocks

We now consider specialized cyclotomic Hecke algebras involving only a single indeterminate, x.

Let K be a number field, stable by complex conjugation $\lambda \mapsto \lambda^*$. Let W be a finite reflection group on a K-vector space V of dimension r.

Let $\Phi(x)$ be a K-cyclotomic polynomial — see Definition 1.48. We assume that the roots of $\Phi(x)$ have order d, and we denote by ζ one of these roots.

1.6.1. Φ-cyclotomic Hecke algebras. —

We recall that we set $m_K = |\boldsymbol{\mu}(K)|$. We choose an indeterminate v such that $v^{m_K} = \zeta^{-1}x$.

Definition 1.70. —

1. *A cyclotomic specialization is a morphism* $\sigma : \mathbb{Z}[(u_{H,i}^{\pm 1})_{H,i}] \to K[v^{\pm 1}]$ *defined as follows:*

   There are

   – *a* W*-equivariant family* $(\zeta_{H,i})_{(H\in\mathcal{A}(W))(i=0,\dots,e_H-1)}$ *of roots of unity in* K*,*

   – *and a* W*-equivariant family* $(m_{H,i})_{(H\in\mathcal{A}(W))(i=0,\dots,e_H-1)}$ *of rational numbers,*

   such that

   (a) $m_K m_{H,i} \in \mathbb{Z}$ *for all* H *and* i*,*

   (b) *the specialization* σ *is of the type* $\sigma : u_{H,i} \mapsto \zeta_{H,i} v^{m_K m_{H,i}}$.

2. *A* Φ*-cyclotomic Hecke algebra of* W *is the algebra*

$$\mathcal{H}_\sigma := K[v^{\pm 1}] \otimes_\sigma \mathcal{H}(W)$$

   defined by applying a cyclotomic specialization $\sigma : \mathbb{Z}[(u_{H,i}^{\pm 1})_{H,i}] \to K[v^{\pm 1}]$ *to the base ring of the generic Hecke algebra of* W*, which satisfies the following conditions:*

   For each $H \in \mathcal{A}(W)$*, the polynomial*

$$P_H(\mathbf{u})(t) = \prod_{i=0}^{e_H-1} (t - u_{H,i}) \in \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}][t]$$

   specializes under σ *to a polynomial* $P_H(t,x)$ *such that*

   (CA1) $P_H(t,x) \in K[t,x]$,

   (CA2) $P_H(t,x) \equiv t^{e_H} - 1 \pmod{\Phi(x)}$.

Remark 1.71. —

1. It follows from Theorem 1.61 that the field $K(v)$ is a splitting field for $\mathcal{H}_\sigma$.
2. Property (2),(CA2) of the preceding definition shows that
   (a) a Φ-cyclotomic Hecke algebra $\mathcal{H}_\sigma$ as above specializes to the group algebra KW through the assignment $v \mapsto 1$ (which implies $x \mapsto \zeta$);
   (b) the specialization of $K[\mathbf{u}, \mathbf{u}^{-1}] \otimes_{\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]} \mathcal{H}(W)$ to the group algebra KW (given by $u_{H,i} \mapsto \zeta_{e_H}^i$ for $0 \le i \le e_H - 1$) factorizes through its specialization to any Φ-cyclotomic algebra.

1.6.2. The Rouquier ring $R_K(v)$. —

Definition 1.72. —

1. *We call* Rouquier ring *of K and denote by $R_K(v)$ the $\mathbb{Z}_K$-subalgebra of $K(v)$*
$$R_K(v) := \mathbb{Z}_K[v, v^{-1}, (v^n - 1)^{-1}_{n\geq 1}] .$$

2. *Let $\sigma : u_{H,j} \mapsto \zeta_{H,j} v^{n_{H,j}}$ be a cyclotomic specialization defining a Φ-cyclotomic Hecke algebra $\mathcal{H}_\sigma$. The* Rouquier blocks *of $\mathcal{H}_\sigma$ are the blocks of the algebra $R_K(v)\mathcal{H}_\sigma$.*

Remark 1.73. —

It has been shown by Rouquier (cf. [**Rou99**, Th.1]), that if W is a Weyl group and $\mathcal{H}_\sigma$ is its ordinary Iwahori-Hecke algebra, then the Rouquier blocks of $\mathcal{H}_\sigma$ coincide with the families of characters defined by Lusztig. In this sense, the Rouquier blocks generalize the notion of "families of characters" to the Φ-cyclotomic Hecke algebras of all complex reflection groups.

Observe that the Rouquier ring $R_K(v)$ is a Dedekind domain (see [**BK03**, §2.B]).

1.6.3. The Schur elements of a cyclotomic Hecke algebra. —

In this section we assume that Conjecture 1.60 holds.

Let us recall the form of the Schur elements of the cyclotomic Hecke algebra $\mathcal{H}_\sigma$ [**BK03**, Prop. 2.5] (see also [**Chl09**, Prop.4.3.5]).

Proposition 1.74. —

If χ is an irreducible character of $K(v)\mathcal{H}_\sigma$, then its Schur element S_χ is of the form
$$S_\chi = m_\chi v^{a_\chi} \prod_{\Psi} \Psi(v)^{n_{\chi,\Psi}}$$
where $m_\chi \in \mathbb{Z}_K$, $a_\chi \in \mathbb{Z}$, Ψ runs over the K-cyclotomic polynomials and $(n_{\chi,\Psi})$ is a family of almost all zero elements of $\mathbb{N}$.

The bad prime ideals of a cyclotomic Hecke algebra have been defined in [**BK03**, Def. 2.6] (see also [**MR03**], and [**Chl09**, Def. 4.4.3]).

Definition 1.75. —

A prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ lying over a prime number p is σ-bad for W, if there exists $\chi \in \mathrm{Irr}(K(v)\mathcal{H}_\sigma)$ with $m_\chi \in \mathfrak{p}$. In this case, p is called a σ-bad prime number for W.

Remark 1.76. —

In the case of the principal series of a split finite reductive group, that is, if W is a Weyl group and $\mathcal{H}_\sigma$ is the usual Hecke algebra of W — the algebra which will be called below (see 3.44) the *"1-cyclotomic special algebra of compact type"* —, it is well kown (this goes back, at least implicitly, to [**Lus79**] and [**Lus82**]) that the corresponding bad prime ideals are the ideals generated by the bad prime numbers (in the usual sense) for W.

1.6.4. Rouquier blocks, central morphisms, and the functions a and A. —

The next two assertions have been proved in [**BK03**, Prop. 2.8 & 2.9] (see also [**Chl09**, §4.4.1]).

Proposition 1.77. —
Let $\chi, \psi \in \mathrm{Irr}(K(v)\mathcal{H}_\sigma)$. The characters χ and ψ are in the same Rouquier block of $\mathcal{H}_\sigma$ if and only if there exist

- *a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(K(v)\mathcal{H}_\sigma)$,*
- *and a finite sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of σ-bad prime ideals for W*

such that

1. *$\chi_0 = \chi$ and $\chi_n = \psi$,*
2. *for all j $(1 \leq j \leq n)$, $\omega_{\chi_{j-1}} \equiv \omega_{\chi_j} \bmod \mathfrak{p}_j R_K(v)$.*

Following the notations of [**BMM99**, §6B], for every element $P(v) \in \mathbb{C}(v)$, we call

- valuation of $P(v)$ at v and denote by $\mathrm{val}_v(P)$ the order of $P(v)$ at 0,
- degree of $P(v)$ at v and denote by $\deg_v(P)$ the negative of the valuation of $P(1/v)$.

Moreover, as $x = \zeta v^{m_K}$, we set

$$\mathrm{val}_x(P(v)) := \frac{\mathrm{val}_v(P)}{|\boldsymbol{\mu}(K)|} \quad \text{and} \quad \deg_x(P(v)) := \frac{\deg_v(P)}{|\boldsymbol{\mu}(K)|} \, .$$

For $\chi \in \mathrm{Irr}(K(v)\mathcal{H}_\sigma)$, we define

$$a_\chi := \mathrm{val}_x(\frac{S_1(v)}{S_\chi(v)}) \quad \text{and} \quad A_\chi := \deg_x(\frac{S_1(v)}{S_\chi(v)}) \, . \tag{1.78}$$

Remark 1.79. —
For W a Weyl group, the integers a_χ and A_χ are just those for the generic character corresponding to χ (see Notation 4.11 below).

The following result is proven in [**BK03**, Prop. 2.9].

Proposition 1.80. —
Let $\chi, \psi \in \mathrm{Irr}(K(v)\mathcal{H}_\sigma)$. If the characters χ and ψ are in the same Rouquier block of $\mathcal{H}_\sigma$, then

$$a_\chi + A_\chi = a_\psi + A_\psi \, .$$

For all Coxeter groups, Lusztig has proved that if χ and ψ belong to the same family, hence (by Remark 1.73) to the same Rouquier block of the Hecke algebra, then $a_\chi = a_\psi$ and $A_\chi = A_\psi$. This assertion has also been generalized by a case by case analysis (see [**BK03**, Prop.4.5], [**MR03**, Th.5.1], [**Chl08**, Th.6.1], and [**Chl09**, §4.4] for detailed references) to the general case.

Theorem 1.81. —

Let W be a complex reflection group, and let $\mathcal{H}_\sigma$ be a cyclotomic Hecke algebra associated with W.

Whenever $\chi, \psi \in \operatorname{Irr}(K(v)\mathcal{H}_\sigma)$ belong to the same Rouquier block of $\mathcal{H}_\sigma$, we have

$$a_\chi = a_\psi \quad \textit{and} \quad A_\chi = A_\psi \, .$$

CHAPTER 2

COMPLEMENTS ON FINITE REDUCTIVE GROUPS

2.1. Notation and hypothesis for finite reductive groups

Before proceeding our development for complex reflection groups, we collect some facts from the theory of finite reductive groups associated to Weyl groups. More precisely, we state a number of results and conjectures about Deligne-Lusztig varieties associated with regular elements, Frobenius eigenvalues of unipotent characters attached to such varieties, actions of some braid groups on these varieties, in connection with the so-called abelian defect group conjectures and their specific formulation in the case of finite reductive groups (see [**Bro90**, §6], and also [**BM96**]).

These results and conjectures will justify and guide most of the definitions and properties given in the following paragraphs about the more general situation where finite reductive groups are replaced by spetsial reflection cosets.

Let $\mathbf{G}$ be a quasi-simple connected reductive group over the algebraic closure of a prime field $\mathbb{F}_p$, endowed with an isogeny F such that $G := \mathbf{G}^F$ is finite.

Our notation is standard:

- δ is the smallest power such that F^δ is a *split Frobenius* (this exists since $\mathbf{G}$ is quasi-simple). We denote by $x \mapsto x.F$ the action of F on elements or subsets of $\mathbf{G}$.
- The real number q ($\sqrt{p}$ raised to an integral power) is defined by the following condition: F^δ defines a rational structure on $\mathbb{F}_{q^\delta}$.
- $\mathbf{T}_1$ is a maximal torus of $\mathbf{G}$ which is stable under F and contained in an F-stable Borel subgroup of $\mathbf{G}$, W is its Weyl group. The action on $V := \mathbb{C} \otimes_{\mathbb{Z}} X(\mathbf{T}_1)$ induced by F is of the form $q\varphi$, where φ is an element of finite order of $N_{\mathrm{GL}(V)}(W)$ which is 1-regular. Thus δ is the order of the element of $N_{\mathrm{GL}(V)}(W)/W$ defined by φ. For $w \in W$ we shall also sometimes note $\varphi(w) := \varphi w \varphi^{-1} = {}^{\varphi}w$.

We also use notation and results from previous work about the braid group of W and the Deligne–Lusztig varieties associated to $\mathbf{G}$ (see [**BM96**], [**DM13**]).

We use freely definitions and notation introduced in §1.4 above. Recall that for $a, d \in \mathbb{N}$, $\zeta_d^a := \exp(2\pi i a/d)$, and let $w\varphi$ be a ζ_d^a-regular element for W.

It is possible to choose a base point x_0 fixed by φ in one of the fundamental chambers of W, which we do. Indeed, φ stabilizes the positive roots corresponding to the F-stable Borel subgroup containing $\mathbf{T}_1$ and consequently fixes their sum.

Since $\mathbf{G}$ is quasi-simple, W acts irreducibly on V, and so ([**Del72**] or [**BS72**]) the center of the pure braid group $\mathbf{P}_W$ is cyclic. We denote by $\boldsymbol{\pi}$ its positive generator.

Following [**DM06**, §3], we choose all our paths satisfying the conditions of [**DM06**, Prop. 3.5]. This allows us to define ([**DM06**, Def. 3.7]) the lift $\mathbf{W} \subset \mathbf{B}_W$ of W, and the monoid $\mathbf{B}_W^+$ generated by $\mathbf{W}$. We denote $w \mapsto \mathbf{w}$ (resp. $\mathbf{w} \mapsto w$) this lift (resp. the projection $\mathbf{W} \to W$).

In particular, we choose a regular eigenvector x_1 of $w\varphi$ associated with the eigenvalue ζ_d^a, and a path γ in V^{reg} from x_0 to x_1 satisfying those conditions.

Then, following §1.4.4 above:

- As in Lemma 1.55 and Lemma 1.57, we lift $w\varphi$ to a path $\mathbf{w}_{\gamma,a/d}$ (abbreviated $\mathbf{w}$) in V^{reg} from the base point x_0 to $(w\varphi)(x_0) = wx_0$, we denote by $\boldsymbol{\varphi}$ the automorphism of $\mathbf{B}_W$ defined by φ, and we have $\mathbf{a}(w\varphi)_{\gamma,a/d} = \mathrm{Ad}(\mathbf{w}) \cdot \mathbf{a}(\varphi)$.
- If
$$\boldsymbol{\rho} := \mathbf{w} \cdot \boldsymbol{\varphi}(\mathbf{w}) \cdot \cdots \cdot \boldsymbol{\varphi}^{\delta-1}(\mathbf{w}),$$
we have $\boldsymbol{\rho} \in \mathbf{B}_W$ and $\boldsymbol{\rho}^d = \boldsymbol{\pi}^{\delta a}$.
- Both $\mathbf{w}$ and $\boldsymbol{\rho}$ belong to $\mathbf{B}_W^+$.

We will work in the semi-direct product $\mathbf{B}_W^+ \rtimes \langle\boldsymbol{\varphi}\rangle$ where we have $(\mathbf{w}\boldsymbol{\varphi})^\delta = \boldsymbol{\rho}$ and $(\mathbf{w}\boldsymbol{\varphi})^d = \boldsymbol{\pi}^a\boldsymbol{\varphi}^d$. We denote by $\mathbf{B}_W(\mathbf{w}\boldsymbol{\varphi})$ (resp. $\mathbf{B}_W^+(\mathbf{w}\boldsymbol{\varphi})$) the centralizer of $\mathbf{w}\boldsymbol{\varphi}$ in $\mathbf{B}_W$ (resp. $\mathbf{B}_W^+$).

We denote by $\mathcal{B}$ the variety of Borel subgroups of $\mathbf{G}$ and by $\mathcal{T}$ the variety of maximal tori. The orbits of $\mathbf{G}$ on $\mathcal{B} \times \mathcal{B}$ are in canonical bijection with W, and we denote $\mathbf{B} \xrightarrow{w} \mathbf{B}'$ the fact that the pair $(\mathbf{B}, \mathbf{B}')$ belongs to the orbit parametrized by $w \in W$.

The Deligne–Lusztig variety $\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}$ is defined as in [**BM96**, Déf. 1.6 and §6] (following [**Del97**]). It is irreducible; indeed, since $\mathbf{w}\boldsymbol{\varphi}$ is a root of $\boldsymbol{\pi}$, the decomposition of $\mathbf{w}$ contains at least one reflection of each orbit of $\boldsymbol{\varphi}$, and the irreducibility follows from [**DM06**, 8.4], (it is also the principal result of [**BR06**]).

Note that if $\mathbf{w} \in \mathbf{W}$ then the Deligne–Lusztig variety $\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}$ associated to the "braid element" $\mathbf{w}\boldsymbol{\varphi}$ is nothing but the classical Deligne–Lusztig variety

$$\mathbf{X}_{w\varphi} = \{\mathbf{B} \in \mathcal{B} \mid \mathbf{B} \xrightarrow{w} \mathbf{B}.F\}$$

associated to the "finite group element" $w\varphi$. When a is prime to δ and $2a \leq d$, by choosing for $\mathbf{w}\boldsymbol{\varphi}$ the a-th power of the lift of a Springer element (see [**BM96**, 3.10 and 6.5]) we may ensure that $\mathbf{w} = \mathbf{w}_{\gamma,a/d} \in \mathbf{W}$.

2.2. Deligne–Lusztig varieties attached to regular elements

We generalize to regular elements a formula of Lusztig giving the product of Frobenius eigenvalues on the cohomology of a Coxeter variety.

In what follows we consider the usual Deligne–Lusztig variety $\mathbf{X}_{w\varphi}$ (attached to $w\varphi \in \mathrm{GL}(V)$).

Proposition 2.1. —
Let $w\varphi$ be a ζ_d-regular element of $W\varphi$. If $\mathbf{T}_{w\varphi}$ is an F-stable maximal torus of $\mathbf{G}$ of type $w\varphi$, we have

$$|\mathbf{X}_{w\varphi}^{F^d}| = \frac{|\mathbf{G}^F|}{|\mathbf{T}_{w\varphi}^F|} .$$

In order to proceed, we reformulate that proposition in terms which can be generalized to complex reflection groups; the proof of that reformulation is then immediate by Proposition 1.47(1).

Corollary 2.2. —
Let $w\varphi$ be a ζ_d-regular element of $W\varphi$. We have

$$|\mathbf{X}_{w\varphi}^{F^d}| = \widetilde{\det}_V(w\varphi)^{-1} q^{N_W^{\mathrm{hyp}}} \mathrm{Deg} R_{w\varphi}^{\mathbf{G}^F} .$$

Proof of 2.1. —
The proposition generalizes [**Lus76**, 3.3(ii)] and our proof is inspired by the argument given there.

We shall establish a bijection

$$\begin{gathered}\mathbf{X}_{w\varphi}^{F^d} = \Big\{ \mathbf{B} \in \mathcal{B} \mid \mathbf{B} \stackrel{w}{\to} \mathbf{B}.F \,,\, \mathbf{B}.F^d = \mathbf{B} \Big\} \\ \stackrel{\sim}{\longleftrightarrow} \\ \Big\{ (\mathbf{T}, \mathbf{B}) \in \mathcal{T} \times \mathcal{B} \mid \mathbf{T}.F = \mathbf{T} \,,\, \mathbf{T} \subset \mathbf{B} \,,\, \mathbf{B} \stackrel{w}{\to} \mathbf{B}.F \Big\} ,\end{gathered} \tag{2.3}$$

and the result will follow from counting the above set of pairs.

Let us check first that last implication. Notice that any maximal F-stable torus $\mathbf{T}$ which is contained in a Borel subgroup $\mathbf{B}$ such that $\mathbf{B} \stackrel{w}{\to} \mathbf{B}.F$ is a torus of type $w\varphi$ (see [**DL76**, §1]). Since all these tori of type $w\varphi$ are $\mathbf{G}^F$-conjugate to $\mathbf{T}$, their number is

$$\frac{|\mathbf{G}^F|}{|N_{\mathbf{G}^F}(\mathbf{T})|} = \frac{1}{|W(w\varphi)|} \frac{|\mathbf{G}^F|}{|\mathbf{T}^F|} .$$

To find the number of pairs $(\mathbf{T}, \mathbf{B})$ as above, we must multiply the above number by the number of Borel subgroups $\mathbf{B}'$ such that $\mathbf{T} \subset \mathbf{B}'$ and $\mathbf{B}' \stackrel{w}{\to} \mathbf{B}'.F$. Given $\mathbf{B}$ such that $\mathbf{T} \subset \mathbf{B}$ and $\mathbf{B} \stackrel{w}{\to} \mathbf{B}.F$, any other $\mathbf{B}'$ containing $\mathbf{T}$ is $\mathbf{B}' = {}^{w'}\mathbf{B}$ for some $w' \in W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$, and $\mathbf{B} \stackrel{w}{\to} \mathbf{B}.F$ if and only if $w' \in W(w\varphi)$. Thus the number of pairs $(\mathbf{T}, \mathbf{B})$ as above is $\frac{|\mathbf{G}^F|}{|\mathbf{T}_{w\varphi}^F|}$.

Now we establish the bijection (2.3).

1. We first show that, whenever $\mathbf{B} \in \mathbf{X}_{w\varphi}^{F^d}$, there exists a unique F-stable maximal torus $\mathbf{T}$ such that, for all $i \geq 0$, $\mathbf{T} \subset \mathbf{B}.F^i$.

Let $\mathbf{B} \in \mathbf{X}_{w\varphi}$, so that $\mathbf{B} \xrightarrow{w} \mathbf{B}.F$). The sequence

$$(\mathbf{B}, \mathbf{B}.F, \ldots, \mathbf{B}.F^d)$$

defines an element of the variety $\mathbf{X}_{\boldsymbol{\pi}\boldsymbol{\varphi}^d}$ (associated with the "braid element $\boldsymbol{\pi}\boldsymbol{\varphi}^d$", and relative to F^d) (see [**BM96**, §1]). Let $\mathbf{B}_1$ be the unique Borel subgroup such that

$$\mathbf{B} \xrightarrow{w_0} \mathbf{B}_1 \xrightarrow{w_0} \mathbf{B}.F^d$$

where w_0 is the longest element of W, and such that $(\mathbf{B}, \mathbf{B}_1, \mathbf{B}.F^d)$ defines the same element of $\mathbf{X}_{\boldsymbol{\pi}\boldsymbol{\varphi}^d}$ as that sequence. Since $\mathbf{B}$ and $\mathbf{B}_1$ are opposed, $\mathbf{T} := \mathbf{B} \cap \mathbf{B}_1$ is a maximal torus.

Let us prove that, for all $i \geq 0$, $\mathbf{T} \subset \mathbf{B}.F^i$.

- Assume first that $i \leq \lfloor d/2 \rfloor$.
  If $v := wF(w) \ldots F^{i-1}(w)$, then the Borel subgroup $\mathbf{B}.F^i$ is the unique Borel subgroup $\mathbf{B}'$ such that
  $$\mathbf{B} \xrightarrow{v} \mathbf{B}' \xrightarrow{v^{-1}w_0} \mathbf{B}_1 \,.$$
  Since such a Borel subgroup can be found among those containing $\mathbf{T}$, *i.e.*, a Borel subgroup ${}^{w'}\mathbf{B}$ for some $w' \in W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$, we have $\mathbf{B}.F^i \supset \mathbf{T}$.
- Now assume $i \geq \lceil d/2 \rceil$. Since $\mathbf{B} = \mathbf{B}.F^d$, if we set now $v := F^i(w)F^{i+1}(w) \ldots F^{d-1}(w)$, we get similarly that the Borel subgroup $\mathbf{B}.F^i$ is the unique Borel subgroup $\mathbf{B}'$ such that
  $$\mathbf{B}_1 \xrightarrow{w_0 v^{-1}} \mathbf{B}' \xrightarrow{v} \mathbf{B}$$
  and by the same reasoning we get that $\mathbf{B}.F^i \supset \mathbf{T}$.

Since $\mathbf{T} = \bigcap_i \mathbf{B}.F^i$, it is clear that $\mathbf{T}$ is F-stable.

2. Conversely, if $(\mathbf{T}, \mathbf{B})$ is a pair such that $\mathbf{T}$ is an F-stable maximal torus of type $w\varphi$ and $\mathbf{B} \supset \mathbf{T}$, then $\mathbf{B} \xrightarrow{w} \mathbf{B}.F$ and $\mathbf{B} \in \mathbf{X}_{w\varphi}^{F^d}$. Indeed, let as above $(\mathbf{B}, \mathbf{B}_1, \mathbf{B}.F^d)$ define the same element of $\mathbf{X}_{\boldsymbol{\pi}\boldsymbol{\varphi}^d}$ as $(\mathbf{B}, \mathbf{B}.F, \ldots, \mathbf{B}.F^d)$. Then each $\mathbf{B}.F^i$ contains $\mathbf{T}$ and by a reasoning similar as above, as either $\mathbf{B}_1 = \mathbf{B}.F^{d/2}$ if d is even, or $\mathbf{B}_1$ is defined by its relative position to its neighbours, in each case $\mathbf{B}_1$ contains $\mathbf{T}$. As both $\mathbf{B}$ and $\mathbf{B}.F^d$ are the unique Borel subgroup which intersect $\mathbf{B}_1$ in $\mathbf{T}$, they coincide. □

2.3. On eigenvalues of Frobenius

2.3.1. The Poincaré duality. —

Let us briefly recall a useful consequence of Poincaré duality (see for example [**Del77**]) in our context.

Here q is a power of a prime number p, and ℓ is a prime number different from p.

Let X be a smooth separated irreducible variety of dimension d, defined over the field $\mathbb{F}_q$ with q elements, with corresponding Frobenius endomorphism F. Its étale cohomology groups $H^i(X, \mathbb{Q}_\ell)$ and $H^i_c(X, \mathbb{Q}_\ell)$ are naturally endowed with an action of F.

The *Poincaré duality* has the following consequence.

Theorem 2.4. —
For $0 \leq i \leq 2d$, there is a natural non degenerate pairing of $\langle F \rangle$-modules

$$H^i_c(X, \mathbb{Q}_\ell) \times H^{2d-i}(X, \mathbb{Q}_\ell) \longrightarrow H^{2d}_c(X, \mathbb{Q}_\ell) .$$

For $n \in \mathbb{Z}$ we denote by $\mathbb{Q}_\ell(n)$ the $\mathbb{Q}_\ell$-vector space of dimension 1 where we let F act by multiplication by q^{-n}.

Since, as a $\mathbb{Q}_\ell\langle F \rangle$-module, we have $H^{2d}_c(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-d)$, the Poincaré duality may be reformulated as follows:

For $0 \leq i \leq 2d$, there is a natural non degenerate pairing of $\langle F \rangle$-modules

$$H^i_c(X, \mathbb{Q}_\ell) \times H^{2d-i}(X, \mathbb{Q}_\ell) \longrightarrow \mathbb{Q}_\ell(-d) .$$

2.3.2. Unipotent characters in position $\mathbf{w}\varphi$. —

In this subsection and the following until 2.9, $\mathbf{w}$ will be any element of $\mathbf{B}^+_W$ such that the variety $\mathbf{X}_{\mathbf{w}\varphi}$ is irreducible.

We will denote by $\mathrm{Un}(\mathbf{G}^F)$ the set of unipotent characters of $\mathbf{G}^F$ and by $\mathrm{Un}(\mathbf{G}^F, \mathbf{w}\boldsymbol{\varphi})$ those appearing in any of the (compact support) cohomology spaces $H^n_c(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$.

We will denote by Id the trivial character of $\mathbf{G}^F$, and by γ^* the complex conjugate of a character γ.

Since the dimension of $\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}$ is equal to $l(\mathbf{w}\boldsymbol{\varphi})$, the Poincaré duality as stated above (2.4) has the following consequences.

Proposition 2.5. —

1. *The set of unipotent characters appearing in some (noncompact support) cohomology spaces $H^n(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$ is $\mathrm{Un}(\mathbf{G}^F, \mathbf{w}\boldsymbol{\varphi})^*$, the set of all complex conjugates of elements of $\mathrm{Un}(\mathbf{G}^F, \mathbf{w}\boldsymbol{\varphi})$.*
2. *For $\gamma \in \mathrm{Un}(\mathbf{G}^F, \mathbf{w}\boldsymbol{\varphi})$, to any eigenvalue λ of F^δ on the γ-isotypic component of $H^i_c(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$ is associated the eigenvalue $q^{\delta l(\mathbf{w}\boldsymbol{\varphi})}/\lambda$ on the γ^*-isotypic component of $H^{2l(\mathbf{w}\boldsymbol{\varphi})-i}(\mathbf{X}, \bar{\mathbb{Q}}_\ell)$.*

Remark 2.6. —

Note that 1 is the unique eigenvalue of minimal module of F^δ on $H^\bullet(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$, corresponding to the case where γ is the trivial character in $H^0(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$.

Indeed, by Poincaré duality it suffices to check that there is a unique eigenvalue of maximal module, equal to $q^{l(w\varphi)}$, in the compact support cohomology $H_c^\bullet(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$. This follows from the fact that only the identity occurs in $H_c^{2l(w\varphi)}(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$ since $\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}$ is irreducible, and that the modules of the eigenvalues in $H_c^n(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$ are less or equal to $q^{n/2}$ (see for example [**DMR07**, 3.3.10(i)]).

The next properties are consequences of results of Lusztig and of Digne–Michel (for the next one, see *e.g.* [**DMR07**, 3.3.4]).

Proposition 2.7. —

Let $\gamma \in \mathrm{Un}(\mathbf{G}^F, \mathbf{w}\boldsymbol{\varphi})$. Let λ be the eigenvalue of F^δ on the γ-isotypic component of $H_c^i(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$. Then $\lambda = q^{f\delta}\lambda_\gamma$, where

- *λ_γ is a root of unity independent of i and of $\mathbf{w}\boldsymbol{\varphi}$*
- *$f = \frac{n}{2}$ for some $n \in \mathbb{N}$, and the image of f in $\mathbb{Q}/\mathbb{Z}$ is independent of i and of $\mathbf{w}\boldsymbol{\varphi}$.*

2.4. Computing numbers of rational points

We denote by $\mathcal{H}(W,x)$ the ordinary Iwahori-Hecke algebra of W defined over $\mathbb{C}[x,x^{-1}]$: using our previous notation, $\mathcal{H}(W,x)$ is the $(x-1)$-cyclotomic Hecke algebra such that, for all reflecting hyperplanes H of W, we have $P_H(t,x) = (t-x)(t+1)$. Notice that $\mathcal{H}(W,x)$ is indeed an $(x-1)$-cyclotomic Hecke algebra for W at the regular element 1.

We choose once for all a square root $\sqrt{x}$ of the indeterminate x. Since the algebra $\mathcal{H}(W,x)$ is split over $\mathbb{C}(\sqrt{x})$, the specialisations

$$\sqrt{x} \mapsto 1 \quad \text{and} \quad \sqrt{x} \mapsto (q^{\delta/2})^{m/\delta}$$

define bijections between (absolutely) irreducible characters:

$$\mathrm{Irr}(\mathcal{H}(W,x)) \longleftrightarrow \mathrm{Irr}(W) \quad \text{and} \quad \mathrm{Irr}(\mathcal{H}(W,x)) \longleftrightarrow \mathrm{Irr}(\mathcal{H}(W,q^m))$$

for all m multiple of δ. As a consequence, we get well-defined bijections

$$\begin{cases} \mathrm{Irr}(W) \longleftrightarrow \mathrm{Irr}(\mathcal{H}(W,q^m))\,, \\ \chi \mapsto \chi_{q^m}\,. \end{cases}$$

The automorphism $\boldsymbol{\varphi}$ of $\mathbf{B}_W$ defined by φ induces an automorphism φ_x of the generic Hecke algebra, and the field $\mathbb{C}(\sqrt{x})$ splits the semidirect product algebra

$\mathcal{H}(W,x) \rtimes \langle\varphi_x\rangle$ (see [**Mal99**]). Hence the above bijections extend to bijections

$$\psi \mapsto \psi_{q^m}$$

between

- extensions to $W \rtimes \langle\varphi\rangle$ of φ-stable characters of W
- and extensions to $\mathcal{H}(W,q^m) \rtimes \langle\varphi_{q^m}\rangle$ of φ_{q^m}-stable characters of $\mathcal{H}(W,q^m)$.

Each character χ of the Hecke algebra $\mathcal{H}(W,x)$ defines, by composition with the natural morphism $\mathbf{B}_W \to \mathcal{H}(W,x)^\times$, a character of $\mathbf{B}_W$. If $\tilde{\chi}$ is a character of $\mathcal{H}(W,x) \rtimes \langle\varphi_x\rangle$, it also defines a character of $\mathbf{B}_W \rtimes \langle\boldsymbol{\varphi}\rangle$; in particular this gives a meaning to the expression $\tilde{\chi}(\mathbf{w}\boldsymbol{\varphi})$ for $\mathbf{w} \in \mathbf{B}_W$.

For χ a φ-stable character of W we choose an extension to $W \rtimes \langle\varphi\rangle$ denoted $\tilde{\chi}$. We define

$$R_{\tilde{\chi}} := |W|^{-1} \sum_{v\in W} \tilde{\chi}(v\varphi) R_{v\varphi}\,,$$

where, for $g \in \mathbf{G}^F$ and $v \in W$,

$$R_{v\varphi}(g) := \sum_i (-1)^i \mathrm{Trace}(g \mid H^i_c(\mathbf{X}_{v\varphi}, \bar{\mathbb{Q}}_\ell))\,.$$

Notice then that, for $\gamma \in \mathrm{Un}(\mathbf{G}^F, \mathbf{w}\boldsymbol{\varphi})$ the expression

$$\langle\gamma, R_{\tilde{\chi}}\rangle_{\mathbf{G}^F} \tilde{\chi}_{q^m}(\mathbf{w}\boldsymbol{\varphi})$$

depends only on χ and on γ, which gives sense to the next result ([**DMR07**, 3.3.7]).

Proposition 2.8. —
For any m multiple of δ and $g \in \mathbf{G}^F$, we have

$$|\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}^{gF^m}| = \sum_{\gamma\in\mathrm{Un}(\mathbf{G}^F,\mathbf{w}\boldsymbol{\varphi})} \lambda_\gamma^{m/\delta}\gamma(g) \sum_{\chi\in\mathrm{Irr}(W)^\varphi} \langle\gamma, R_{\tilde{\chi}}\rangle_{\mathbf{G}^F} \tilde{\chi}_{q^m}(\mathbf{w}\boldsymbol{\varphi})\,.$$

Let us draw some consequences of the last proposition when $\mathbf{w}$ satisfies the assumptions of Section 2.1 (so that $\mathbf{w} = \mathbf{w}_{\gamma,a/d}$).

Since by assumption we have $(\mathbf{w}\boldsymbol{\varphi})^d = \boldsymbol{\pi}^a\varphi^d$, it follows that

$$\mathbf{w}^{\mathrm{lcm}(d,\delta)} = \boldsymbol{\pi}^{a.\mathrm{lcm}(d,\delta)/d}\,. \tag{2.9}$$

By [**BMM99**, 6.7], we know that

$$\chi_{q^m}(\boldsymbol{\pi}) = q^{m(l(\boldsymbol{\pi})-(a_\chi+A_\chi))} = q^{ml(\boldsymbol{\pi})(1-(a_\chi+A_\chi)/l(\boldsymbol{\pi}))} \tag{2.10}$$

where, as usual, a_χ and A_χ are the valuation and the degree of the generic degree of χ (see Section 1.6.4). It follows that

$$\tilde{\chi}_{q^m}(\mathbf{w}\boldsymbol{\varphi}) = \tilde{\chi}(w\varphi) q^{ml(\mathbf{w})(1-(a_\chi+A_\chi)/l(\boldsymbol{\pi}))} \tag{2.11}$$

(the power of q is given by the above equation and the constant in front by specialization).

For all χ such that $\langle\gamma, R_{\tilde{\chi}}\rangle_{\mathbf{G}^F} \neq 0$ since the functions a and A are constant on families, we have $a_\chi = a_\gamma$ and $A_\chi = A_\gamma$. So

$$|\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}^{gF^m}| = \sum_{\gamma\in\mathrm{Un}(\mathbf{G}^F,\mathbf{w}\boldsymbol{\varphi})} \langle\gamma, R_{\mathbf{w}\boldsymbol{\varphi}}\rangle_{\mathbf{G}^F} \lambda_\gamma^{m/\delta}\gamma(g) q^{ml(\mathbf{w}\boldsymbol{\varphi})(1-(a_\chi+A_\chi)/l(\boldsymbol{\pi}))}. \tag{2.12}$$

2.5. Some consequences of abelian defect group conjectures

The next conjectures are special cases of abelian defect group conjectures for finite reductive groups (see for example [**Bro01**]).

Conjecture 2.13. —

1. $H_c^{\mathrm{odd}} := \bigoplus_{i\,\mathrm{odd}} H_c^i(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$ *and* $H_c^{\mathrm{even}} := \bigoplus_{i\,\mathrm{even}} H_c^i(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$ *are disjoint as* $\mathbf{G}^F$*-modules.*
2. F^δ *is semi-simple on* $H_c^i(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$ *for all* $i \geq 0$.

Let us then set

$$\mathcal{H}_c(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}) := \mathrm{End}_{\mathbf{G}^F}\left(\bigoplus_i H_c^i(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)\right).$$

Comparing the Lefschetz formula

$$|\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}^{gF^m}| = \sum_i (-1)^i \mathrm{Trace}(gF^m \mid H_c^i(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell))$$

with (2.12), we see that the preceding Conjecture 2.13 implies

1. there is a single eigenvalue $\mathrm{Fr}_\gamma^c := \lambda_\gamma q^{\delta l(\mathbf{w}\boldsymbol{\varphi})(1-(a_\gamma+A_\gamma)/l(\boldsymbol{\pi}))}$ of F^δ on the γ-isotypic part of $\bigoplus_i H_c^i(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$,
2. F^δ is central in $\mathcal{H}_c(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$.

Since $|\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}^{gF^m}| \in \mathbb{Q}$, the conjecture implies also:

$$\forall\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}),\ \begin{cases}\sigma(\mathrm{Un}(\mathbf{G}^F, \mathbf{w}\boldsymbol{\varphi})) = \mathrm{Un}(\mathbf{G}^F, \mathbf{w}\boldsymbol{\varphi})\\ \mathrm{Fr}_{\sigma(\gamma)}^c = \sigma(\mathrm{Fr}_\gamma^c).\end{cases}$$

By Poincaré duality, there is similarly a single eigenvalue Fr_γ of F^δ attached to γ on $H^\bullet(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$ and since by Proposition 2.5(2) $\mathrm{Fr}_\gamma \mathrm{Fr}_{\gamma^*}^c = q^{\delta l(\mathbf{w}\boldsymbol{\varphi})}$ we get

$$\mathrm{Fr}_\gamma = \lambda_\gamma q^{\delta l(\mathbf{w}\boldsymbol{\varphi})(a_\gamma+A_\gamma)/l(\boldsymbol{\pi})}. \tag{2.14}$$

We get similarly that

$$F^\delta \text{ is central in } \mathcal{H}(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}) := \mathrm{End}_{\mathbf{G}^F}\left(\bigoplus_i H^i(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)\right).$$

The next conjecture may be found in [**BM93**] (see also [**BM96**], [**Bro01**]).

As in §2.4 above, we denote by $w\varphi$ a ζ_d^a-regular element for W, which we lift as in §3.1 above to an element $\mathbf{w}\boldsymbol{\varphi}$ such that $\mathbf{w}$ and $\boldsymbol{\rho} := (\mathbf{w}\boldsymbol{\varphi})^\delta$ belong to $\mathbf{B}_W^+$, and $\boldsymbol{\rho}^d = \boldsymbol{\pi}^{a\delta}$. We denote by Φ the d-th cyclotomic polynomial (thus $\Phi \in \mathbb{Z}[x]$).

Conjecture 2.15. —
The algebra $\mathcal{H}_c(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$ is the specialization at $x = q$ of a Φ-cyclotomic Hecke algebra $\mathcal{H}_W(w\varphi)_c(x)$ for $W(w\varphi)$ over $\mathbb{Q}$, with the following properties:

1. *Let us denote by τ_q the corresponding specialization of the canonical trace of $\mathcal{H}_W(w\varphi)_c(x)$ to the algebra $\mathcal{H}_c(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$. Then for $\mathbf{v} \in \mathbf{B}_{W(w\varphi)}$, we have*
$$\sum_i (-1)^i \mathrm{tr}(\mathbf{v}, H^i(\mathbf{X}_{w\varphi}, \mathbb{Q}_\ell)) = \mathrm{Deg}(R_{w\varphi}^{\mathbf{G}})\tau_q(\mathbf{v}) \,.$$
2. *The central element $\boldsymbol{\pi}^{a.\mathrm{lcm}(d,\delta)/d} = \boldsymbol{\rho}^{\mathrm{lcm}(d,\delta)/\delta}$ corresponds to the action of $F^{\mathrm{lcm}(d,\delta)}$.*

Let us draw some consequences of Conjecture 2.15.

2.5.1. Consequence of 2.15: Computation of $\tau_q(\boldsymbol{\pi})$. —

The following proposition makes Conjecture 2.15 more precise, and justifies Conjecture 2.21 below.

Proposition 2.16. —

1. *Assume that 2.15 holds. Assume moreover that $a = 1$ and that d is a multiple of δ. Then*
$$\tau_q(\boldsymbol{\pi}) = \widetilde{\det}_V(w\varphi)^{-1} q^{N_W^{\mathrm{hyp}}} = (\zeta^{-1}q)^{N_W^{\mathrm{hyp}}} \,.$$
2. *Assume that, for $I \in \mathcal{A}_W(w\varphi)$, the minimal polynomial of $\mathbf{s}_I$ on $\mathcal{H}_c(\mathbf{X}_{w\varphi})$ is $P_I(t)$. Then*
$$\prod_{I \in \mathcal{A}_W(w\varphi)} P_I(0) = (-1)^{N_{W(w\varphi)}^{\mathrm{hyp}}} \widetilde{\det}_V(w\varphi)^{-1} q^{N_W^{\mathrm{hyp}}} = (-1)^{N_{W(w\varphi)}^{\mathrm{hyp}}} (\zeta^{-1}q)^{N_W^{\mathrm{hyp}}} \,.$$

Proof. —
(1) By 2.15(2), the element $\boldsymbol{\pi}$ acts as F^d on the algebra $\mathcal{H}_c(\mathbf{X}_{w\varphi})$. Hence by the Lefschetz formula, we have
$$|\mathbf{X}_{w\varphi}^{F^d}| = \sum_i (-1)^i \mathrm{tr}(\boldsymbol{\pi}, H^i(\mathbf{X}_{w\varphi}, \mathbb{Q}_\ell)) \,,$$
and hence by 2.15 we find
$$|\mathbf{X}_{w\varphi}^{F^d}| = \mathrm{Deg}(R_{w\varphi}^{\mathbf{G}})\tau_q(\boldsymbol{\pi}) \,.$$
Now by 2.1, we have
$$|\mathbf{X}_{w\varphi}^{F^d}| = \widetilde{\det}_V(w\varphi)^{-1} q^{N_W^{\mathrm{hyp}}} \mathrm{Deg}(R_{w\varphi}^{\mathbf{G}}) \,,$$
which implies the formula.

(2) By [**BMM99**, 2.1(2)(b)], we know that

$$\tau_q(\boldsymbol{\pi}) = (-1)^{N^{\mathrm{hyp}}_{W(w\varphi)}} \prod_{I\in\mathcal{A}_W(w\varphi)} P_I(0)\,,$$

which implies the result. □

2.5.2. Consequence of 2.15: Computation of Frobenius eigenvalues. —

Recall that there is an extension $L(v)$ of $\mathbb{Q}(x)$ which splits the algebra $\mathcal{H}_W(w\varphi)_c(x)$, where L is an abelian extension of $\mathbb{Q}$ and $v^k = \zeta^{-1}x$ for some integer k.

Choose a complex number $(\zeta^{-1}q)^{1/k}$.

Assume Conjecture 2.15 holds. Then all unipotent characters in $\mathrm{Un}(\mathbf{G}^F, \mathbf{w}\boldsymbol{\varphi})$ are defined over $L[(\zeta^{-1}q)^{1/k}]$, and the specializations

$$v \mapsto 1 \quad\text{and}\quad v \mapsto (\zeta^{-1}q)^{1/k}$$

define bijections

$$(2.17)\qquad \begin{cases} \mathrm{Irr}(W(w\varphi)) \longleftrightarrow \mathrm{Irr}(\mathcal{H}(\mathbf{X}_{w\varphi}, \bar{\mathbb{Q}}_\ell)) \longleftrightarrow \mathrm{Un}(\mathbf{G}^F, \mathbf{w}\boldsymbol{\varphi}) \\ \chi \mapsto \chi_q \mapsto \gamma_\chi \end{cases}$$

Remark 2.18. —

It is known from the work of Lusztig (see *e.g.* [**Gec05**] and the references therein) that the unipotent characters of $\mathbf{G}^F$ are defined over an extension of the form $L(q^{1/2})$ where L is an abelian extension of $\mathbb{Q}$.

Recall that $\mathrm{Fr}_{\gamma_\chi}$ denotes the eigenvalue of F^δ on the γ_χ-isotypic component of $H^\bullet(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$. By 2.15(2), we have

$$\mathrm{Fr}_{\gamma_\chi}^{\mathrm{lcm}(d,\delta)/\delta} = \omega_{\chi_q}(\boldsymbol{\rho})^{\mathrm{lcm}(d,\delta)/\delta}\,.$$

Since the algebra $\mathcal{H}_W(w\varphi)_c(x)$ specializes

- for $x = \zeta$, to the group algebra of $W(w\varphi)$,
- and for $x = q$, to the algebra $\mathcal{H}_c(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$,

we also have

$$\omega_{\chi_q}(\boldsymbol{\rho}) = \omega_\chi(\rho)(\zeta^{-1}q)^{c_\chi}$$

for some $c_\chi \in \mathbb{N}/2$.

Comparing with formula 2.14 we find

$$c_\chi = \delta l(\mathbf{w}\boldsymbol{\varphi})(1 - (a_{\gamma_\chi} + A_{\gamma_\chi})/l(\boldsymbol{\pi})) = (e_W - (a_{\gamma_\chi} + A_{\gamma_\chi}))\delta a/d\,,$$

which proves:

Proposition 2.19. —

Whenever $w\varphi$ is ζ_d^a-regular and $\chi \in \mathrm{Irr}(W(w\varphi))$, if γ_χ denotes the corresponding element in $\mathrm{Un}(\mathbf{G}^F, \mathbf{w}\boldsymbol{\varphi})$ we have

$$\lambda_{\gamma_\chi}^{\mathrm{lcm}(d,\delta)/\delta} = \omega_\chi(\rho)^{\mathrm{lcm}(d,\delta)/\delta}\zeta^{-(e_W - (a_{\gamma_\chi} + A_{\gamma_\chi}))\mathrm{lcm}(d,\delta)a/d}\,.$$

2.6. Actions of some braids

Now we turn to the equivalences of étale sites defined in [**BM96**] and studied also in [**DM13**]. For the definition of the operators $D_{\mathbf{v}}$ we refer the reader to [**BM96**].

Theorem 2.20. — *There is a morphism*

$$\begin{cases} \mathbf{B}_W^+(\mathbf{w}\boldsymbol{\varphi}) \to \operatorname{End}_{\mathbf{G}^F}(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}) \\ \mathbf{v} \mapsto D_{\mathbf{v}} \end{cases}$$

such that:

1. *The operators $D_{\mathbf{v}}$ are equivalences of étale sites on $\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}$.*

   *The next assertion has only been proved for the cases where W is of type A, B or D_4 [**DM06**]. It is conjectural in the general case.*

2. *The map $\mathbf{v} \mapsto D_{\mathbf{v}}$ induces representations*

$$\rho_c : \bar{\mathbb{Q}}_\ell \mathbf{B}_W(\mathbf{w}\boldsymbol{\varphi}) \to \mathcal{H}_c(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}) \quad \textit{and} \quad \rho : \bar{\mathbb{Q}}_\ell \mathbf{B}_W(\mathbf{w}\boldsymbol{\varphi}) \to \mathcal{H}(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})\,.$$

3. $D_{(\mathbf{w}\boldsymbol{\varphi})^\delta} = F^\delta$.

2.6.1. More precise conjectures. —

The next conjecture is also part of abelian defect group conjectures for finite reductive groups (see for example [**Bro01**]). It makes conjecture 2.15 much more precise.

Results similar to (CS) are proved in [**DMR07**] and [**DM06**] for several cases.

Conjecture 2.21. —

COMPACT SUPPORT CONJECTURE (CS)

1. *The morphism $\rho_c : \bar{\mathbb{Q}}_\ell \mathbf{B}_W(\mathbf{w}\boldsymbol{\varphi}) \to \mathcal{H}_c(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$ is surjective.*
2. *It induces an isomorphism between $\mathcal{H}_c(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$ and the specialization at $x = q$ of a Φ-cyclotomic Hecke algebra $\mathcal{H}_W(w\varphi)_c(x)$ over $\mathbb{Q}$ at $w\varphi$ for $W(w\varphi)$.*

NONCOMPACT SUPPORT CONJECTURE (NCS)

1. *The morphism $\rho : \bar{\mathbb{Q}}_\ell \mathbf{B}_W(\mathbf{w}\boldsymbol{\varphi}) \to \mathcal{H}(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$ is surjective.*
2. *It turns $\mathcal{H}(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$ into the specialization at $x = q$ of a Φ-cyclotomic Hecke algebra $\mathcal{H}_W(w\varphi)(x)$ over $\mathbb{Q}$ at $w\varphi$ for $W(w\varphi)$.*

2.6.2. Consequence of 2.21: noncompact support and characters in $\mathrm{Un}(\mathbf{G}^F, \mathbf{w}\boldsymbol{\varphi})$. —

We refer the reader to notation introduced in 2.5.2, in particular to bijections 2.17.

We can establish some evidence towards identifying $\mathcal{H}(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$ with a cyclotomic Hecke algebra of noncompact type. For example, we have the following lemma. We denote by $\mathcal{A}(w\varphi)$ the set of reflecting hyperplanes of $W(w\varphi)$ in its action on $V(w\varphi)$.

Lemma 2.22. —

Assume that $\mathcal{H}(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$ is the specialization at $x = q$ of a cyclotomic Hecke algebra for the group $W(w\varphi)$.

Then whenever $I \in \mathcal{A}(w\varphi)$, the corresponding polynomial $P_I(t,x)$ has only one root of degree 0 in x, namely 1. If $\mathbf{s}_I$ is the corresponding braid reflection in $\mathbf{B}_W(\mathbf{w}\boldsymbol{\varphi})$, this root is the eigenvalue of $D_{\mathbf{s}_I}$ on $H^0(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$.

Proof. —

Let $\chi \in \mathrm{Irr}(W(w\varphi))$ be a linear character, thus corresponding *via* 2.17 to a linear character χ_q of $\mathcal{H}(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$. Let $I \in \mathcal{A}(w\varphi)$, and let $\mathbf{s}_I$ be the corresponding braid reflection in $\mathbf{B}_W(\mathbf{w}\boldsymbol{\varphi})$. Let us set $u_{I,j_I} := \chi_q(\mathbf{s}_I)$ so that $u_{I,j_I} = \xi_{I,j_I} q^{m_{I,j_I}}$ where ξ_{I,j_I} is a root of unity and m_{I,j_I} is a rational number.

The element $\boldsymbol{\rho} = (\mathbf{w}\boldsymbol{\varphi})^\delta$ is central in $\mathbf{B}_W(\mathbf{w}\boldsymbol{\varphi})$. Since $W(w\varphi)$ is irreducible (see for example [**Bro10**, Th. 5.6, 6]) it follows that there exists some $a \in \mathbb{Q}$ such that

$$\chi_q(\boldsymbol{\rho}) = \chi(\rho) \prod_I u_{I,j_I}^{a e_I} .$$

Now $\chi_q(\boldsymbol{\rho})$ is the eigenvalue of F^δ on the γ_χ-isotypic component of $H^\bullet(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$, and we know (see Remark 2.6 above) that there is a unique such eigenvalue of minimum module, which is 1, corresponding to the case where γ is the trivial character in $H^0(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$.

It follows that there is a unique linear character χ of $W(w\varphi)$ such that $\chi_q(\mathbf{s})$ has minimal module for each braid reflection $\mathbf{s}$. Since $D_\mathbf{s}$ acts trivially on $H^0(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$ we have $\chi_q(\mathbf{s}) = 1$, so the unique minimal m_{I,j_I} is 0. □

2.7. Is there a stronger Poincaré duality ?

We will see now how (CS) and (NCS) are connected, under some conjectural extension of Poincaré duality.

Conjecture 2.23. —

For any $\mathbf{v} \in \mathbf{B}_W(\mathbf{w}\boldsymbol{\varphi})$ and any n large enough multiple of δ, Poincaré duality holds for $D_{\mathbf{v}(\mathbf{w}\varphi)^n}$, i.e., we have a perfect pairing of $D_{\mathbf{v}(\mathbf{w}\varphi)^n}$-modules:

$$H^i_c(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell) \times H^{2l(\mathbf{w}\boldsymbol{\varphi})-i}(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell) \to H^{2l(\mathbf{w}\boldsymbol{\varphi})}_c(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell) .$$

Remarks 2.24. —

1. First note that for n large enough $\mathbf{v}(\mathbf{w}\varphi)^n \in \mathbf{B}_W^+(\mathbf{w}\varphi)$, so there is a well-defined endomorphism $D_{\mathbf{v}(\mathbf{w}\varphi)^n}$. Indeed, since $(\mathbf{w}\varphi)^n$ is a power of $\boldsymbol{\pi}$ for n divisible enough, the element $\mathbf{v}(\mathbf{w}\boldsymbol{\varphi})^n$ is positive for n large enough.
2. The Lefschetz formula which would be implied by 2.23 at least holds. Indeed, Fujiwara's theorem (see [**DMR07**, 2.2.7]) states that when D is a finite morphism and F a Frobenius, then for n sufficiently large DF^n satisfies Lefschetz's trace formula ; this implies that, for n large enough and multiple of δ, the endomorphism $D_{\mathbf{v}(\mathbf{w}\varphi)^n}$ satisfies Lefschetz's trace formula, since $D_{(\mathbf{w}\boldsymbol{\varphi})^\delta} = F^\delta$ is a Frobenius.
3. Conjecture 2.23 implies that $D_{\mathbf{v}}$ acts trivially on $H^0(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$.

The following theorem is a consequence of what precedes. It refers to the definitions introduced below (see Def. 3.7), and the reader is invited to read them before reading this theorem.

Theorem 2.25. —

Assuming conjectures 2.13, 2.21, 2.23,

1. *the algebra $\mathcal{H}_c(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$ is a spetsial Φ-cyclotomic algebra of W at $w\varphi$ of compact type, and the algebra $\mathcal{H}(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$ is a spetsial Φ-cyclotomic algebra of W at $w\varphi$ of noncompact type,*
2. *$\mathcal{H}_c(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$ is the compactification of $\mathcal{H}(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$, and $\mathcal{H}(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$ is the noncompactification of $\mathcal{H}_c(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$.*

We can give a small precision about Conjectures 2.21 (which will be reflected in Definition 3.7 below) using now the strong Poincaré Conjecture 2.23.

Lemma 2.26. —

Assume 2.23 and 2.13, and assume that $\mathcal{H}_c(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$ is the specialization at $x \mapsto q$ of a ζ-cyclotomic Hecke algebra of $W(w\varphi)$.

Let I be a reflecting hyperplane for $W(w\varphi)$, and if $\mathbf{s}_I$ denotes the corresponding braid reflection in $\mathbf{B}_W(\mathbf{w}\boldsymbol{\varphi})$, let us denote by ν_I the eigenvalue of $D_{\mathbf{s}}$ on $H_c^{2l(\mathbf{w}\boldsymbol{\varphi})}(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$. Assume that $\nu_I = \lambda_I q^{m_I}$ where λ_I is a complex number of module 1 and $m_I \in \mathbb{Q}$ is independent of q.

Then $\lambda_I = 1$.

Proof. —

From 2.23 and 2.22 we get that ν_I is the unique eigenvalue of maximal module of $D_{\mathbf{s}_I}$ on $H_c^\bullet(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}})$; the eigenvalue of F^δ on $H_c^{2l(w)}(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$ is also the unique eigenvalue of maximal module (equal to $q^{\delta l(w)}$).

As remarked in 2.24 2., for sufficiently large n multiple of δ the endomorphism $D_{\mathbf{s}_I(\mathbf{w}\boldsymbol{\varphi})^n}$ satisfies the Lefschetz fixed point formula. Its eigenvalue on $H_c^{2l(w)}(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}, \bar{\mathbb{Q}}_\ell)$ is $\lambda_I q^{m_I} q^{\delta n l(w)}$, and this is the dominant term in the Lefschetz formula. Since the formula sums to an integer, this term must be a real number, thus $\lambda_I = 1$. □

Remark 2.27. —

Note that the assumption of the previous lemma on the shape of ν_I is reasonable since we believe that it suffices to prove it in the case where $W(w\varphi)$ is cyclic, and in the latter case $D_{\mathbf{s}}$ is a root of F.

Incidently, assume that $W(w\varphi)$ is cyclic of order c, and let $\mathbf{s}$ be the positive generator of $\mathbf{B}_W(\mathbf{w}\boldsymbol{\varphi})$. Since $(\mathbf{w}\boldsymbol{\varphi})^\delta$ is a power of $\mathbf{s}$, we get (comparing lengths):

$$\mathbf{s}^{\frac{ac\delta}{d}} = (\mathbf{w}\boldsymbol{\varphi})^\delta\,.$$

CHAPTER 3

SPETSIAL Φ-CYCLOTOMIC HECKE ALGEBRAS

Leaning on properties and conjectures stated in the previous paragraph, we define in this section the special type of cyclotomic Hecke algebras which should occur as building blocks of the spetses: these algebras (called "spetsial cyclotomic Hecke algebras") satisfy properties which generalize properties of algebras occurring as commuting algebras of cohomology of Deligne-Lusztig varieties attached to regular elements (see §3 above).

Let K be a number field which is stable under complex conjugation, denoted by $\lambda \mapsto \lambda^*$. Let $\mathbb{Z}_K$ be the ring of integers of K.

Let V be an r-dimensional vector space over K.

Let W be a finite reflection subgroup of $\mathrm{GL}(V)$ and $\varphi \in N_{\mathrm{GL}(V)}(W)$ be an element of finite order. We set $\mathbb{G} := (V, W\varphi)$.

3.1. Prerequisites and notation

Throughout, $w\varphi \in W\varphi$ denotes a regular element. If $w\varphi$ is ζ-regular for a root of unity ζ with irreducible polynomial Φ over K, we say that $w\varphi$ is Φ-*regular*.

We set the following notation:

- $V(w\varphi) := \ker \Phi(w\varphi)$ as a $K[x]/(\Phi)$-vector space,
- $W(w\varphi) := C_W(w\varphi)$, a reflection group on $V(w\varphi)$ (see above Theorem 1.50(5)),
- $\mathcal{A}(w\varphi)$ is the set of reflecting hyperplanes of $W(w\varphi)$ in its action on $V(w\varphi)$,
- $e_W(w\varphi) := e_{W(w\varphi)}$.

Note that $K[x]/(\Phi)$ contains $\mathbb{Q}_{W(w\varphi)}$.

The next theorem follows from Springer–Lehrer's theory of regular elements (see *e.g.* [**Bro10**, Th. 5.6]).

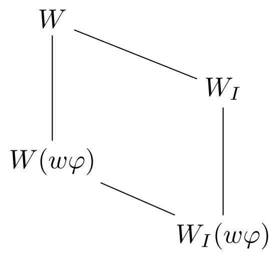
Theorem 3.1. —

Assume that W is irreducible. If $w\varphi$ is regular, then $W(w\varphi)$ acts irreducibly on $V(w\varphi)$.

3.2. Reduction to the cyclic case

We relate data for $W(w\varphi)$ with local data for $W_I(w\varphi)$, $I \in \mathcal{A}(w\varphi)$.

So let $I \in \mathcal{A}(w\varphi)$. We denote by W_I the fixator of I in W, a parabolic subgroup of W. The element $w\varphi$ normalises the group W_I (since it acts by scalar multiplication on I), and it is also a Φ-regular element for W_I. Moreover, the group $W_I(w\varphi)$, the fixator of the hyperplane I in $W(w\varphi)$, is cyclic.



Thus we have a reflection coset

$$\mathbb{G}_I := (V, W_I w\varphi)\,.$$

Note that

$$\mathcal{A}(W_I) = \{H \in \mathcal{A}(W) \mid H \cap V(w\varphi) = I\,\}\,.$$

The "reduction to the cyclic case" is expressed first in a couple of simple formulæ relating "global data" for W to the collection of "local data" for the family $(W_I)_{I\in\mathcal{A}(w\varphi)}$, such as:

$$(3.2)\qquad \begin{cases} \mathcal{A}(W) = \bigsqcup_{I\in\mathcal{A}(w\varphi)} \mathcal{A}(W_I)\,, \text{ from which follow} \\ N_W^{\mathrm{ref}} = \sum_{I\in\mathcal{A}(w\varphi)} N_{W_I}^{\mathrm{ref}}\,,\ N_W^{\mathrm{hyp}} = \sum_{I\in\mathcal{A}(w\varphi)} N_{W_I}^{\mathrm{hyp}}\,,\ e_W = \sum_{I\in\mathcal{A}(w\varphi)} e_{W_I}\,. \end{cases}$$

$$(3.3)\qquad \begin{cases} J_W = \prod_{I\in\mathcal{A}(w\varphi)} J_{W_I}\,, \text{ from which follows} \\ \widetilde{\det}_V^{(W)}(w\varphi) = \prod_{I\in\mathcal{A}(w\varphi)} \widetilde{\det}_V^{(W_I)}(w\varphi)\,. \end{cases}$$

Notice the following consequence of (3.2), where we set the following piece of notation:

$$e_{W_I}(w\varphi) := e_{W_I(w\varphi)} \,,$$

which we often abbreviate e_I.

Lemma 3.4. —

1. *Whenever $I \in \mathcal{A}(w\varphi)$, $e_{W_I}(w\varphi)$ divides e_{W_I}.*
2. *Assume that there is a single orbit of reflecting hyperplanes for $W(w\varphi)$. Then $e_W(w\varphi)$ divides e_W.*

Proof. —
(1) Consider the discriminant for the contragredient representation of W on the symmetric algebra $S(V^*)$ of the dual of V (see 1.1.2 above), which we denote by Disc_W^*. With obvious notation, we have

$$\mathrm{Disc}_W^* = J_W^* {J_W^*}^\vee = \prod_{H\in\mathcal{A}(W)} (j_H^*)^{e_H} \,.$$

By restriction to the subspace $V(w\varphi)$, we get

$$\mathrm{Disc}_W^*|_{V(w\varphi)} = \prod_{I\in\mathcal{A}(w\varphi)} (j_I^*)^{e_{W_I}} \,.$$

Since Disc_W^* is fixed under W, $\mathrm{Disc}_W^*|_{V(w\varphi)}$ is fixed under $W(w\varphi)$. Since Disc_W^* is a monomial in the j_I^*'s, it follows from [**Bro10**, Prop.3.11,2] that $\mathrm{Disc}_W^*|_{V(w\varphi)}$ must be a power of the discriminant of $W_I(w\varphi)$, which shows that $e_{W_I}(w\varphi)$ divides e_{W_I}.

(2) If all reflecting hyperplanes of $W(w\varphi)$ are in the same orbit as I, the relation $e_W = \sum_{I\in\mathcal{A}(w\varphi)} e_{W_I}$ may be written in W and in $W(w\varphi)$ as

$$e_W = N^{\mathrm{hyp}}_{W(w\varphi)} e_{W_I} \quad\text{and}\quad e_W(w\varphi) = N^{\mathrm{hyp}}_{W(w\varphi)} e_{W_I}(w\varphi) \,,$$

from which it follows that

$$\frac{e_W}{e_W(w\varphi)} = \frac{e_{W_I}}{e_{W_I}(w\varphi)} \in \mathbb{N} \,.$$

□

Remark 3.5. —
The conclusion in (2) of Lemma 3.4 need not be true in general. For example, consider the case where $W = G_{25}$ (in Shephard–Todd notation), and let w be a 2-regular element of W. Then $W(w) = G_5$. It follows that $e_W = 36$ and $e_W(w) = 24$, so $e_W(w)$ does not divide e_W. Note that G_5 has two orbits of reflecting hyperplanes.

Remark 3.6. —

As a special case, assume that we started with a split coset *i.e.*, $\varphi \in W$, and assume that W acts irreducibly on V. Let us denote by d the order of the regular element w of W.

Then by Theorem 3.1 $W(w)$ acts irreducibly on $V(w)$, hence its center $ZW(w)$ is cyclic. Let us choose a generator s of that center. Then we have

1. s is regular (since s acts as scalar multiplication on $V(w)$)
2. $W(s) = W(w)$.

3.3. Spetsial Φ-cyclotomic Hecke algebras at $w\varphi$

3.3.1. A long definition. —

We still denote by Φ the K-cyclotomic polynomial such that $\Phi(\zeta) = 0$, where $w\varphi$ is ζ-regular and ζ has order d.

Notice that, by definition

– $w\varphi$ acts on $V(w\varphi)$ as a scalar whose minimal polynomial over K is Φ,
– $K[x]/(\Phi) = K_{W(w\varphi)}$.

Definition 3.7. —

A spetsial Φ-cyclotomic Hecke algebra for W at $w\varphi$ *is a* $\overline{K}[x, x^{-1}]$*-algebra denoted* $\mathcal{H}_W(w\varphi)$*, specialization of the generic Hecke algebra of* $W(w\varphi)$ *through a morphism* σ *and subject to supplementary conditions listed below:*

There are

- *a* $W(w\varphi)$*-equivariant family* $(\zeta_{I,j})_{I\in\mathcal{A}(w\varphi),j=0,\ldots,e_I-1}$ *of elements of* $\boldsymbol{\mu}_{e_I}$,
- *a* $W(w\varphi)$*-equivariant family* $(m_{I,j})_{I\in\mathcal{A}(w\varphi),j=0,\ldots,e_I-1}$ *of nonnegative elements of* $\dfrac{1}{|ZW|}\mathbb{Z}$,

such that $\sigma : u_{I,j} \mapsto \zeta_{I,j} v^{|ZW|m_{I,j}}$ *where* v *is an indeterminate such that* $v^{|ZW|} = \zeta^{-1}x$*, with the following properties.*

For each $I \in \mathcal{A}(w\varphi)$*, the polynomial* $\prod_{j=0}^{e_I-1}(t - u_{I,j})$ *specialises to a polynomial* $P_I(t, x)$ *satisfying the conditions:*

(CA1) $P_I(t, x) \in K_{W(w\varphi)}[t, x]$,
(CA2) $P_I(t, x) \equiv t^{e_I} - 1 \pmod{\Phi(x)}$,

and the following supplementary conditions.

Global conditions

(RA) *The algebra* $\mathcal{H}_W(w\varphi)$ *splits over* $K_{W(w\varphi)}(v)$.
(SC1) *All Schur elements of irreducible characters of* $\mathcal{H}_W(w\varphi)$ *belong to* $\mathbb{Z}_K[x, x^{-1}]$.

(SC2) *There is a unique irreducible character χ_0 of $\mathcal{H}_W(w\varphi)$ with the following property: Whenever χ is an irreducible character of $\mathcal{H}_W(w\varphi)$ with Schur element S_χ, we have $S_{\chi_0}/S_\chi \in K[x]$. Moreover, χ_0 is linear.*

(SC3) *Whenever χ is an irreducible character of $\mathcal{H}_W(w\varphi)$ its Schur element S_χ divides $\mathrm{Feg}_{\mathbb{G}}(R_{w\varphi}^{\mathbb{G}})$ in $K[x, x^{-1}]$.*

For χ an irreducible character of $\mathcal{H}_W(w\varphi)$, we call generic degree *of χ the element of $K[x]$ defined by*

$$\mathrm{Deg}(\chi) := \frac{\mathrm{Feg}_{\mathbb{G}}(R_{w\varphi}^{\mathbb{G}})}{S_\chi}\,.$$

LOCAL CONDITIONS

Whenever $I \in \mathcal{A}(w\varphi)$, let us denote by $\mathcal{H}_{W_I}(w\varphi)$ the parabolic subalgebra of $\mathcal{H}_W(w\varphi)$ corresponding to the minimal parabolic subgroup $W_I(w\varphi)$ of $W(w\varphi)$. We set $\mathbb{G}_I := (V, W_I w\varphi)$. The following conditions concern the collection of parabolic subalgebras $\mathcal{H}_{W_I}(w\varphi)$ ($I \in \mathcal{A}(w\varphi)$).

The algebras $\mathcal{H}_{W_I}(w\varphi)$ have to satisfy all previous conditions (CA1), (CA2), *as well as conditions* (RA), (SC1), (SC2), (SC3) *that we state again now, plus*

- *for the* noncompact support type, *conditions* (NCS1), (NCS2), (NCS3) *stated below,*
- *for the* compact support type, *conditions* (CS1), (CS2), (CS3) *stated below.*

COMMON LOCAL CONDITIONS

Notice that the following conditions impose some properties of rationality to the local algebra $\mathcal{H}_{W_I}(w\varphi)$ coming from the global datum $\mathbb{G} = (V, W\varphi)$.

(RA_I) *The algebra $\mathcal{H}_{W_I}(w\varphi)$ splits over $K_{W(w\varphi)}(v)$, (where v is an indeterminate such that $v^{|ZW|} = \zeta^{-1}x$).*

($\mathrm{SC1}_I$) *All Schur elements of irreducible characters of $\mathcal{H}_{W_I}(w\varphi)$ belong to $\mathbb{Z}_K[x, x^{-1}]$.*

($\mathrm{SC2}_I$) *There is a unique irreducible character χ_0^I of $\mathcal{H}_{W_I}(w\varphi)$ with the following property: Whenever χ is an irreducible character of $\mathcal{H}_{W_I}(w\varphi)$ with Schur element S_χ, we have $S_{\chi_0^I}/S_\chi \in K[x]$. Note that since $W_I(w\varphi)$ is cyclic, χ_0^I is linear.*

($\mathrm{SC3}_I$) *Whenever χ is an irreducible character of $\mathcal{H}_{W_I}(w\varphi)$ its Schur element divides $\mathrm{Feg}_{\mathbb{G}}(R_{w\varphi}^{\mathbb{G}_I})$.*

We set $e := e_I = e_{W_I}(w\varphi)$.

Let us define $a_1(x), \dots, a_e(x) \in K_{W(w\varphi)}[x]$ (the $a_j(x)$ depend on I) by

$$P_I(t, x) = t^e - a_1(x)t^{e-1} + \cdots + (-1)^e a_e(x)\,.$$

NONCOMPACT SUPPORT TYPE (NCS)

We say that the algebra is of noncompact support type *if*

(NCS0) $P_I(t, x) \in K[t, x]$,

(NCS1) *1 is a root of $P_I(t,x)$ (as a polynomial in t) and it is the only root which has degree 0 in x. In particular $a_1(0) = 1$.*

(NCS2) *The unique character χ_0^I defined by condition* ($\textsc{sc}2_I$) *above is the restriction of χ_0 to $\mathcal{H}_{W_I}(w\varphi)$, and is defined by*

$$\chi_0^I(\mathbf{s}_I) = 1\,.$$

In other words, χ_0 defines the trivial character *on $\mathbf{B}_W(w\varphi)$.*

(NCS2') *We have*

$$S_{\chi_0}(x) = (\zeta^{-1}x)^{-N_W^{\mathrm{ref}}}\,\mathrm{Feg}(R_{w\varphi}^{\mathbb{G}})(x)\,,$$
$$S_{\chi_0^I}(x) = (\zeta^{-1}x)^{-N_{W_I}^{\mathrm{ref}}}\,\mathrm{Feg}(R_{w\varphi}^{\mathbb{G}_I})(x)\,,$$

and in particular

$$\mathrm{Deg}(\chi_0)(x) = (\zeta^{-1}x)^{N_W^{\mathrm{ref}}}\,.$$

(NCS3) $P_I(0,x) = (-1)^e a_e(x) = -(\zeta^{-1}x)^{N_{W_I}^{\mathrm{ref}}}\,.$

Compact support type (cs)

We say that the algebra is of compact support type *if*

(CS0) *For $j = 1,\dots,e$, we have $\zeta^{jm_I}a_j(x) \in K[x]$.*

(CS1) *There is only one root (as a polynomial in t, and in some field extension of $K(x)$) of $P_I(t,x)$ of highest degree in x, namely $(\zeta^{-1}x)^{\frac{e_{W_I}}{e_I}}$.*

(CS2) *The unique character χ_0^I defined by condition* ($\textsc{sc}2_I$) *above is the restriction of χ_0 to $\mathcal{H}_{W_I}(w\varphi)$, and is defined by*

$$\chi_0^I(\mathbf{s}_I) = (\zeta^{-1}x)^{\frac{e_{W_I}}{e_I}}\,.$$

(CS2') *We have*

$$S_{\chi_0} = \mathrm{Feg}(R_{w\varphi}^{\mathbb{G}})\,,$$
$$S_{\chi_0^I} = \mathrm{Feg}(R_{w\varphi}^{\mathbb{G}_I})\,,$$

and in particular

$$\mathrm{Deg}(\chi_0)(x) = 1\,.$$

(CS3) $P_I(0,x) = (-1)^e a_e(x) = -(\zeta^{-1}x)^{N_{W_I}^{\mathrm{hyp}}}\,.$

3.3.2. From compact type to noncompact type and vice versa. —

Let us first state some elementary facts about polynomials.

Let $P(t,x) = t^e - a_1(x)t^{e-1} + \cdots + (-1)^e a_e(x) \in K[t,x]$ such that $P(t,x) = \prod_{j=0}^{e-1}(t-\lambda_j)$, where the λ_j are nonzero elements in a suitable extension of $K(x)$.

Assume that P is "ζ-cyclotomic", *i.e.*, that $P(t,\zeta) = t^e - 1$.

Choose an integer m and consider the polynomial

$$P^{[m]}(t,x) := \frac{t^e}{P(0,x)}P(x^m t^{-1}, x)\,.$$

Then

$$P^{[m]}(t,x) = \prod_{j=0}^{e-1}(t - x^m\lambda_j^{-1})\,,$$

and

$$P^{[m]}(t,\zeta) = \zeta^{me}((\zeta^{-m}t)^e - 1)\,.$$

Define

$$P^{[m,\zeta]}(t,x) := \zeta^{-me}P^{[m]}(\zeta^m t, x) = \frac{t^e}{P(0,x)}P((\zeta^{-1}x)^m t^{-1}, x)\,.$$

We have

$$P^{[m,\zeta]}(t,x) = \prod_{j=0}^{e-1}(t - (\zeta^{-1}x)^m\lambda_j^{-1})\,,$$

and

$$P^{[m,\zeta]}(t,\zeta) = t^e - 1\,.$$

Note that $P(t,x) \mapsto P^{[m,\zeta]}(t,x)$ is an involution. Write $P^{[m,\zeta]}(t,x) = t^e - b_1(x)t^{e-1} + \cdots + (-1)^e b_e(x)$.

Remarks 3.8. —

1. If the highest degree term of $a_1(x)$ is $(\zeta^{-1}x)^m$, then $b_1(0) = 1$, and if $a_1(0) = 1$ then the highest degree term of $b_1(x)$ is $(\zeta^{-1}x)^m$.
2. If $P_I(0,x) = -(\zeta^{-1}x)^{N^{\mathrm{ref}}_{W_I}}$, then $P_I^{[e_{W_I}/e_I,\zeta]}(0,x) = -(\zeta^{-1}x)^{N^{\mathrm{hyp}}_{W_I}}$, and *vice versa*.

Let us prove (2). By definition of $P^{[m,\zeta]}(t,x)$, we have $P^{[m,\zeta]}(0,x) = (\zeta^{-1}x)^{em}P(0,x)$, whence

$$P^{[e_{W_I}/e_I,\zeta]}(0,x) = (\zeta^{-1}x)^{e_{W_I}}P(0,x)\,.$$

Now $e_{W_I} = N^{\mathrm{ref}}_{W_I} + N^{\mathrm{hyp}}_{W_I}$, so if $P(0,x) = -(\zeta^{-1}x)^{N^{\mathrm{ref}}_{W_I}}$ (resp. if $P(0,x) = -(\zeta^{-1}x)^{N^{\mathrm{hyp}}_{W_I}}$), we see that $P^{[e_{W_I}/e_I,\zeta]}(0,x) = -(\zeta^{-1}x)^{N^{\mathrm{hyp}}_{W_I}}$ (respectively $P^{[e_{W_I}/e_I,\zeta]}(0,x) = -(\zeta^{-1}x)^{N^{\mathrm{ref}}_{W_I}}$).

The following lemma is then easy to prove. It is also a definition.

Lemma 3.9. — *Assume given a $W(w\varphi)$-equivariant family of polynomials $(P_I(t,x))_{I\in\mathcal{A}(w\varphi)}$ in $K[t,x]$.*

For $I \in \mathcal{A}(w\varphi)$, set $m_I := \dfrac{e_{W_I}}{e_I}$. Then:

1. *If the family $(P_I(t,x))_{I\in\mathcal{A}(w\varphi)}$ defines a "spetsial Φ-cyclotomic Hecke algebra $\mathcal{H}_W(w\varphi)$ of W at $w\varphi$ of compact support type" then the family $(P_I^{[m_I,\zeta]}(t,x))_{I\in\mathcal{A}(w\varphi)}$ defines a "spetsial Φ-cyclotomic Hecke algebra $\mathcal{H}_W^{\mathrm{nc}}(w\varphi)$ of W at $w\varphi$ of noncompact support type", called the "noncompactification of $\mathcal{H}_W(w\varphi)$".*

2. *If the family* $(P_I(t,x))_{I\in\mathcal{A}(w\varphi)}$ *defines a "spetsial* Φ*-cyclotomic Hecke algebra* $\mathcal{H}_W(w\varphi)$ *of* W *at* $w\varphi$ *of noncompact support type" then the family* $(P_I^{[m_I,\zeta]}(t,x))_{I\in\mathcal{A}(w\varphi)}$ *defines a "spetsial* Φ*-cyclotomic Hecke algebra* $\mathcal{H}_W^c(w\varphi)$ *of* W *at* $w\varphi$ *of compact support type", called the "compactification of* $\mathcal{H}_W(w\varphi)$*".*

3.3.3. A normalization. —

Let $\mathcal{H}_W(w\varphi)$ be a spetsial Φ-cyclotomic Hecke algebra of W at $w\varphi$ of noncompact type, defined by a family of polynomials $(P_I(t,x))_{I\in\mathcal{A}(w\varphi)}$. We denote by $\mathcal{H}_W^c(w\varphi)$ its compactification, defined by the family $(P_I^{[m_I,\zeta]}(t,x))_{I\in\mathcal{A}(w\varphi)}$.

In the case where W is a Weyl group, the spetsial Φ-cyclotomic algebras $\mathcal{H}_W(w\varphi)$ and $\mathcal{H}_W^c(w\varphi)$ should have the following interpretation for every choice of a prime power q (see §3 above).

There is an appropriate Deligne–Lusztig variety $\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}}$, endowed with an action of the braid group $\mathbf{B}_W(w\varphi)$ as automorphisms of étale sites, such that

– (*Noncompact type*) the element $\mathbf{s}_I \in \mathbf{B}_W(w\varphi)$ has minimal polynomial $P_I(t,q)$ when acting on $H^\bullet(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}},\mathbb{Q}_\ell)$,
– (*Compact type*) the element $\zeta^{-m_I}\mathbf{s}_I$ has minimal polynomial $P_I^{[m_I,\zeta]}(t,q)$ when acting on $H_c^\bullet(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}},\mathbb{Q}_\ell)$.

Remark 3.10. — It results from (CA2) in definition 3.7 that the set $\{\zeta_{I,j}\}_j$ is equal to $\boldsymbol{\mu}_{e_I}$, but we have not yet chosen a specific bijection, which is how the data may appear in practice — see the second step of algorithm 5.4. We now make the specific choice that $\zeta_{I,j} = \zeta_{e_I}^j$; such a choice determines the indexation of the characters of $\mathcal{H}_W(w\phi)$ by those of $W(w\phi)$.

With the above choice, we have $P_I(t,x) = (t-1)\prod_{j=1}^{e_I-1}\left(t-\zeta_{e_I}^j(\zeta^{-1}x)^{m_{I,j}}\right)$ (where $m_{I,j}>0$ for all $j=1,\dots,e_I-1$, see (NCS1)), and we see that the minimal polynomial of $\mathbf{s}_I$ on $H_c^\bullet(\mathbf{X}_{\mathbf{w}\boldsymbol{\varphi}},\mathbb{Q}_\ell)$ is then

$$\widetilde{P}_I(t,x) := (t-x^{m_I})\prod_{j=1}^{e_I-1}\left(t-\zeta^{m_I}\zeta_{e_I}^{-j}(\zeta^{-1}x)^{m_I-m_{I,j}}\right)\,.$$

The polynomial $\widetilde{P}_I(t,x)$ is cyclotomic (*i.e.*, reduces to $t^{e_I}-1$ when $x\mapsto\zeta$) if and only if $\zeta^{m_I}\in\boldsymbol{\mu}_{e_I}$. That last condition is equivalent to

$$\zeta^{ew_I}=1 \quad i.e., \quad \Delta_{W_I}(w\varphi)=1\,.$$

Let us denote by $\widetilde{\mathcal{H}}_W^c(w\varphi)$ the specialisation of the generic Hecke algebra of $W(w\varphi)$ defined by the above polynomials $\widetilde{P}_I(t,x)$.

The following property results from Lemma 1.10.

Lemma 3.11. —
If $\mathbb{G}=(V,W\varphi)$ *is real, then the algebra* $\widetilde{\mathcal{H}}_W^c(w\varphi)$ *is* Φ*-cyclotomic.*

3.3.4. Rationality questions. —

A spetsial Φ-cyclotomic Hecke algebra for $W(w\varphi)$ over K is split over $K(\zeta)(v)$ where v is an indeterminate such that $v^{|ZW|} = \zeta^{-1}x$, by Def. 3.7 (RA).

Since $\boldsymbol{\mu}_{|ZW|} \subseteq K$, the extensions $K(\zeta, v)/K(x)$ and $K(v)/K(\zeta^{-1}x)$ are Galois.

For $I \in \mathcal{A}(w\varphi)$, let us set

$$P_I(t,x) = \prod_{j=0}^{e_I-1}(t - \zeta_{e_I}^j(\zeta^{-1}x)^{m_{I,j}}) = \prod_{j=0}^{e_I-1}(t - \zeta_{e_I}^j v^{n_{I,j}})\,,$$

with

$$m_{I,j} = \frac{n_{I,j}}{|ZW|} \quad \text{where} \quad n_{I,j} \in \mathbb{Z}\,.$$

Since $P_I(t,x) \in K(x)[t]$, its roots are permuted by the Galois group $\mathrm{Gal}(K(\zeta,v)/K(\zeta,x))$. Let us denote by $g \in \mathrm{Gal}(K(\zeta,v)/K(\zeta,x))$ the element defined by $g(v) = \zeta_{|ZW|}v$. Since g permutes the roots of P_I, there is a permutation σ of $\{0,\dots,e_I-1\}$ such that

$$g(\zeta_{e_I}^j v^{n_{I,j}}) = \zeta_{e_I}^j \zeta_{|ZW|}^{n_{I,j}} v^{n_{I,j}} = \zeta_{e_I}^{\sigma(j)} v^{n_{I,\sigma(j)}}\,,$$

and so

$$n_{I,\sigma(j)} = n_{I,j} \quad \text{and} \quad \zeta_{e_I}^{\sigma(j)} = \zeta_{e_I}^j \zeta_{|ZW|}^{n_{I,j}}\,. \tag{3.12}$$

Remark 3.13. — By Equation 3.12, we see that if j is such that $m_{I,j} \neq m_{I,j'}$ for all $j' \neq j$, then $\sigma(j) = j$, which implies that $\zeta_{e_I}^j = \zeta_{e_I}^j \zeta_{|ZW|}^{n_{I,j}}$, hence that $n_{I,j}$ is a multiple of $|ZW|$, and so $m_{I,j} \in \mathbb{Z}$.

By (SC1), the Schur elements of $\mathcal{H}_{W_I}(w\varphi)$ (see 1.68)

$$S_j := \frac{1}{P(0,x)}\left(t\frac{d}{dt}P(t,x)\right)|_{t=\zeta_{e_I}^j v^{n_{I,j}}}$$

belong to $K[x]$, hence are fixed by $\mathrm{Gal}(K(\zeta,v)/K(\zeta,x))$, *i.e.*, we have $S_{\sigma(j)} = S_j$, or, in other words

$$\left(t\frac{d}{dt}P(t,x)\right)|_{t=\zeta_{e_I}^j \zeta_{|ZW|}^{n_{I,j}} v^{n_{I,j}}} = \left(t\frac{d}{dt}P(t,x)\right)|_{t=\zeta_{e_I}^j v^{n_{I,j}}}\,. \tag{3.14}$$

3.3.5. Ennola twist. —

Let us choose an element in $W \cap Z\mathrm{GL}(V)$, the scalar multiplication by $\varepsilon \in \boldsymbol{\mu}(K)$. Then the element $\varepsilon w\varphi$ is $\Phi(\varepsilon^{-1}x)$-regular, and we obviously have $W(\varepsilon w\varphi) = W(w\varphi)$.

Assume given a $W(w\varphi)$-equivariant family $(P_I(t,x))_{I\in\mathcal{A}(w\varphi)}$ of polynomials in $K[t,x]$. For $I \in \mathcal{A}(w\varphi)$, set

$$(\varepsilon.P_I)(t,x) := P_I(t,\varepsilon^{-1}x)\,.$$

Note that the map $P \mapsto \varepsilon.P$ is an operation of order the order of ε.

The following lemma is also a definition. Its proof is straightforward.

Lemma 3.15. —

(cs) *Assume that the family* $(P_I(t,x))_{I\in\mathcal{A}(w\varphi)}$ *defines a spetsial* Φ*-cyclotomic Hecke algebra* $\mathcal{H}_W(w\varphi)$ *of* W *at* $w\varphi$ *of compact support type.*

The family $((\varepsilon.P_I)(t,x))_{I\in\mathcal{A}(w\varphi)}$ *defines a spetsial* $\Phi(\varepsilon^{-1}x)$*-cyclotomic Hecke algebra of* W *at* $\varepsilon w\varphi$ *of compact support type, denoted* $\varepsilon.\mathcal{H}_W(w\varphi)$ *and called the Ennola* ε*-twist of* $\mathcal{H}_W(w\varphi)$*.*

(ncs) *Assume that the family* $(P_I(t,x))_{I\in\mathcal{A}(w\varphi)}$ *defines a spetsial* Φ*-cyclotomic Hecke algebra* $\mathcal{H}_W(w\varphi)$ *of* W *at* $w\varphi$ *of noncompact support type .*

The family $((\varepsilon.P_I)(t,x))_{I\in\mathcal{A}(w\varphi)}$ *defines a spetsial* $\Phi(\varepsilon^{-1}x)$*-cyclotomic Hecke algebra of* W *at* $\varepsilon w\varphi$ *of noncompact support type, denoted by* $\varepsilon.\mathcal{H}_W(w\varphi)$ *and called the Ennola* ε*-twist of* $\mathcal{H}_W(w\varphi)$*.*

Remark 3.16. —

Assume that $\mathcal{H}_W(w\varphi)$ is split over $K(\zeta)(v)$ for some k such that $k|m_K$ and $v^k = \zeta^{-1}x$. Then we see that $\varepsilon.\mathcal{H}_W(w\varphi)$ is split over $K(\varepsilon\zeta)(v_\varepsilon)$ if $v_\varepsilon^k = \varepsilon^{-1}\zeta^{-1}x$.

Thus the field $K(\varepsilon^{1/k}, v)$ splits both $\mathcal{H}_W(w\varphi)$ and $\varepsilon.\mathcal{H}_W(w\varphi)$.

3.4. More on spetsial Φ-cyclotomic Hecke algebras

Let $\mathcal{H}_W(w\varphi)$ be a spetsial Φ-cyclotomic Hecke algebra attached to the ζ-regular element $w\varphi$.

Note that, unless specified, $\mathcal{H}_W(w\varphi)$ may be of noncompact type or of compact type.

3.4.1. Computation of $\omega_\chi(\boldsymbol{\pi})$ and applications. —

Choose a positive integer h and an indeterminate v such that $v^h = \zeta^{-1}x$ and such that $\mathcal{H}_W(w\varphi)$ splits over $\overline{K}(v)$.

Whenever χ is an (absolutely) irreducible character of $\mathcal{H}_W(w\varphi)$ over $\overline{K}(v)$, we denote by $\chi_{v=1}$ the irreducible character of $W(w\varphi)$ defined by the specialization $v \mapsto 1$.

We denote by σ_χ the sum of the valuation and the degree of the Schur element (a Laurent polynomial) $S_\chi(x)$.

Since $S_\chi(x)$ is semi-palindromic (see [**BMM99**, §6.B]), we have

$$S_\chi(x)^\vee = (\text{Constant}).x^{-\sigma_\chi} S_\chi(x)\,.$$

We have (see Lemma 1.64, assuming 1.60)

$$S_\chi(x)^\vee = \frac{\tau(\boldsymbol{\pi})}{\omega_\chi(\boldsymbol{\pi})} S_\chi(x)\,.$$

From what precedes and by comparing with the specialization $v \mapsto 1$ we get

$$\omega_\chi(\boldsymbol{\pi}) = v^{h\sigma_\chi}\tau(\boldsymbol{\pi}) = (\zeta^{-1}x)^{\sigma_\chi}\tau(\boldsymbol{\pi})\,. \tag{3.17}$$

– Now in the NCS case we have $\tau(\boldsymbol{\pi}) = (\zeta^{-1}x)^{N_W^{\mathrm{ref}}} = v^{hN_W^{\mathrm{ref}}}$
– while in the CS case we have $\tau(\boldsymbol{\pi}) = (\zeta^{-1}x)^{N_W^{\mathrm{hyp}}} = v^{hN_W^{\mathrm{hyp}}}$

from which we deduce

Proposition 3.18. —

1. *In the NCS case,* $\omega_\chi(\boldsymbol{\pi}) = (\zeta^{-1}x)^{N_W^{\mathrm{ref}}+\sigma_\chi} = v^{h(N_W^{\mathrm{ref}}+\sigma_\chi)}$.
2. *In the CS case,* $\omega_\chi(\boldsymbol{\pi}) = (\zeta^{-1}x)^{N_W^{\mathrm{hyp}}+\sigma_\chi} = v^{h(N_W^{\mathrm{hyp}}+\sigma_\chi)}$.

Assume that the algebra $\mathcal{H}_W(w\varphi)$ is defined over $K(x)$ by the family of polynomials

$$\left(P_I(t,x) := \prod_{j=0}^{e_I-1}(t-\zeta_{e_I}^j v^{hm_{I,j}})\right)_{I\in\mathcal{A}_W(w\varphi)}$$

of $K_{W(w\varphi)}[t,x]$ and that it splits over $\overline{K}(v)$. We set

$$N_W := \begin{cases} N_W^{\mathrm{ref}} & \text{if } \mathcal{H}_W(w\varphi) \text{ is of noncompact type,} \\ N_W^{\mathrm{hyp}} & \text{if } \mathcal{H}_W(w\varphi) \text{ is of compact type.} \end{cases}$$

Any specialization of the type $v \mapsto \lambda$ where λ is an h-th root of unity defines a bijection

$$\begin{cases} \mathrm{Irr}\left(\mathcal{H}_W(w\varphi)\right) \to \mathrm{Irr}\left(W(w\varphi)\right), \\ \chi \mapsto \chi_{v=\lambda}, \end{cases}$$

whose inverse is denoted

$$\begin{cases} \mathrm{Irr}\left(W(w\varphi)\right) \to \mathrm{Irr}\left(\mathcal{H}_W(w\varphi)\right), \\ \theta \mapsto \theta^{(\lambda^{-1}v)}. \end{cases}$$

Lemma 3.19. —

Let $\chi \in \mathrm{Irr}(\mathcal{H}_W(w\varphi))$.

1. *Let* $\boldsymbol{\rho} \in Z\mathbf{B}_W(w\varphi)$ *such that* $\boldsymbol{\rho}^n = \boldsymbol{\pi}^a$ *for some* $a, n \in \mathbb{N}$. *Then*
$$h(N_W+\sigma_\chi)a/n \in \mathbb{Z}.$$
If moreover χ *is rational over* $\overline{K}(x)$, *we have*
$$(N_W+\sigma_\chi)a/n \in \mathbb{Z}.$$
2. *Whenever* λ *is an* h*-th root of unity, then*
$$\omega_\chi(\boldsymbol{\rho}) = \omega_{\chi_{v=\lambda}}(\rho)(\lambda^{-1}v)^{h(N_W+\sigma_\chi)a/n}.$$
In particular,
$$\omega_\chi(\boldsymbol{\rho}) = \omega_{\chi_{v=1}}(\rho)v^{h(N_W+\sigma_\chi)a/n}.$$
3. *We have*
$$\omega_{\chi_{v=\lambda}}(\rho) = \lambda^{h(N_W+\sigma_\chi)a/n}\omega_{\chi_{v=1}}(\rho).$$

Proof of 3.19. —

By Proposition 3.18,

$$\omega_\chi(\boldsymbol{\pi}) = v^{h(N_W+\sigma_\chi)}\,.$$

Since $\boldsymbol{\rho}^n = \boldsymbol{\pi}^a$, it follows that, whenever $\lambda \in \overline{K}$, we have

$$(*) \qquad \omega_\chi(\boldsymbol{\rho}) = \kappa(\lambda^{-1}v)^{h(N_W+\sigma_\chi)a/n} \quad \text{for some } \kappa \in \overline{K}\,.$$

Since the character χ is rational over $\overline{K}(v)$, $\omega_\chi(\boldsymbol{\rho}) \in \overline{K}(v)$, which implies $h(N_W + \sigma_\chi)a/n \in \mathbb{Z}$.

If χ is rational over $\overline{K}(x)$ we have $\omega_\chi(\boldsymbol{\rho}) \in \overline{K}(x)$, which implies $(N_W+\sigma_\chi)a/n \in \mathbb{Z}$. This proves (1).

As $\lambda^h = 1$, by specializing $v \mapsto \lambda$ in (*) we find $\kappa = \omega_{\chi_{v=\lambda}}(\rho)$.

Assertion (3) follows from the equality

$$\omega_\chi(\boldsymbol{\rho}) = \omega_{\chi_{v=\lambda}}(\rho)(\lambda^{-1}v)^{h(N_W+\sigma_\chi)a/n} = \omega_{\chi_{v=1}}(\rho)v^{h(N_W+\sigma_\chi)a/n}\,.$$

□

3.4.2. Compactification and conventions. —

Assume now that $\mathcal{H}_W(w\varphi)$ is a spetsial Φ-cyclotomic Hecke algebra attached to the ζ-regular element $w\varphi$, *of noncompact type*, defined by the family of polynomials $(P_I(t,x))_{I\in\mathcal{A}_W(w\varphi)}$ with

$$P_I(t,x) = \prod_{j=0}^{e_I-1}(t - \zeta_{e_I}^j(\zeta^{-1}x)^{m_{I,j}})\,.$$

Some notation.

For $I \in \mathcal{A}_W(w\varphi)$, we denote by $\mathbf{s}_I$ the braid reflection around I in $\mathbf{B}_W(w\varphi)$, and by T_I the image of $\mathbf{s}_I$ in $\mathcal{H}_W(w\varphi)$. Thus we have

$$\prod_{j=0}^{e_I-1}(T_I - \zeta_{e_I}^j(\zeta^{-1}x)^{m_{I,j}}) = 0\,.$$

The map $\mathbf{s}_I \mapsto T_I$ extends to a group morphism

$$\mathbf{B}_W(w\varphi) \to \mathcal{H}_W(w\varphi)^\times\,, \quad b \mapsto T_b\,.$$

We denote by χ_0 the unique linear character of $\mathcal{H}_W(w\varphi)$ such that (see 3.3.1) $S_{\chi_0}(x) = (\zeta^{-1}x)^{-N_W^{\mathrm{ref}}}\mathrm{Feg}(R_{w\varphi}^{\mathbb{G}})(x)$.

Let us denote by $\mathcal{H}_W^{\mathrm{c}}(w\varphi)$ the compactification of $\mathcal{H}_W(w\varphi)$. By definition, $\mathcal{H}_W^{\mathrm{c}}(w\varphi)$ is generated by a family of elements $(T_I^{\mathrm{c}})_{I\in\mathcal{A}_W(w\varphi)}$ satisfying

$$\prod_{j=0}^{j=e_I-1}(T_I^{\mathrm{c}} - \zeta_{e_I}^{-j}(\zeta^{-1}x)^{m_I - m_{I,j}}) = 0\,,$$

where $m_I := e_{W_I}/e_I$. The map $\mathbf{s}_I \mapsto T_I^{\mathrm{c}}$ extends to a group morphism

$$\mathbf{B}_W(w\varphi) \to \mathcal{H}_W^{\mathrm{c}}(w\varphi)^\times \,, \quad b \mapsto T_b^{\mathrm{c}} \,.$$

We denote by $\chi_{0,\mathrm{c}}$ the unique linear character of $\mathcal{H}_W^{\mathrm{c}}(w\varphi)$ such that (see 3.3.1) $S_{\chi_{0,\mathrm{c}}} = \mathrm{Feg}(R_{w\varphi})$.

For the definition of the opposite algebra $\mathcal{H}_W^{\mathrm{c}}(w\varphi)^{\mathrm{op}}$, the reader may refer to [**BMM99**, 1.30].

Finally, we recall (see introduction of Section 2) that for $P(x) \in \overline{K}[x, x^{-1}]$, we set $P(x)^\vee := P(x^{-1})^*$.

Proposition 3.20 **(Relation between $\mathcal{H}_W(w\varphi)$ and $\mathcal{H}_W^{\mathrm{c}}(w\varphi)$)**

1. *The algebra morphism*

$$\mathcal{H}_W(w\varphi) \to \mathcal{H}_W^{\mathrm{c}}(w\varphi)^{\mathrm{op}}$$

defined by

$$T_I \mapsto (\zeta^{-1}x)^{m_I}(T_I^{\mathrm{c}})^{-1}$$

is an isomorphism of algebras.

2. *There is a bijection*

$$\mathrm{Irr}\,\mathcal{H}_W^{\mathrm{c}}(w\varphi) \xrightarrow{\sim} \mathrm{Irr}\,\mathcal{H}_W(w\varphi) \,, \quad \chi \mapsto \chi^{\mathrm{nc}} \,,$$

defined by

$$\chi^{\mathrm{nc}}(T_b) := \chi_{0,\mathrm{c}}(T_b^{\mathrm{c}})\chi(T_{b^{-1}}^{\mathrm{c}}) \quad \textit{whenever } b \in \mathbf{B}_W(w\varphi) \,.$$

We have $(\chi_{0,\mathrm{c}})^{\mathrm{nc}} = \chi_0$.

3. *By specialisation $v \mapsto 1$, χ and χ^{nc} become dual characters of the group $W(w\varphi)$:*

$$\chi^{\mathrm{nc}}|_{v=1} = (\chi|_{v=1})^* \,.$$

4. *Assuming 1.60, we have*
   (a) $S_{\chi^{\mathrm{nc}}}(x) = S_\chi(x)^\vee = (\zeta^{-1}x)^{-\sigma_\chi} S_\chi(x)$,
   (b) $\sigma_{\chi^{\mathrm{nc}}} + \sigma_\chi = 0$.

Proof. —

(1) and (2) are immediate consequences of the definition of $\mathcal{H}_W^{\mathrm{c}}(w\varphi)$. (3) follows immediately from the definition of the correspondence, since $\chi_{0,\mathrm{c}}$ specialises to the trivial character.

By Theorem–Conjecture 1.60, the generic Schur elements are multihomogeneous of degree 0 (we recall that this is proven, for example, for all imprimitive irreducible complex reflection groups — see 1.65 and 1.66 above), so by construction of the compactification we have $S_{\chi^{\mathrm{nc}}}(x) = S_\chi(x)^\vee$. The assertion (4)(a) follows from 3.17 and the assertion (b) is obvious. □

Let us recall (see 3.3.1) that

$$\mathrm{Deg}(\chi) := \frac{\mathrm{Feg}(R_{w\varphi})}{S_\chi}\,,$$
$$\mathrm{Deg}(\chi^{\mathrm{nc}}) = \frac{\mathrm{Feg}(R_{w\varphi})}{S_{\chi^{\mathrm{nc}}}}\,.$$

We denote by δ_χ (resp. $\delta_{\chi^{\mathrm{nc}}}$) the sum of the valuation and of the degree of $\mathrm{Deg}(\chi)(x)$ (resp. of $\mathrm{Deg}(\chi^{\mathrm{nc}})(x)$).

Lemma 3.21. —

1. *We have*

$$\delta_\chi = N_W^{\mathrm{ref}} - \sigma_\chi \qquad \textit{and} \quad \mathrm{Deg}(\chi)(x)^\vee = (\zeta^{-1}x)^{-\delta_\chi}\mathrm{Deg}(\chi)(x)\,,$$
$$\delta_{\chi^{\mathrm{nc}}} = N_W^{\mathrm{ref}} - \sigma_{\chi^{\mathrm{nc}}} \qquad \textit{and} \quad \mathrm{Deg}(\chi^{\mathrm{nc}})(x)^\vee = (\zeta^{-1}x)^{-\delta_{\chi^{\mathrm{nc}}}}\mathrm{Deg}(\chi^{\mathrm{nc}})(x)\,.$$

2. *We have*

$$\mathrm{Deg}(\chi^{\mathrm{nc}})(x) = (\zeta^{-1}x)^{N_W^{\mathrm{ref}}-\delta_\chi}\mathrm{Deg}(\chi)(x) = (\zeta^{-1}x)^{\sigma_\chi}\mathrm{Deg}(\chi)(x)\,.$$

Proof. —

(1) is immediate. To prove (2), notice that

$$\mathrm{Deg}(\chi)(x)^\vee = \frac{\mathrm{Feg}(R_{w\varphi})(x)^\vee}{S_\chi(x)^\vee} = \frac{(\zeta^{-1}x)^{-N_W^{\mathrm{ref}}}\mathrm{Feg}(R_{w\varphi})(x)}{S_{\chi^{\mathrm{nc}}}(x)}$$
$$= (\zeta^{-1}x)^{-N_W^{\mathrm{ref}}}\mathrm{Deg}(\chi^{\mathrm{nc}})(x)$$

and (2) results from (1). □

The case $W(w\varphi)$ cyclic

We assume now that $W(w\varphi)$ is cyclic of order e. Let s be its distinguished generator, and let $\mathbf{s}$ be the corresponding braid reflection in $\mathbf{B}_W(w\varphi)$.

Let $\mathcal{H}_W(w\varphi)$ be a spetsial Φ-cyclotomic Hecke algebra of compact type associated with $w\varphi$, defined by the polynomial

$$\prod_{j=0}^{e-1}(t - \zeta_e^j(\zeta^{-1}x)^{m_j})\,,$$

where m_j are nonnegative rational numbers such that $em_j \in \mathbb{N}$. We have $m_0 = e_W/e > m_j$ for all j.

Let us denote by v an indeterminate such that $v^e = \zeta^{-1}x$, so that the algebra $\mathcal{H}_W(w\varphi)$ splits over $K(v)$.

For each j, we denote by θ_j the character of $W(w\varphi)$ defined by $\theta_j(s) = \zeta_e^j$.

The specialization $v \mapsto 1$ defines a bijection

$$\begin{cases} \mathrm{Irr}\,(W(w\varphi)) \to \mathrm{Irr}\,(\mathcal{H}_W(w\varphi))\,, \\ \theta_j \mapsto \theta_j^{(v)}\,, \end{cases}$$

by the condition $\theta_j^{(v)}(\mathbf{s}) := \zeta_e^j v^{em_j}$. We set $\sigma_j := \sigma_{\theta_j^{(v)}}$.

Let $\mathcal{H}_W^{\mathrm{nc}}(w\varphi)$ be the noncompactification of $\mathcal{H}_W(w\varphi)$, defined by the polynomial

$$\prod_{j=0}^{e-1}(t-\zeta_e^{-j}(\zeta^{-1}x)^{m-m_j})\,.$$

We denote by $\theta_j^{(v),\mathrm{nc}}$ the character of $\mathcal{H}_W^{\mathrm{nc}}(w\varphi)$ which specializes to θ_{-j} for $v=1$. So $\theta_j^{(v),\mathrm{nc}}(\mathbf{s}) = \zeta_e^{-j}v^{e(m-m_j)}$. We set $\sigma_j^{\mathrm{nc}} := \sigma_{\theta_j^{(v),\mathrm{nc}}}$.

Lemma 3.22. —

$$\begin{cases} \sigma_j = em_j - N_W^{\mathrm{ref}}\,, \\ \sigma_j^{\mathrm{nc}} = e(m-m_j) - N_W^{\mathrm{hyp}} = N_W^{\mathrm{ref}} - em_j\,. \end{cases}$$

Proof. —

Let S_j be the Schur element of $\mathcal{H}_W(w\varphi)$ corresponding to $\theta_j^{(v)}$. By definition, the integer σ_j is defined by an equation

$$S_j(x)^\vee = \lambda x^{-\sigma_j} S_j(x)$$

for some complex number λ.

On the other hand, it results from 1.69 and from Definition 3.3.1, (NCS3) and (CS3), that

$$\begin{cases} S_j(x)^\vee = v^{-em_j+N_W^{\mathrm{ref}}} S_j(x)\,, \\ S_j^{\mathrm{nc}}(x)^\vee = v^{-e(m-m_j)+N_W^{\mathrm{hyp}}} S_j^{\mathrm{nc}}(x)\,, \end{cases}$$

proving that $\sigma_j = em_j - N_W^{\mathrm{ref}}$ and $\sigma_j^{\mathrm{nc}} = e(m-m_j) - N_W^{\mathrm{hyp}}$. Since $em = N_W^{\mathrm{hyp}} + N_W^{\mathrm{ref}}$, we deduce that $\sigma_j^{\mathrm{nc}} = N_W^{\mathrm{ref}} - em_j$. □

A generalization of the cyclic case

We shall present now a generalization of Lemma 3.22 to the general case, where $w\varphi$ is a ζ-regular element for W, and $W(w\varphi)$ not necessarily cyclic.

Let $\mathcal{H}_W(w\varphi)$ we a spetsial Φ-cyclotomic Hecke algebra (either of compact type or of noncompact type) attached to $w\varphi$, defined by a family of polynomials

$$\left(P_I(t,x) = \prod_{j=0}^{e_I-1}(t-\zeta_{e_I}^j(\zeta^{-1}x)^{m_{I,j}})\right)_{I\in\mathcal{A}_W(w\varphi)}.$$

Any linear character χ of $\mathcal{H}_W(w\varphi)$ is defined by a family $(j_{I,\chi})$ where $I\in\mathcal{A}_W(w\varphi)$ and $0\le j_{I,\chi}\le e_I-1$ such that, if $\mathbf{s}_I$ denotes the braid reflection attached to I, we have

$$\chi(\mathbf{s}_I) = \zeta_{e_I}^{j_{I,\chi}}(\zeta^{-1}x)^{m_{I,j_{I,\chi}}}\,.$$

Whenever $I\in\mathcal{A}_W(w\varphi)$, we denote by ν_I the cardinality of the orbit of I under $W(w\varphi)$.

Note that the second assertion of the following Lemma reduces to Lemma 3.22 in the case where $W(w\varphi)$ is cyclic.

Lemma 3.23. —

1. *Whenever χ is a linear character of $\mathcal{H}_W(w\varphi)$, we have*
$$\chi(\boldsymbol{\pi}) = \prod_{I\in\mathcal{A}_W(w\varphi)} (\zeta^{-1}x)^{e_I m_{I,j_{I,\chi}}} = \prod_{I\in\mathcal{A}_W(w\varphi)/W(w\varphi)} (\zeta^{-1}x)^{\nu_I e_I m_{I,j_{I,\chi}}} .$$
2. *We have*
$$N_W + \sigma_\chi = \sum_{I\in\mathcal{A}_W(w\varphi)} e_I m_{I,j_{I,\chi}} = \sum_{I\in\mathcal{A}_W(w\varphi)/W(w\varphi)} \nu_I e_I m_{I,j_{I,\chi}} .$$

Proof. —

The assertion (2) follows from (1) and from 3.18. Let us prove (1).

From [**BMR98**, 2.26], we know that in the abelianized braid group $\mathbf{B}_W/[\mathbf{B}_W,\mathbf{B}_W]$, we have $\boldsymbol{\pi} = \prod_{I\in\mathcal{A}_W(w\varphi)} \mathbf{s}_I^{e_I}$, which implies (1). □

3.4.3. Ennola action. —

If $\mathbf{G}^F$ is a finite reductive group, with Weyl group W of type B_n, C_n, D_{2n}, E_7, E_8, F_4, G_2, then "changing x into $-x$" in the generic degrees formulæ corresponds to a permutation on the set of unipotent characters, which we call *Ennola transform*. The Ennola transform permutes the generalized d-Harish–Chandra series (see [**BMM93**]), sending the d-series (corresponding to the cyclotomic polynomial $\Phi_d(x)$) to the series corresponding to the cyclotomic polynomial $\Phi_d(-x)$. We shall now introduce appropriate tools to generalize the notion of Ennola transform to the setting of "spetses".

Throughout this paragraph, we assume that the reflection group W *acts irreducibly on* V. Its center ZW is cyclic and acts by scalar multiplications on V. By abuse of notation, for $z \in ZW$ we still denote by z the scalar by which z acts on V. We set $c := |ZW|$.

We define an operation of $Z\mathbf{B}_W$ on the disjoint union
$$\bigsqcup_{z\in ZW} \operatorname{Irr}\left(\mathcal{H}_W(zw\varphi)\right) .$$

Let $\mathbf{z}_0$ be the positive generator of $Z\mathbf{B}_W$. For $\mathbf{z} \in Z\mathbf{B}_W$, we denote by z its image in ZW. There is a unique n $(0 \le n \le c-1)$ such that $z = \zeta_c^n$.

The element $w'\varphi := zw\varphi$ is then $\zeta' := z\zeta$-regular. We have $K(\zeta) = K(\zeta')$ and the algebra $\mathcal{H}_W(zw\varphi)$ is split over $K(\zeta)(v')$ where $v' := \zeta_{hc}^{-n}v$. We have ${v'}^h = {\zeta'}^{-1}x$, and thus $\overline{K}(v) = \overline{K}(v')$.

Consider the character ξ defined on $Z\mathbf{B}_W$ by the condition
$$\xi : \mathbf{z}_0 \mapsto \zeta_{hc} = \exp\left(2\pi i/hc\right) ,$$

so that $\xi(\mathbf{z}) = \zeta_{hc}^{n}$. Note that $\xi(\mathbf{z}^h) = z$.

The group $Z\mathbf{B}_W$ acts on $\overline{K}(v)$ as a Galois group, as follows:

$$\varepsilon : \begin{cases} Z\mathbf{B}_W \to \mathrm{Gal}\left(\overline{K}(v)/\overline{K}(v^{hc})\right) , \\ \mathbf{z} \mapsto \left(v \mapsto \xi(\mathbf{z})^{-1}v\right) . \end{cases}$$

Notice that if $\mathcal{H}_W(w\varphi)$ is a spetsial $\Phi(x)$-cyclotomic Hecke algebra attached to a regular element $w\varphi \in W\varphi$, then the algebra $z.\mathcal{H}_W(w\varphi)$ (see Lemma 3.15) is a spetsial $\Phi(z^{-1}x)$-cyclotomic Hecke algebra attached to the regular element $zw\varphi \in W\varphi$.

Definition 3.24. —

For $\chi \in \mathrm{Irr}\,(\mathcal{H}_W(w\varphi))$, $\mathbf{z} \in Z\mathbf{B}_W$ and so $z = \xi(\mathbf{z}^h)$, we denote by $\mathbf{z}.\chi$ (the Ennola image of χ under $\mathbf{z}$) the irreducible character of $\mathcal{H}_W(zw\varphi)$ over $\overline{K}(v)$ defined by the following condition:

$$(\mathbf{z}.\chi)_{v=\xi(\mathbf{z})} = \chi_{v=1} \, .$$

In other words, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Irr}(\mathcal{H}_W(w\varphi)) & \xrightarrow{\ \mathbf{z}\cdot\ } & \mathrm{Irr}(\mathcal{H}_W(zw\varphi)) \\ & {\scriptstyle v\mapsto 1}\searrow \quad \swarrow{\scriptstyle v\mapsto\xi(\mathbf{z})} & \\ & \mathrm{Irr}(W) & \end{array}$$

In particular, the element $\boldsymbol{\pi} = \mathbf{z}_0^c$ defines the following permutation

$$\begin{cases} \mathrm{Irr}\,(\mathcal{H}_W(w\varphi)) \to \mathrm{Irr}\,(\mathcal{H}_W(w\varphi)) \\ \chi \mapsto \boldsymbol{\pi}.\chi \quad \text{where} \quad (\boldsymbol{\pi}.\chi)_{v=\zeta_h} = \chi_{v=1} \, , \end{cases}$$

so it acts on $\mathrm{Irr}(\mathcal{H}_W(w\varphi))$ as a generator of $\mathrm{Gal}(\overline{K}(v)/\overline{K}(x))$.

Lemma 3.25. —

Let $\boldsymbol{\rho}$ be an element of $Z\mathbf{B}_W(w\varphi)$ (hence of $Z\mathbf{B}_W(zw\varphi)$) such that $\boldsymbol{\rho}^n = \boldsymbol{\pi}^a$ for some $a, n \in \mathbb{N}$. Then for $\mathbf{z} \in Z\mathbf{B}_W$ and $\chi \in \mathrm{Irr}\,(\mathcal{H}_W(w\varphi))$ we have

1. $\omega_{\mathbf{z}\cdot\chi_{v=1}}(\rho) = \xi(\mathbf{z})^{-h(N_W^{\mathrm{hyp}}+\sigma_\chi)a/n}\omega_{\chi_{v=1}}(\rho)$,
2. $\omega_{\mathbf{z}.\chi}(\boldsymbol{\rho}) = \omega_\chi(\boldsymbol{\rho})\xi(\mathbf{z})^{-h(N_W^{\mathrm{hyp}}+\sigma_\chi)a/n}$.

Proof. —

By Definition 3.24, we know that

$$\omega_{\chi_{v=1}} = \omega_{\mathbf{z}\cdot\chi_{v=\xi(\mathbf{z})}} \, .$$

Thus by Lemma 3.19(2), we get

$$\omega_{\chi_{v=1}}(\rho) = \omega_{\mathbf{z}\cdot\chi_{v=\xi(\mathbf{z})}}(\rho) = \xi(\mathbf{z})^{h(N_W^{\mathrm{hyp}}+\sigma_\chi)a/n}\omega_{\mathbf{z}\cdot\chi_{v=1}}(\rho)$$

from which (1) follows.

Now (2) follows from (1) and from Lemma 3.19(1). □

Proposition 3.26. — *Let* $\chi \in \mathrm{Irr}\,(\mathcal{H}_W(w\varphi))$.

1. *The character* χ *takes its values in* $\overline{K}(x)$ *if and only if* $\boldsymbol{\pi}\cdot\chi = \chi$.
2. *Assume* χ *is such that* $\boldsymbol{\pi}\cdot\chi = \chi$. *Then for all* $\boldsymbol{\rho} \in Z\mathbf{B}_W(w\varphi)$ *such that* $\boldsymbol{\rho}^n = \boldsymbol{\pi}^a$, *we have*
$$(N_W^{\mathrm{hyp}} + \sigma_\chi)\frac{a}{n} \in \mathbb{Z}\,.$$

Remark 3.27. — Consider for example the case of the Weyl group of type E_7, and choose $K = \mathbb{Q}$, $\zeta = 1$, $w\varphi = 1$. The algebra $\mathcal{H}_W(1)$ is then the "usual" Hecke algebra over $\mathbb{Z}[x, x^{-1}]$, and we have $h = 2$. We set $x = v^2$.

All the irreducible characters of $\mathcal{H}_W(1)$ are $\overline{\mathbb{Q}}(x)$-rational, except for the two characters of dimension 512 denoted $\phi_{512,11}$ and $\phi_{512,12}$, which take their value in $\mathbb{Q}[v]$. The Galois action of $\boldsymbol{\pi}$ is given by $v \mapsto -v$, and we have $\boldsymbol{\pi}\cdot\phi_{512,11} = \phi_{512,12}$.

Proof of 3.26. — (1) follows as $\boldsymbol{\pi}$ acts as a generator of $\mathrm{Gal}(\overline{K}(v)/\overline{K}(x))$.

(2) Applying Lemma 3.25 above to the case $\mathbf{z} = \boldsymbol{\pi}$ gives
$$\omega_{\boldsymbol{\pi}.\chi}(\boldsymbol{\rho}) = \omega_\chi(\boldsymbol{\rho})\zeta_h^{-h(N_W^{\mathrm{hyp}}+\sigma_\chi)a/n}\,, \quad \text{hence } \zeta_h^{-h(N_W^{\mathrm{hyp}}+\sigma_\chi)a/n} = 1\,,$$
from which the claim follows. □

Remark 3.28. — By Lemma 3.21(1), we know that $\sigma_\chi = N_W^{\mathrm{ref}} - \delta_\chi$, which implies that
$$(N_W^{\mathrm{hyp}} + \sigma_\chi)\frac{a}{n} = (e_W - \delta_\chi)\frac{a}{n}\,.$$
Since $e_W\dfrac{a}{n}$ is the length of $\boldsymbol{\rho}$, it lies in $\mathbb{Z}$, and so, as elements of $\mathbb{Q}/\mathbb{Z}$, we have
$$(N_W^{\mathrm{hyp}} + \sigma_\chi)\frac{a}{n} = -\delta_\chi\frac{a}{n}\,.$$
In other words, the knowledge of the element $(N_W^{\mathrm{hyp}} + \sigma_\chi)\dfrac{a}{n}$ as an element of $\mathbb{Q}/\mathbb{Z}$ is the same as the knowledge of the root of unity $\zeta_n^{-\delta_\chi a}$.

3.4.4. Frobenius eigenvalues. —

Here we develop tools necessary to generalize to reflection cosets Proposition 2.19, where we compute the Frobenius eigenvalues for unipotent characters in a principal $w\varphi$-series.

We resume the notation from §1.4.4:

– $w\varphi$ is a ζ-regular element, Φ is the minimal polynomial of ζ over K, and we have $\zeta = \exp(2\pi i a/d)$,

– we have an element $\boldsymbol{\rho}_{\gamma,a/d}$ of the center of the braid group $\mathbf{B}_W(w\varphi)$ which satisfies $\boldsymbol{\rho}_{\gamma,a/d}^d = \boldsymbol{\pi}^{a\delta}$.

We still denote by $\mathcal{H}_W(w\varphi)$ a spetsial Φ-cyclotomic Hecke algebra associated with $w\varphi$, either of compact or of noncompact type.

Notation 3.29. —

Whenever χ is an (absolutely) irreducible character of $\mathcal{H}_W(w\varphi)$ over $\overline{K}(v)$, we define a monomial $\mathrm{Fr}_{w\varphi}^{(\boldsymbol{\rho}_{\gamma,a/d})}(\chi)$ in v by the following formulae:

$$\mathrm{Fr}_{w\varphi}^{(\boldsymbol{\rho}_{\gamma,a/d})}(\chi) := \begin{cases} \zeta^{l(\boldsymbol{\rho}_{\gamma,a/d})}\omega_\chi(\boldsymbol{\rho}_{\gamma,a/d}) & \text{for } \mathcal{H}_W(w\varphi) \text{ of compact type,} \\ \omega_\chi(\boldsymbol{\rho}_{\gamma,a/d}) & \text{for } \mathcal{H}_W(w\varphi) \text{ of noncompact type.} \end{cases}$$

When there is no ambiguity about the ambient algebra, we shall note $\mathrm{Fr}^{(\boldsymbol{\rho}_{\gamma,a/d})}(\chi)$ instead of $\mathrm{Fr}_{w\varphi}^{(\boldsymbol{\rho}_{\gamma,a/d})}(\chi)$.

Since $\boldsymbol{\rho}_{\gamma,a/d}^d = \boldsymbol{\pi}^{\delta a}$, the following lemma is an immediate corollary of Lemma 3.19. Recall that we denote by h the integer such that $\zeta^{-1}x = v^h$.

Lemma 3.30. —

Whenever λ is an h-th root of unity we have

$$\mathrm{Fr}^{(\boldsymbol{\rho}_{\gamma,a/d})}(\chi) = \begin{cases} \zeta^{e_W\frac{\delta a}{d}}\omega_{\chi_{v=\lambda}}(\rho)(\lambda^{-1}v)^{h(N_W^{\mathrm{hyp}}+\sigma_\chi)\frac{\delta a}{d}} & \textit{(compact type),} \\ \omega_{\chi_{v=\lambda}}(\rho)(\lambda^{-1}v)^{h(N_W^{\mathrm{ref}}+\sigma_\chi)\frac{\delta a}{d}} & \textit{(noncompact type),} \end{cases}$$

and in particular

$$\mathrm{Fr}^{(\boldsymbol{\rho}_{\gamma,a/d})}(\chi) = \begin{cases} \zeta^{e_W\frac{\delta a}{d}}\omega_{\chi_{v=1}}(\rho)v^{h(N_W^{\mathrm{hyp}}+\sigma_\chi)\frac{\delta a}{d}} & \textit{(compact type),} \\ \omega_{\chi_{v=1}}(\rho)v^{h(N_W^{\mathrm{ref}}+\sigma_\chi)\frac{\delta a}{d}} & \textit{(noncompact type).} \end{cases}$$

The above lemma shows that the value of $\mathrm{Fr}^{(\boldsymbol{\rho}_{\gamma,a/d})}(\chi)$ does not depend on the choice of γ.

If we change a/d (in other words, if we replace $\boldsymbol{\rho}_{\gamma,a/d}$ by $\boldsymbol{\rho}_{\gamma,a/d}\boldsymbol{\pi}^{n\delta}$), we get (in the compact case):

$$\mathrm{Fr}^{(\boldsymbol{\rho}_{\gamma,a/d}\boldsymbol{\pi}^{n\delta})}(\chi) = \zeta^{-(N_W^{\mathrm{hyp}}+\sigma_\chi)n\delta}\mathrm{Fr}^{(\boldsymbol{\rho}_{\gamma,a/d})}(\chi) \tag{3.31}$$

If we force $\boldsymbol{\rho}_{\gamma,a/d}$ to be as short as possible (*i.e.*, if we assume $0 \leq a < d$), the monomial $\mathrm{Fr}^{(\boldsymbol{\rho}_{\gamma,a/d})}(\chi)$ depends only on χ. In that case we denote it by $\mathrm{Fr}_{w\varphi}(\chi)$.

Notice that if χ is $\overline{K}(x)$-rational, or equivalently if $\boldsymbol{\pi}\cdot\chi = \chi$ (see 3.26), we have $\omega_\chi(\boldsymbol{\rho}_{\gamma,a/d}) \in \overline{K}(x)$, and $\mathrm{Fr}^{(\boldsymbol{\rho}_{\gamma,a/d})}(\chi)$ is a monomial in x.

Definition 3.32. —

Assume χ is $\overline{K}(x)$-rational. Then the Frobenius eigenvalue of χ is the root of unity defined by

$$\mathrm{fr}(\chi) := \mathrm{Fr}_{w\varphi}(\chi)|_{x=1}\,.$$

Proposition 3.33. —

Let χ be an irreducible character of the algebra $\mathcal{H}_W^{\mathrm{c}}(w\varphi)$ of compact type. Assume that χ is $\overline{K}(x)$-rational.

1. *We have*
$$\mathrm{fr}(\chi) = \zeta^{\frac{\delta a}{d}(N_W^{\mathrm{ref}} - \sigma_\chi)} \omega_{\chi_{v=1}}(\rho) = \zeta^{\frac{\delta a}{d}\delta_\chi} \omega_{\chi_{v=1}}(\rho)\,.$$
2. *For the corresponding character χ^{nc} of the associated algebra of noncompact type, we have*
$$\mathrm{fr}(\chi^{\mathrm{nc}}) = \zeta^{-\frac{\delta a}{d}(N_W^{\mathrm{ref}} + \sigma_{\chi^{\mathrm{nc}}})} \omega_{\chi^{\mathrm{nc}}_{v=1}}(\rho) = \zeta^{-\frac{\delta a}{d}\delta_\chi} \omega_{\chi^{\mathrm{nc}}_{v=1}}(\rho) = \mathrm{fr}(\chi)^*\,.$$

Proof. —

It is direct from Lemma 3.30. □

A definition for general characters. —

Comment 3.34. —

In the case of a character χ which is not $\overline{K}(x)$-rational, we can only attach a set of roots of unity to the orbit of χ under $\mathrm{Gal}(\overline{K}(v)/\overline{K}(x))$.

Consider for example the case of the Weyl group of type E_7, and choose $w\varphi := w_0$, the longest element. Then the algebra $\mathcal{H}_W(w_0)$ has two irrational characters, say χ_1 and χ_2, which correspond to two unipotent cuspidal characters of the associated finite reductive groups (these characters belong to the same Lusztig family as the principal series unipotent characters $\rho_{\chi_{512,11}}$ and $\rho_{\chi_{512,12}}$).

These two unipotent cuspidal characters can be distinguished by their Frobenius eigenvalues, which are i and $-i$.

Here we shall only attach to the Galois orbit $\{\chi_1, \chi_2\}$ the set of two roots of unity $\{i, -i\}$.

From now on, in order to make the exposition simpler, we assume that $\mathcal{H}_W(w\varphi)$ *is of compact type.*

Let $\chi \in \mathrm{Irr}(\mathcal{H}_W(w\varphi))$. Let k denote the length of the orbit of χ under $\boldsymbol{\pi}$. It follows from 3.25 that
$$\omega_{\boldsymbol{\pi}.\chi}(\boldsymbol{\rho}) = \omega_\chi(\boldsymbol{\rho})\zeta^{(N_W^{\mathrm{hyp}} + \sigma_\chi)\delta}\,,$$
hence d divides $k(N_W^{\mathrm{hyp}} + \sigma_\chi)\delta$. Thus $(N_W^{\mathrm{hyp}} + \sigma_\chi)\delta a/d$ defines an element of $\mathbb{Q}/\mathbb{Z}$ of order k' dividing k.

We shall attach to the orbit of χ under $\boldsymbol{\pi}$ an orbit of roots of unity under the action of the group $\boldsymbol{\mu}_{k'}$, as follows.

Choose a k-th root ζ_0 of ζ, and set $x_0 := \zeta_0 v^{h/k}$ so that $v^{h/k} = \zeta_0^{-1} x_0$. Then we have
$$v^{h(N_W^{\mathrm{hyp}} + \sigma_\chi)\delta a/d} = (\zeta_0^{-1} x_0)^{k(N_W^{\mathrm{hyp}} + \sigma_\chi)\delta a/d}$$
where $k(N_W^{\mathrm{hyp}} + \sigma_\chi)\delta a/d \in \mathbb{Z}$.

By Lemma 3.30, we see that $\mathrm{Fr}_{w\varphi}(\chi)$ is a monomial in x_0.

We recall (see Remark 3.28 above) that, as an element of $\mathbb{Q}/\mathbb{Z}$, we have $(N_W^{\text{hyp}}+\sigma_\chi)\frac{\delta a}{d} = -\delta_\chi\frac{\delta a}{d}$, and that it is defined by the root of unity $\zeta^{-\delta_\chi\delta}$.

Definition 3.35. —
If χ has an orbit of length k under $\boldsymbol{\pi}$, we attach to that orbit the set of roots of unity defined as

$$\mathrm{fr}(\chi) = \{\mathrm{Fr}_{w\varphi}(\chi)|_{x_0=1}\,\lambda^{k(N_W^{\text{hyp}}+\sigma_\chi)\frac{\delta a}{d}}\ ;\ (\lambda\in\boldsymbol{\mu}_k)\} = \mathrm{Fr}_{w\varphi}(\chi)|_{x_0=1}\,\boldsymbol{\mu}_{k'}\,,$$

with k' the order of the element of $\mathbb{Q}/\mathbb{Z}$ defined by

$$(N_W^{\text{hyp}}+\sigma_\chi)\frac{\delta a}{d} = -\delta_\chi\frac{\delta a}{d}\,,$$

(in other words, k' is the order of $\zeta^{-\delta_\chi\delta}$).

Remark 3.36. —
The set $\mathrm{fr}(\chi)$ is just the set of all k'-th roots of $(\mathrm{Fr}_{w\varphi}(\chi)|_{x_0=1})^{k'}$.

A computation similar to the computation made above for the rational case gives

Proposition 3.37. —
Let χ be an irreducible character of the algebra $\mathcal{H}_W^{\mathrm{c}}(w\varphi)$ of compact type. Assume that χ has an orbit of length k under $\boldsymbol{\pi}$, and that $(N_W^{\text{hyp}}+\sigma_\chi)\frac{\delta a}{d}$ has order k' in $\mathbb{Q}/\mathbb{Z}$.

The set of Frobenius eigenvalues attached to that orbit is the set of all k'-th roots of

$$\zeta^{k'\frac{\delta a}{d}\delta_\chi}(\omega_{\chi_{v=1}}(\rho))^{k'}\,.$$

Definition 3.38. —
For χ an irreducible character of the compact type algebra $\mathcal{H}_W^{\mathrm{c}}(w\varphi)$, the set of Frobenius eigenvalues attached to the orbit of the corresponding character χ^{nc} of the associated noncompact type algebra is

$$\mathrm{fr}(\chi^{\mathrm{nc}}) = \mathrm{fr}(\chi)^*\,,$$

the set of complex conjugates of elements of $\mathrm{fr}(\chi)$.

3.4.5. Ennola action and Frobenius eigenvalues. —

Let us now compute the effect of the Ennola action on Frobenius eigenvalues. Recall that we assume $\mathcal{H}_W(w\varphi)$ to be of compact type.

We start by studying the special case of the action of the permutation defined by $\boldsymbol{\pi}$ on $\mathrm{Irr}\,(\mathcal{H}_W(w\varphi))$.

Lemma 3.39. —
For $\boldsymbol{\rho} = \boldsymbol{\rho}_{\gamma,a/d}$ as above, whenever $\chi\in\mathrm{Irr}\,(\mathcal{H}_W(w\varphi))$, we have

$$\mathrm{Fr}^{\boldsymbol{\rho}\boldsymbol{\pi}^\delta}(\chi) = \mathrm{Fr}^{\boldsymbol{\rho}}(\boldsymbol{\pi}.\chi)x^{\delta(N_W^{\text{hyp}}+\sigma_\chi)}\,.$$

Proof of 3.39. —

On the one hand, by Definitions 3.24 and 3.29 we have

$$\begin{aligned}\mathrm{Fr}^{\boldsymbol{\rho}\boldsymbol{\pi}^\delta}(\chi) &= \zeta^{l(\boldsymbol{\rho}\boldsymbol{\pi}^\delta)}\omega_\chi(\boldsymbol{\rho}\boldsymbol{\pi}^\delta) = \zeta^{l(\boldsymbol{\rho})}\zeta^{l(\boldsymbol{\pi}^\delta)}\omega_\chi(\boldsymbol{\rho})\omega_\chi(\boldsymbol{\pi}^\delta)\\ &= \left(\zeta^{l(\boldsymbol{\rho})}\omega_\chi(\boldsymbol{\rho})\right)\left(\zeta^{l(\boldsymbol{\pi}^\delta)}\omega_\chi(\boldsymbol{\pi}^\delta)\right).\end{aligned}$$

Proposition 3.18 gives that

$$\omega_\chi(\boldsymbol{\pi}^\delta) = v^{h(N_W^{\mathrm{hyp}}+\sigma_\chi)\delta}.$$

Morever, $\zeta^{l(\boldsymbol{\pi}^\delta)} = (\zeta^\delta)^{l(\boldsymbol{\pi})} = 1$ since $(w\varphi)^\delta$ is a ζ^δ-regular element of W (see the second remark following 1.3). It follows that

$$\mathrm{Fr}^{\boldsymbol{\rho}\boldsymbol{\pi}^\delta}(\chi) = \left(\zeta^{l(\boldsymbol{\rho})}\omega_\chi(\boldsymbol{\rho})\right) v^{h(N_W^{\mathrm{hyp}}+\sigma_\chi)\delta}.$$

On the other hand, by definition

$$\mathrm{Fr}^{\boldsymbol{\rho}}(\boldsymbol{\pi}.\chi) = \zeta^{l(\boldsymbol{\rho})}\omega_{\boldsymbol{\pi}.\chi}(\boldsymbol{\rho}).$$

By Lemma 3.25, and since $\xi(\boldsymbol{\pi}) = \xi(\mathbf{z}_0^c) = \exp(2\pi i/h)$, this yields

$$\omega_{\boldsymbol{\pi}.\chi}(\boldsymbol{\rho}) = \omega_\chi(\boldsymbol{\rho})\exp(-2\pi i(N_W^{\mathrm{hyp}}+\sigma_\chi)\delta a/d) = \omega_\chi(\boldsymbol{\rho})\zeta^{-(N_W^{\mathrm{hyp}}+\sigma_\chi)\delta},$$

hence

$$\mathrm{Fr}^{\boldsymbol{\rho}}(\boldsymbol{\pi}.\chi) = \left(\zeta^{l(\boldsymbol{\rho})}\omega_\chi(\boldsymbol{\rho})\right)\zeta^{(N_W^{\mathrm{hyp}}+\sigma_\chi)\delta}.$$

The lemma follows. □

Now we know by (3.31) that

$$\mathrm{Fr}^{\boldsymbol{\rho}\boldsymbol{\pi}^\delta}(\chi) = \zeta^{-(N_W^{\mathrm{hyp}}+\sigma_\chi)\delta}\mathrm{Fr}^{\boldsymbol{\rho}}(\chi),$$

which implies by Lemma 3.39

$$\mathrm{Fr}^{\boldsymbol{\rho}}(\boldsymbol{\pi}.\chi) = \zeta^{(N_W^{\mathrm{hyp}}+\sigma_\chi)\delta}\mathrm{Fr}^{\boldsymbol{\rho}}(\chi)x^{-\delta(N_W^{\mathrm{hyp}}+\sigma_\chi)}.$$

The following proposition is now immediate. Note that its statement contains a slight abuse of notation, since for χ a $\overline{K}(x)$-rational character, $\mathrm{fr}(\chi)$ is not a set — it has then to be considered as a singleton.

Proposition 3.40. —

$$\mathrm{fr}(\boldsymbol{\pi}\cdot\chi) = \{\zeta^{(N_W^{\mathrm{hyp}}+\sigma_\chi)\delta}\lambda\,;\,(\lambda\in\mathrm{fr}(\chi))\}.$$

Let us now consider the general case of Ennola action by an element $\mathbf{z}\in Z\mathbf{B}_W$. By Lemma 3.25, we have

$$\begin{aligned}\mathrm{Fr}_{zw\varphi}^{\mathbf{z}^\delta\boldsymbol{\rho}}(\mathbf{z}\cdot\chi) &= (z\zeta)^{l(\mathbf{z}^\delta\boldsymbol{\rho})}\omega_{\mathbf{z}\cdot\chi}(\mathbf{z}^\delta\boldsymbol{\rho})\\ &= (z\zeta)^{l(\mathbf{z}^\delta\boldsymbol{\rho})}\omega_\chi(\mathbf{z}^\delta\boldsymbol{\rho})\xi(\mathbf{z})^{-h(N_W^{\mathrm{hyp}}+\sigma_\chi)l(\mathbf{z}^\delta\boldsymbol{\rho})/l(\boldsymbol{\pi})}\\ &= (z\zeta)^{l(\mathbf{z}^\delta\boldsymbol{\rho})}(z\zeta)^{-(N_W^{\mathrm{hyp}}+\sigma_\chi)\delta}\omega_\chi(\mathbf{z}^\delta\boldsymbol{\rho}),\end{aligned}$$

hence

$$\frac{\mathrm{Fr}^{\mathbf{z}^\delta\boldsymbol{\rho}}_{zw\varphi}(\mathbf{z}\cdot\chi)}{\mathrm{Fr}^{\boldsymbol{\rho}}_{w\varphi}(\chi)} = (z\zeta)^{l(\mathbf{z}^\delta)} z^{l(\boldsymbol{\rho})} (z\zeta)^{-(N_W^{\mathrm{hyp}}+\sigma_\chi)\delta}\omega_\chi(\mathbf{z}^\delta)$$
$$= (z\zeta)^{l(\mathbf{z}^\delta)} z^{l(\boldsymbol{\rho})} (z\zeta)^{-(N_W^{\mathrm{hyp}}+\sigma_\chi)\delta}\omega_{\chi_{v=1}}(z^\delta) v^{h(N_W^{\mathrm{hyp}}+\sigma_\chi)l(\mathbf{z}^\delta)/l(\boldsymbol{\pi})}\,.$$

Since $\mathrm{Fr}^{\mathbf{z}^\delta\boldsymbol{\rho}}_{zw\varphi}(\mathbf{z}\cdot\chi) = \mathrm{Fr}_{zw\varphi}(\mathbf{z}\cdot\chi)$ up to an integral power of $(z\zeta)^{(N_W^{\mathrm{hyp}}+\sigma_\chi)\delta}$ (see (3.31)), what precedes proves the following proposition.

Proposition 3.41. —
Up to an integral power of $(z\zeta)^{(N_W^{\mathrm{hyp}}+\sigma_\chi)\delta}$, we have

$$\frac{\mathrm{Fr}^{\mathbf{z}^\delta\boldsymbol{\rho}}_{zw\varphi}(\mathbf{z}\cdot\chi)}{\mathrm{Fr}^{\boldsymbol{\rho}}_{w\varphi}(\chi)} = (z\zeta)^{l(\mathbf{z}^\delta)} z^{l(\boldsymbol{\rho})} \omega_{\chi_{v=1}}(z^\delta) v^{h(N_W^{\mathrm{hyp}}+\sigma_\chi)l(\mathbf{z}^\delta)/l(\boldsymbol{\pi})}\,.$$

3.5. Spetsial data at a regular element, spetsial groups

3.5.1. Spetsial data at a regular element. —

Definition 3.42. — *Let $\mathbb{G} = (V, W, \overline{\varphi})$ be a reflection coset and let $w\varphi$ be a Φ-regular element of $W\varphi$.*

We say that $\mathbb{G} = (V, W, \overline{\varphi})$ is spetsial at Φ *(or* spetsial at $w\varphi$*) if there exists a spetsial Φ-cyclotomic Hecke algebra of W at $w\varphi$.*

We are not able at the moment to classify the irreducible reflection cosets which are spetsial at an arbitrary cyclotomic polynomial Φ. But:

- One can classify the irreducible split reflection cosets which are spetsial at $x-1$: they are precisely the spetsial reflection groups (see Proposition 3.44 below).
- If $\mathbb{G} = (V, W)$ is a split spetsial reflection coset on K, and if w is a Φ-regular element of W, then $(V(w), W(w))$ is spetsial at Φ : for W primitive, this will be a consequence of the construction of the spets data associated with $\mathbb{G}$ (see §5 below).

3.5.2. The 1-spetsial algebra $\mathcal{H}_W$. —

Let us now consider the special case where $w\varphi = \mathrm{Id}_V$.

Definition 3.43. —
Let W be a reflection group. We denote by $\mathcal{H}_W$ (resp. $\mathcal{H}_W^{\mathrm{nc}}$) the algebra defined by the collection of polynomials $P_H(t,x)_{H\in\mathcal{A}(W)}$ where

$$P_H(t,x) = (t-x)(t^{e_H-1}+\cdots+t+1)$$
$$(\textit{resp.}\quad P_H(t,x) = (t-1)(t^{e_H-1}+\cdots+tx^{e_H-2}+x^{e_H-1})\ )\,.$$

Proposition 3.44. —
Let $\mathbb{G} = (V, W)$ be a split reflection coset.

1. *There is at most one* 1*-cyclotomic spetsial Hecke algebra* $\mathcal{H}_W(\mathrm{Id}_V)$ *of compact type (resp. of noncompact type), namely the algebra* $\mathcal{H}_W$ *(resp* $\mathcal{H}_W^{\mathrm{nc}}$*).*
2. (V, W) *is spetsial at* 1 *if and only if it is spetsial according to* [**Mal98**, §3.9].

Proof. —
We only consider the compact type case.

(1) Since $w\varphi = 1$, we have $W(\varphi) = W$, $\mathcal{A}(w\varphi) = \mathcal{A}$, and for each $H \in \mathcal{A}$ we have $e_{W_H}/e_H = 1$.

Hence by Definition 3.7, (CA1), we have $P_H(t, x) \in K[t, x]$, and by Definition 3.7, (CS1), we see that $P_H(t, x)$ is divisible by $(t - x)$ and

$$P_H(t, x) = (t - x)Q_H(t)$$

for some $Q_H(t) \in K[t]$.

By Definition 3.7, (CA2), we see that $Q_H(t) = (t^{e_H - 1} + \cdots + t + 1)$.

(2) If the algebra $\mathcal{H}_W$ is a 1-cyclotomic spetsial Hecke algebra, it follows from Definition 3.3.1, Global conditions, (SC1), that its Schur elements belong to $K(x)$. This shows that (V, W) is spetsial according to [**Mal98**, §3.9] by condition (ii) of [**Mal98**, Prop. 3.10].

Reciprocally, assume that (V, W) is spetsial according to [**Mal98**, §3.9]. Then we know that all parabolic subgroups of W are still spetsial according to [**Mal98**, §3.9] (see *e.g.* [**Mal00**, Prop. 7.2]). The only properties which are not straightforward to check among the list of assertions in Definition 3.3.1 are the properties concerning the splitting fields of algebras. Since the spetsial groups according to [**Mal98**, §3.9] are all well–generated, these properties hold by [**Mal99**, Cor. 4.2]. □

3.5.3. Spetsial reflection groups. —

Let (V, W) be a reflection group on $\mathbb{C}$. Assume that the corresponding reflection representation is defined over a number field K (so that $\mathbb{Q}_W \subseteq K$).

The algebra $\mathcal{H}_W$ has been defined above (3.43).

Let v be such that $v^{|\mu(K)|} = x$. Whenever χ is an absolutely irreducible character of the spetsial algebra $\mathcal{H}_W$,

- we denote by S_χ the corresponding Schur element (so $S_\chi \in K[v, v^{-1}]$),
- We denote by 1 the unique character of $\mathcal{H}_W$ whose Schur element is the Poincaré polynomial of W (see Definition 3.7, (CS2')). The degree of a character χ of $\mathcal{H}_W$ is (see Definition 3.7, (SC3))

$$\mathrm{Deg}(\chi) = \frac{\mathrm{Feg}(R_1^{\mathbb{G}})}{S_\chi}$$

and in particular $\mathrm{Deg}(1) = 1$.

The following theorem (see [**Mal00**, Prop. 8.1]) has been proved by a case-by-case analysis.

Theorem 3.45. —
Assume Theorem–Conjecture 1.60 holds.

1. *The following assertions are equivalent.*
   (i) *For all* $\chi \in \mathrm{Irr}(\mathcal{H}_W)$*, we have* $\mathrm{Deg}(\chi)(v) \in K(x)$.
   (ii) W *is a product of some of the following reflection groups:*
   - $G(d,1,r)$ $(d,r \geq 1)$, $G(e,e,r)$ $(e,r \geq 2)$,
   - *one of the well-generated exceptional groups* G_i *with* $4 \leq i \leq 37$ *generated by true reflections,*
   - G_4, G_6, G_8, G_{14}, G_{25}, G_{26}, G_{32}.
2. *If the preceding properties hold, then the specialization* $S_1(x)$ *of the generic Poincaré polynomial equals the "ordinary" Poincaré polynomial of* W*:*

$$S_1(x) = \prod_{j=1}^{r} (1 + x + \cdots + x^{d_j - 1}) \,,$$

where $r = \mathrm{dim} V$ *and* $d_1, \ldots, d_r$ *are the degrees of* W *(see 1.1.5).*

Remark 3.46. —

- A spetsial group of rank r is well-generated, but not all well-generated reflection groups of rank r are spetsial.
- From the classification of spetsial groups, it follows that all parabolic subgroups of a spetsial group are spetsial.

3.5.4. Rouquier blocks of the spetsial algebra: special characters. —

Let $\mathbb{G} = (V, W)$ be a spetsial split reflection coset on K.

For $\theta \in \mathrm{Irr}(W)$, we recall (see 1.25) that the fake degree $\mathrm{Feg}_{\mathbb{G}}(R_\theta)$ of the class function R_θ on W (which in the split case coincides with θ) is the graded multiplicity of θ in the graded regular representation KW^{gr}.

Let $\mathcal{H}_W$ be the 1-cyclotomic spetsial Hecke algebra (see Proposition 3.44). Let us choose an indeterminate v such that $v^{|ZW|} = x$. We denote by

$$\mathrm{Irr}(W) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}_W) \ , \ \theta \mapsto \chi_\theta \,,$$

the bijection defined by the specialization $v \mapsto 1$.

Notation 3.47. —
For $\theta \in \mathrm{Irr}(W)$ we define the following:

1. a_θ and A_θ :
   - we denote by a_θ the valuation of $\mathrm{Deg}(\chi_\theta)$ (*i.e.*, the largest integer such that $x^{-a_\theta}\mathrm{Deg}(\chi_\theta)$ is a polynomial),
   - and by A_θ the degree of $\mathrm{Deg}(\chi_\theta)$,

2. b_θ and B_θ :
   - we denote by b_θ the valuation of $\mathrm{Feg}_{\mathbb{G}}(R_\theta)$,
   - and by B_θ the degree of $\mathrm{Feg}_{\mathbb{G}}(R_\theta)$.

Definition 3.48. —
We say that $\theta \in \mathrm{Irr}(W)$ is special *if $a_\theta = b_\theta$.*

Let us recall (see Theorem 1.81) that a_θ and A_θ are constant if χ_θ runs over the set of characters in a given Rouquier block of $\mathrm{Irr}(\mathcal{H}_W)$. Then if $\mathcal{B}$ is a Rouquier block, we denote by $a_{\mathcal{B}}$ and $A_{\mathcal{B}}$ the common value for a_θ and A_θ for $\chi_\theta \in \mathcal{B}$.

The following result is proved in [**MR03**, §5] (under certain assumptions for some of the exceptional spetsial groups), using essential tools from [**Mal00**, §8].

Theorem 3.49. —
Assume that W is spetsial. Let $\mathcal{B}$ be a Rouquier block ("family") of the 1-spetsial algebra $\mathcal{H}_W$.

1. *$\mathcal{B}$ contains a unique special character χ_{θ_0}.*
2. *For all θ such that $\chi_\theta \in \mathcal{B}$, we have*

$$a_{\mathcal{B}} \leq b_\theta \quad \text{and} \quad B_\theta \leq A_{\mathcal{B}}\,.$$

CHAPTER 4

AXIOMS FOR SPETSES

Our goal is to attach to $\mathbb{G} = (V, W\varphi)$ (where W is a spetsial reflection group) an abstract set of *unipotent characters of* $\mathbb{G}$, to each element of which we associate a *degree* and an *eigenvalue of Frobenius*. In the case where $\mathbb{G}$ is rational, these are the set of unipotent characters with their generic degrees and corresponding eigenvalues of Frobenius attached to the associated reductive groups.

These data have to satisfy certain axioms that we proceed to give below.

We hope to give a general construction satisfying these axioms in a subsequent paper. For the time being, we only know how to attach that data to the particular case where $\mathbb{G}$ has a split semi simple part (see below §6); the construction in that case is the object of §6.

4.1. Axioms used in §6

4.1.1. Compact and non compact types: Unipotent characters, degrees and eigenvalues. —

Given $\mathbb{G}$ as above, we shall construct two finite sets:

– the set $\mathrm{Uch}^{\mathrm{c}}(\mathbb{G})$ of unipotent characters of compact type,

– the set $\mathrm{Uch}(\mathbb{G}_{\mathrm{nc}})$ of unipotent characters of noncompact type,

each of them (denoted $\mathrm{Uch}(\mathbb{G})$ below), endowed with two maps

– the map called *degree*

$$\mathrm{Deg} : \mathrm{Uch}(\mathbb{G}) \to K[x]\ ,\ \rho \mapsto \mathrm{Deg}(\rho)\,,$$

– the map called *Frobenius eigenvalue* and denoted Fr, which associates to each element $\rho \in \mathrm{Uch}(\mathbb{G})$ a monomial of the shape $\mathrm{Fr}(\rho) = \lambda_\rho x^{\nu_\rho}$ where
  - λ_ρ is a root of unity,

- ν_ρ is an element of $\mathbb{Q}/\mathbb{Z}$,

with a bijection

$$\mathrm{Uch}^{\mathrm{c}}(\mathbb{G}) \xrightarrow{\sim} \mathrm{Uch}(\mathbb{G}_{\mathrm{nc}}) \ , \ \rho \mapsto \rho^{\mathrm{nc}} \, ,$$

such that

1. $\mathrm{Deg}(\rho^{\mathrm{nc}}) = x^{N_W^{\mathrm{ref}}} \mathrm{Deg}(\rho)^\vee$,
2. $\mathrm{Fr}(\rho)\mathrm{Fr}(\rho^{\mathrm{nc}}) = 1$,

and subject to many further axioms to be given below.

In what follows, we shall construct the "compact type case" $\mathrm{Uch}^{\mathrm{c}}(\mathbb{G})$ (which will be denoted simply $\mathrm{Uch}(\mathbb{G})$). The noncompact type case can be obtained using the above properties of the bijection $\rho \mapsto \rho^{\mathrm{nc}}$.

4.1.2. Basic axioms. —

Axioms 4.1. —

1. *If* $\mathbb{G} = \mathbb{G}_1 \times \mathbb{G}_2$ *(with split semi-simple parts) then we have* $\mathrm{Uch}(\mathbb{G}) \simeq \mathrm{Uch}(\mathbb{G}_1) \times \mathrm{Uch}(\mathbb{G}_2)$. *If we write* $\rho = \rho_1 \otimes \rho_2$ *this product decomposition, then* $\mathrm{Fr}(\rho) = \mathrm{Fr}(\rho_1)\mathrm{Fr}(\rho_2)$ *and* $\mathrm{Deg}(\rho) = \mathrm{Deg}(\rho_1)\mathrm{Deg}(\rho_2)$.
2. *A torus has a unique unipotent character* Id, *with* $\mathrm{Deg}(\mathrm{Id}) = \mathrm{Fr}(\mathrm{Id}) = 1$.

Axiom 4.2. —
For all $\rho \in \mathrm{Uch}^{\mathrm{c}}(\mathbb{G})$, $\mathrm{Deg}(\rho)$ *divides* $|\mathbb{G}|_{\mathrm{c}}$.
For all $\rho \in \mathrm{Uch}(\mathbb{G}_{\mathrm{nc}})$, $\mathrm{Deg}(\rho)$ *divides* $|\mathbb{G}|_{\mathrm{nc}}$

Axiom 4.3. —
There is an action of $N_{\mathrm{GL}(V)}(W\varphi)/W$ *on* $\mathrm{Uch}(\mathbb{G})$ *in a way which preserves* Deg *and* Fr.

The action mentioned above will be determined more precisely below by some further axioms (see 4.16(2)(a)).

Remark 4.4. —
We recall that parabolic subgroups of spetsial groups are spetsial (see 3.46 above). For a Levi $\mathbb{L}$ of $\mathbb{G}$, we set $W_{\mathbb{G}}(\mathbb{L}) := N_W(\mathbb{L})/W_{\mathbb{L}}$. We have by 4.3 a well-defined action of $W_{\mathbb{G}}(\mathbb{L})$ on $\mathrm{Uch}(\mathbb{L})$, which allows us to define for $\lambda \in \mathrm{Uch}(\mathbb{L})$ its stabilizer $W_{\mathbb{G}}(\mathbb{L}, \lambda)$.

4.1.3. Axioms for the principal ζ-series. —

Definition 4.5. —
Let $\zeta \in \boldsymbol{\mu}$ and let Φ its minimal polynomial on K (a K-cyclotomic polynomial).

1. *The* ζ-principal series *is the set of unipotent characters of* $\mathbb{G}$ *defined by*

$$\mathrm{Uch}(\mathbb{G},\zeta) := \{\rho \in \mathrm{Uch}(\mathbb{G}) \mid \mathrm{Deg}(\rho)(\zeta) \neq 0\}\,.$$

2. *An element* $\rho \in \mathrm{Uch}(\mathbb{G})$ *is said to be* ζ-cuspidal *or* Φ-cuspidal *if*

$$\mathrm{Deg}(\rho)_\Phi = \frac{|\mathbb{G}|_\Phi}{|Z\mathbb{G}|_\Phi}\,.$$

Let us recall that, given a spetsial ζ-cyclotomic Hecke algebra (either of compact or of noncompact type) $\mathcal{H}_W(w\varphi)$ associated with a regular element $w\varphi$, each irreducible character χ of $\mathcal{H}_W(w\varphi)$ comes equipped with a degree $\mathrm{Deg}(\chi)$ and a Frobenius eigenvalue $\mathrm{Fr}(\chi)$ (see §4 above).

Axiom 4.6. —
Let $w\varphi \in W\varphi$ be ζ-regular. There is a spetsial ζ-cyclotomic Hecke algebra of compact type $\mathcal{H}_W(w\varphi)$ associated with $w\varphi$, a bijection

$$\mathrm{Irr}(\mathcal{H}_W(w\varphi)) \xrightarrow{\sim} \mathrm{Uch}^{\mathrm{c}}(\mathbb{G},\zeta)\ ,\ \chi \mapsto \rho_\chi\,,$$

and a collection of signs $(\varepsilon_\chi)_{\chi\in\mathrm{Irr}(\mathcal{H}_W(w\varphi))}$ such that

1. *that bijection is invariant under the action of $N_{\mathrm{GL}(V)}(W\varphi)/W$,*
2. $\mathrm{Deg}(\rho_\chi) = \varepsilon_\chi \mathrm{Deg}(\chi)\,,$
3. $\mathrm{Fr}(\rho_\chi) \equiv \mathrm{Fr}(\chi) \mod x^{\mathbb{Z}}\,.$

Remark 4.7. —
By 3.7(sc3), we see that condition (2) above is equivalent to

$$\mathrm{Deg}(\rho_\chi) = \epsilon_\chi \frac{\mathrm{Feg}(R_{w\varphi)}}{S_\chi}\,.$$

4.1.4. On Frobenius eigenvalues. —

For what follows, we use freely §4, and in particular §3.4.4.

We denote by v an indeterminate such that $\mathcal{H}_W(w\varphi)$ splits over $\mathbb{C}(v)$ and such that $v^h = \zeta^{-1}x$ for some integer h. Then the specialization $v \mapsto 1$ induces a bijection

$$\mathrm{Irr}(W(w\varphi)) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}_W(w\varphi))\ ,\ \theta \mapsto \chi_\theta\,.$$

Assume $\zeta = \exp(2\pi i a/d)$ and let $\boldsymbol{\rho} \in Z\mathbf{B}_W(w\varphi)$ be such that $\boldsymbol{\rho}^d = \boldsymbol{\pi}^{a\delta}$. Then we have the following equality modulo integral powers of x :

$$\begin{aligned}\operatorname{Fr}(\rho_{\chi_\theta}) &= \operatorname{Fr}^{\boldsymbol{\rho}}(\chi_\theta) = \zeta^{l(\boldsymbol{\rho})}\omega_{\chi_\theta}(\boldsymbol{\rho}) \\ &= \zeta^{l(\boldsymbol{\rho})}\omega_\theta(\rho)(\zeta^{-1}x)^{l(\boldsymbol{\rho})-(a_{\rho_{\chi_\theta}}+A_{\rho_{\chi_\theta}})\frac{l(\boldsymbol{\rho})}{l(\boldsymbol{\pi})}} \\ &= \omega_\theta(\rho)(\zeta^{-1}x)^{-(a_{\rho_{\chi_\theta}}+A_{\rho_{\chi_\theta}})\frac{l(\boldsymbol{\rho})}{l(\boldsymbol{\pi})}}.\end{aligned} \tag{4.8}$$

The last formula should be interpreted as follows: for the power of x, one should take $-(a_{\rho_{\chi_\theta}} + A_{\rho_{\chi_\theta}})\frac{l(\boldsymbol{\rho})}{l(\boldsymbol{\pi})}$ modulo 1, and if χ_θ has an orbit under $\boldsymbol{\pi}$ of length k, then $\zeta^{(a_{\rho_{\chi_\theta}}+A_{\rho_{\chi_\theta}})\frac{l(\boldsymbol{\rho})}{l(\boldsymbol{\pi})}}$ should be interpreted as attributing to the elements of the orbit of χ_θ the k-th roots of $\zeta^{k(a_{\rho_{\chi_\theta}}+A_{\rho_{\chi_\theta}})\frac{l(\boldsymbol{\rho})}{l(\boldsymbol{\pi})}}$, a well-defined expression since the exponent is integral.

4.1.5. Some consequences of the axioms. —

1. Let us recall (see 4.7 above) that

$$\operatorname{Deg}(\chi_\theta) = \frac{\operatorname{Feg}(R_{w\varphi})}{S_{\chi_\theta}}\,.$$

Since $\mathbb{C}\mathcal{H}_W(w\varphi)$ specializes to $\mathbb{C}W(w\varphi)$ for $v \mapsto 1$, we have

$$S_{\chi_\theta}(\zeta) = \frac{|W(w\varphi)|}{\theta(1)}\,.$$

We also have $\operatorname{Feg}(R_{w\varphi})(\zeta) = |W(w\varphi)|$ (see Proposition 1.53, (2)). Thus we get

$$\operatorname{Deg}(\rho_{\chi_\theta})(\zeta) = \varepsilon_{\chi_\theta}\theta(1)\,. \tag{4.9}$$

2. Let us denote by $\tau_W(w\varphi)$ the canonical trace form of the algebra $\mathcal{H}_W(w\varphi)$.

By definition of spetsial cyclotomic Hecke algebras (see Definition 3.7, (SC3)), we have the following equality between linear forms on $\mathbb{C}(x)\mathcal{H}_W(w\varphi)$:

$$\operatorname{Feg}(R_{w\varphi})\tau_W(w\varphi) = \sum_{\chi\in\operatorname{Irr}(\mathcal{H}_W(w\varphi))} \operatorname{Deg}(\chi)\chi\,,$$

and taking the value at 1 we get

$$\begin{aligned}\operatorname{Feg}(R_{w\varphi}) &= \sum_{\chi\in\operatorname{Irr}(\mathcal{H}_W(w\varphi))} \chi(1)\operatorname{Deg}(\chi) \\ &= \sum_{\chi\in\operatorname{Irr}(\mathcal{H}_W(w\varphi))} \varepsilon_\chi\chi(1)\operatorname{Deg}(\rho_\chi)\,.\end{aligned}$$

Using the bijection

$$\operatorname{Irr}(W(w\varphi)) \xrightarrow{\sim} \operatorname{Irr}(\mathcal{H}_W(w\varphi)) \quad , \quad \theta \mapsto \chi_\theta\,,$$

and setting $\varepsilon_\theta := \varepsilon_{\chi_\theta}$ for $\theta \in \operatorname{Irr}(W(w\varphi))$, we get

$$\operatorname{Feg}(R_{w\varphi}) = \sum_{\theta\in\operatorname{Irr}(W(w\varphi))} \varepsilon_\theta \theta(1)\operatorname{Deg}(\rho_{\chi_\theta})\,. \tag{4.10}$$

Finally, we note yet another numerical consequence of Axiom 4.6.

Notation 4.11. —
For $\rho \in \operatorname{Uch}(\mathbb{G})$, let us denote (see above 1.78) by a_ρ and A_ρ respectively the valuation and the degree of $\operatorname{Deg}(\rho)$ as a polynomial in x. We set $\delta_\rho := a_\rho + A_\rho$.

Corollary 4.12. —
Let $\zeta_1, \zeta_2 \in \boldsymbol{\mu}$. If $\rho \in \operatorname{Uch}(\mathbb{G}, \zeta_1) \cap \operatorname{Uch}(\mathbb{G}, \zeta_2)$, then $\zeta_1^{\delta_\rho} = \zeta_2^{\delta_\rho}$.

Proof. —
By Lemma 3.21, (1), we see that whenever $\rho \in \operatorname{Uch}(\mathbb{G}, \zeta)$, we have $\operatorname{Deg}(\rho)^\vee = (\zeta^{-1}x)^{-\delta_\rho}\operatorname{Deg}(\rho)$, which implies the corollary. □

4.1.6. Ennola transform. —

Axiom 4.13. —
For $\mathbf{z} \in Z(\mathbf{B}_W)$ with image $z \in Z(W)$, the algebra $\mathcal{H}_W(zw\varphi)$ is the image of $\mathcal{H}_W(w\varphi)$ by the Ennola transform explained in 2.23. If $\xi\zeta$ and ζ are the corresponding regular eigenvalues, this defines a correspondence $E_{\mathbf{z}}$ (a well-defined bijection except for irrational characters) between $\operatorname{Uch}(\mathbb{G}, \zeta)$ and $\operatorname{Uch}(\mathbb{G}, \mathrm{Id})$, such that

$$\operatorname{Deg}(E_{\mathbf{z}}(\rho))(x) = \pm\operatorname{Deg}(\rho)(z^{-1}x)$$

and $\operatorname{Fr}(E_{\mathbf{z}}(\rho))/\operatorname{Fr}(\rho)$ is given by 3.41 taken modulo $x^{\mathbb{Z}}$.

4.1.7. Harish-Chandra series. —

Here we define a particular case of what will be called "Φ-Harish-Chandra series" in the next section.

Definition 4.14. —
We call cuspidal pair for $\mathbb{G}$ a pair $(\mathbb{L}, \lambda)$ where
- *$\mathbb{L}$ is Levi subcoset of $\mathbb{G}$ of type $\mathbb{L} = (V, W_{\mathbb{L}}\varphi)$ ($W_{\mathbb{L}}$ is a parabolic subgroup of W), and*
- *$\lambda \in \operatorname{Uch}(\mathbb{L})$ is 1-cuspidal.*

Remark 4.15. —
- By remark 3.46 a parabolic subgroup of a spetsial group is spetsial thus it makes sense to consider $\operatorname{Uch}(\mathbb{L})$.
- A Levi subcoset $\mathbb{L}$ has type $(V, W_{\mathbb{L}}\varphi)$ if and only if it is the centralizer of the 1-Sylow subcoset of its center $Z\mathbb{L}$.

– It can be checked case by case that whenever $(\mathbb{L},\lambda)$ is a cuspidal pair for $\mathbb{G}$, then $W_{\mathbb{G}}(\mathbb{L},\lambda)$ is a reflection group on the orthogonal of the intersection of the hyperplanes of $W_{\mathbb{L}}$, which gives a meaning to (2) below.

***Axioms 4.16* (Harish–Chandra theory).** —

1. *There is a partition*

$$\mathrm{Uch}(\mathbb{G}) = \bigsqcup_{(\mathbb{L},\lambda)} \mathrm{Uch}_{\mathbb{G}}(\mathbb{L},\lambda)$$

where $(\mathbb{L},\lambda)$ runs over a complete set of representatives of the orbits of W on cuspidal pairs for $\mathbb{G}$.

2. *For each cuspidal pair $(\mathbb{L},\lambda)$, there is a 1-cyclotomic Hecke algebra $\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda)$ associated to $W_{\mathbb{G}}(\mathbb{L},\lambda)$, an associated bijection*

$$\mathrm{Irr}(\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda)) \xrightarrow{\sim} \mathrm{Uch}_{\mathbb{G}}(\mathbb{L},\lambda)\ ,\ \chi\mapsto\rho_\chi\,,$$

with the following properties.

   (a) *Those bijections are invariant under $N_{\mathrm{GL}(V)}(W\varphi)/W$.*

   (b) *If we denote by S_χ the Schur element of the character χ of $\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda)$, we have*

$$\mathrm{Deg}(\rho_\chi) = \frac{\mathrm{Deg}(\lambda)(|\mathbb{G}|/|\mathbb{L}|)_{x'}}{S_\chi}\,.$$

   (c) *If $\mathbb{G}$ is assumed to have a split semisimple part (see below §6), for $\mathbb{T}_\varphi := (V,\varphi)$ the corresponding maximal torus, the algebra $\mathcal{H}_{\mathbb{G}}(\mathbb{T}_\varphi,\mathrm{Id})$ is the 1-cyclotomic spetsial Hecke algebra $\mathcal{H}_W$ and $\mathrm{Uch}_{\mathbb{G}}(\mathbb{T}_\varphi,\mathrm{Id}) = \mathrm{Uch}(\mathbb{G},1)$. The bijection $\chi\mapsto\rho_\chi$ is the same as that in 4.6, in particular the signs ε_χ in 4.6 are 1 when $\zeta=1$.*

   (d) *For all $\chi\in\mathrm{Irr}(\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda))$, we have $\mathrm{Fr}(\rho_\chi)=\mathrm{Fr}(\lambda)$.*

3. *What precedes is compatible with a product decomposition as in 4.1(1).*

***Remark 4.17*.** —

Since the canonical trace form τ of $\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda)$ satisfies the formula

$$\tau = \sum_\chi \frac{1}{S_\chi}\chi\,,$$

it follows from formula (2)(b) above that

$$\mathrm{Deg}(\lambda)\frac{|\mathbb{G}|_{x'}}{|\mathbb{L}|_{x'}} = \sum_\chi \mathrm{Deg}(\rho_\chi)\chi(1)\,. \tag{4.18}$$

4.1.8. Reduction to the cyclic case. —

Assume that $\mathbb{L} = (V, W_{\mathbb{L}}\varphi)$, and let H be a reflecting hyperplane for $W_{\mathbb{G}}(\mathbb{L}, \lambda)$. We denote by $\mathbb{G}_H$ the "parabolic reflection subcoset" of $\mathbb{G}$ defined by $\mathbb{G}_H := (V, W_H\varphi)$ where W_H is the fixator (pointwise stabilizer) of H. Then $W_{\mathbb{G}_H}(\mathbb{L}, \lambda)$ is cyclic and contains a unique distinguished reflection (see 1.1.1) of $W_{\mathbb{G}}(\mathbb{L}, \lambda)$.

Axiom 4.19. —
In the above situation the parameters of $\mathcal{H}_{\mathbb{G}_H}(\mathbb{L}, \lambda)$ are the same as the parameters corresponding to H in $\mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda)$.

This allows us to reduce the determination of the parameters of $\mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda)$ to the case where $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ is cyclic.

4.1.9. Families of unipotent characters. —

Axioms 4.20. —
There is a partition

$$\mathrm{Uch}(\mathbb{G}) = \bigsqcup_{\mathcal{F} \in \mathrm{Fam}(\mathbb{G})} \mathcal{F}$$

with the following properties.

1. *Let $\mathcal{F} \in \mathrm{Fam}(\mathbb{G})$. Whenever $\rho, \rho' \in \mathcal{F}$, we have*
$$a_\rho = a_{\rho'} \quad \textit{and} \quad A_\rho = A_{\rho'}\,.$$
2. *Assume that $\mathbb{G}$ has a split semisimple part. For $\mathcal{F} \in \mathrm{Fam}(\mathbb{G})$, let us denote by $\mathcal{B}_{\mathcal{F}}$ the Rouquier block of the 1-cyclotomic spetsial Hecke algebra $\mathcal{H}_W$ defined by $\mathcal{F} \cap \mathrm{Uch}(\mathbb{G}, 1)$. Then*
$$\sum_{\rho \in \mathcal{F}} \mathrm{Deg}(\rho)(x)\mathrm{Deg}(\rho^*)(y) = \sum_{\theta \in \mathrm{Irr}(W)\,|\,\chi_\theta \in \mathcal{B}_{\mathcal{F}}} \mathrm{Feg}_{\mathbb{G}}(R_\theta)(x)\mathrm{Feg}_{\mathbb{G}}(R_\theta)(y)\,.$$
3. *The partition of $\mathrm{Uch}(\mathbb{G})$ is globally stable by Ennola transforms.*

Remark 4.21. —
Evaluating 4.20(2) at the eigenvalue ζ of a regular element $w\varphi$, we get using 4.5 (1) and 4.9

$$\sum_{\theta \in \mathrm{Irr}(W)|\chi_\theta \in \mathcal{B}_{\mathcal{F}}} |\mathrm{Feg}_{\mathbb{G}}(R_\theta)(\zeta)|^2 = \sum_{\{\theta \in \mathrm{Irr}(W(w\varphi))\,|\,\rho_{\chi_\theta} \in \mathcal{F}\}} \theta(1)^2\,. \tag{4.22}$$

Remark 4.23 **(When $W(w\varphi)$ has only one class of hyperplanes)**

If $W(w\varphi)$ has only one class of hyperplanes, the algebra $\mathcal{H}_W(w\varphi)$ is defined by a family of parameters $\zeta_j v^{hm_j}$, which are in bijection with the linear characters of

$W(w\varphi)$. If θ is a linear character of $W(w\varphi)$, let m_θ be the corresponding m_j. We have from Lemma 3.23, (2) that

$$N_W^{\mathrm{ref}} + N_W^{\mathrm{hyp}} - a_{\rho_{\chi_\theta}} - A_{\rho_{\chi_\theta}} = e_{W(w\varphi)} m_\theta \,.$$

If $\mathcal{F}$ is the family of ρ_χ, this can be written

$$m_\theta = (N_W^{\mathrm{ref}} + N_W^{\mathrm{hyp}} - \delta_{\mathcal{F}})/e_{W(w\varphi)} \,.$$

Thus formula 4.22, since we know its left-hand side, gives a majoration (a precise value when $W(w\varphi)$ is cyclic) of the number of m_θ with a given value (equal to the number of θ such that $\rho_{\chi_\theta} \in \mathcal{F}$ with a given $\delta_{\mathcal{F}}$).

4.2. Supplementary axioms for spetses

In this section, we state some supplementary axioms which should be true for the data (unipotent characters, degrees and Frobenius eigenvalues, families, Ennola transforms) that we hope to construct for any reflection coset $\mathbb{G} = (V, W\varphi)$ where W is spetsial.

On the data presented in §6 and appendix below we have checked 4.24.

4.2.1. General axioms. —

Axiom 4.24. —
The Frobenius eigenvalues are globally invariant under the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$.

Axiom 4.25. —
Let $w\varphi \in W\varphi$ *be* ζ*-regular. There is a bijection*

$$\mathrm{Irr}(\mathcal{H}_W^{\mathrm{nc}}(w\varphi)) \xrightarrow{\sim} \mathrm{Uch}(\mathbb{G}_{\mathrm{nc}}, \zeta) \ , \ \chi \mapsto \rho_\chi^{\mathrm{nc}}$$

and a collection of signs $(\varepsilon_\chi^{\mathrm{nc}})_{\chi \in \mathrm{Irr}(\mathcal{H}_W(w\varphi))}$ *such that*

1. *that bijection is invariant under the action of* $N_{\mathrm{GL}(V)}(W\varphi)/W$.
2. $\mathrm{Deg}(\rho_\chi^{\mathrm{nc}}) = \varepsilon_\chi^{\mathrm{nc}} \zeta^{N_W^{\mathrm{hyp}}} \mathrm{Deg}(\chi^{\mathrm{nc}})$,
3. $\mathrm{Fr}(\rho_\chi^{\mathrm{nc}}) \equiv \mathrm{Fr}(\chi^{\mathrm{nc}}) \pmod{x^{\mathbb{Z}}}$.

***Remark 4.26* (The real case)**. —
By 1.10, if $\mathbb{G}$ is defined over $\mathbb{R}$, then $\zeta^{N_W^{\mathrm{hyp}}} = \pm 1$, hence we have $\mathrm{Deg}(\rho_\chi^{\mathrm{nc}})(x) = \pm \mathrm{Deg}(\chi^{\mathrm{nc}})(x)$.

4.2.2. Alvis–Curtis duality. —

Axiom 4.27. —

1. *The map*

$$\mathrm{Uch}(\mathbb{G}_{\mathrm{c}}) \xrightarrow{\sim} \mathrm{Uch}(\mathbb{G}_{\mathrm{nc}}) \ , \ \rho \mapsto \rho^{\mathrm{nc}} \, ,$$

is stable under the action of $N_{\mathrm{GL}(V)}(W\varphi)/W$.

2. *In the case where* $\mathbb{G} = (V, W)$ *is split and* W *is generated by true reflections, we have (by formulae 1.44)* $|\mathbb{G}|_{\mathrm{nc}} = |\mathbb{G}|_{\mathrm{c}}$. *In that case*

$$\mathrm{Uch}(\mathbb{G}_{\mathrm{nc}}) = \mathrm{Uch}(\mathbb{G}_{\mathrm{c}}) \, ,$$

a set which we denote $\mathrm{Uch}(\mathbb{G})$, *and the map*

$$D_{\mathbb{G}} : \begin{cases} \mathrm{Uch}(\mathbb{G}) \xrightarrow{\sim} \mathrm{Uch}(\mathbb{G}) \\ \rho \mapsto \rho^{\mathrm{nc}} \, , \end{cases}$$

is an involutive permutation such that
   (a) $\mathrm{Deg}(D_{\mathbb{G}}(\rho))(x) = x^{N_W^{\mathrm{ref}}} \mathrm{Deg}(\rho)(x)^{\vee} \, ,$
   (b) $\mathrm{Fr}(\rho)\mathrm{Fr}(D_{\mathbb{G}}(\rho)) = 1 \, ,$

called the Alvis–Curtis duality.

By definition of the ζ-series, given the property connecting the degree of ρ^{nc} with the degree of ρ, it is clear that the map $\rho \mapsto \rho^{\mathrm{nc}}$ induces a bijection

$$\mathrm{Uch}(\mathbb{G}_{\mathrm{c}}, \zeta) \xrightarrow{\sim} \mathrm{Uch}(\mathbb{G}_{\mathrm{nc}}, \zeta) \, .$$

In particular, if $\mathbb{G} = (V, W)$ is split and W is generated by true reflections, the Alvis–Curtis duality (see 4.27, (2)) induces an involutive permutation of $\mathrm{Uch}(\mathbb{G}, \zeta)$.

This is expressed by a property of the corresponding spetsial ζ-cyclotomic Hecke algebra.

Axiom 4.28. —

Assume $\mathbb{G} = (V, W)$ *is split and* W *is generated by true reflections. Let* $w \in W$ *be a* ζ*-regular element, let* $\mathcal{H}_W(w)$ *be the associated spetsial* ζ*-cyclotomic Hecke algebra.*

1. *There is an involutive permutation*

$$D_W(w) : \mathrm{Irr}(\mathcal{H}_W(w)) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}_W(w))$$

with the following properties, for all $\chi \in \mathrm{Irr}(\mathcal{H}_W(w))$*:*
   (a) $\mathrm{Deg}(D_W(w)(\chi)) = x^{N_W^{\mathrm{ref}}} \mathrm{Deg}(\chi) \, ,$
   (b) $\mathrm{Fr}(D_W(w)(\chi))\mathrm{Fr}(\chi) = 1 \, .$

2. *This is a consequence of the following properties of the parameters of* $\mathcal{H}_W(w)$. *Assume that* $\mathcal{H}_W(w)$ *is defined by the family of polynomials*

$$\left(P_I(t, x) = \prod_{j=0}^{j=e_I - 1} \left(t - \zeta_{e_I}^j (\zeta^{-1} x)^{m_{I,j}} \right) \right)_{I \in \mathcal{A}_W(w)} \, .$$

Then for all $I \in \mathcal{A}_W(w)$, there is a unique j_0 ($0 \leq j_0 \leq e_I - 1$) such that $m_{I,j_0} = 0$, and for all j with $0 \leq j \leq e_I - 1$, we have

$$m_{I,j} + m_{I,j_0-j} = m_I \,.$$

4.2.3. Φ-Harish-Chandra series. —

Let Φ be a K-cyclotomic polynomial.

Definition 4.29. —
We call Φ-cuspidal pair for $\mathbb{G}$ a pair $(\mathbb{L}, \lambda)$ where
- *$\mathbb{L}$ is the centralizer of the Φ-Sylow subdatum of its center $Z\mathbb{L}$,*
- *λ a Φ-cuspidal unipotent character of $\mathbb{L}$, i.e., (see 4.5)*

$$\mathrm{Deg}(\lambda)_\Phi = \frac{|\mathbb{L}|_\Phi}{|Z\mathbb{L}|_\Phi} \,.$$

Axiom 4.30. —
Whenever $(\mathbb{L}, \lambda)$ is a Φ-cuspidal pair for $\mathbb{G}$, then $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ is a reflection group on the orthogonal of the intersection of the hyperplanes of $W_{\mathbb{L}}$.

***Axioms 4.31* (Φ-Harish–Chandra theory)**. —

1. *There is a partition*

$$\mathrm{Uch}(\mathbb{G}) = \bigsqcup_{(\mathbb{L},\lambda)} \mathrm{Uch}_{\mathbb{G}}(\mathbb{L}, \lambda)$$

   where $(\mathbb{L}, \lambda)$ runs over a complete set of representatives of the orbits of W on Φ-cuspidal pairs of $\mathbb{G}$.
2. *For each Φ-cuspidal pair $(\mathbb{L}, \lambda)$, there is a Φ-cyclotomic Hecke algebra $\mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda)$, an associated bijection*

$$\mathrm{Irr}(\mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda)) \xrightarrow{\sim} \mathrm{Uch}_{\mathbb{G}}(\mathbb{L}, \lambda) \ , \ \chi \mapsto \rho_\chi \,,$$

   and a collection $(\varepsilon_\chi)_{\chi \in \mathrm{Irr}(\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda))}$ of signs, with the following properties.
   (a) *Those bijections are invariant under $N_{\mathrm{GL}(V)}(W\varphi)/W$.*
   (b) *If we denote by S_χ the Schur element of the character χ of $\mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda)$, we have*

$$\mathrm{Deg}(\rho_\chi) = \varepsilon_\chi \frac{\mathrm{Deg}(\lambda)(|\mathbb{G}|/|\mathbb{L}|)_{x'}}{S_\chi} \,.$$

   (c) *Assume that a root ζ of P is regular for $W\varphi$, and let $w\varphi$ be ζ-regular. For $\mathbb{T}_{w\varphi} := (V, w\varphi)$ the corresponding maximal torus, the algebra $\mathcal{H}_{\mathbb{G}}(\mathbb{T}_{w\varphi}, \mathrm{Id})$ is a ζ-cyclotomic spetsial Hecke algebra $\mathcal{H}_W(w\varphi)$.*
   (d) *For all $\chi \in \mathrm{Irr}(\mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda))$, $\mathrm{Fr}(\rho_\chi)$ only depends on $\mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda)$ and $\mathrm{Fr}(\lambda)$.*
3. *What precedes is compatible with a product decomposition as in 4.1(1).*

4.2.4. Reduction to the cyclic case. —

Assume that $\mathbb{L} = (V, W_{\mathbb{L}} w\varphi)$ is a Φ-cuspidal pair, and let H be a reflecting hyperplane for $W_{\mathbb{G}}(\mathbb{L}, \lambda)$. We denote by $\mathbb{G}_H$ the "parabolic reflection subdatum" of $\mathbb{G}$ defined by $\mathbb{G}_H := (V, W_H w\varphi)$ where W_H is the fixator (pointwise stabilizer) of H. Then $W_{\mathbb{G}_H}(\mathbb{L}, \lambda)$ is cyclic and contains a unique distinguished reflection (see 1.1.1) of $W_{\mathbb{G}}(\mathbb{L}, \lambda)$.

Axiom 4.32. —

In the above situation the parameters of $\mathcal{H}_{\mathbb{G}_H}(\mathbb{L}, \lambda)$ *are the same as the parameters corresponding to* H *in* $\mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda)$.

4.2.5. Families, Φ-Harish-Chandra series, Rouquier blocks. —

Her we refer the reader to 4.20 above.

Axiom 4.33. —

For each Φ*-cuspidal pair* $(\mathbb{L}, \lambda)$ *of* $\mathbb{G}$*, the partition*

$$\mathrm{Uch}_{\mathbb{G}}(\mathbb{L}, \lambda) = \bigsqcup_{\mathcal{F} \in \mathrm{Fam}(\mathbb{G})} (\mathcal{F} \cap \mathrm{Uch}_{\mathbb{G}}(\mathbb{L}, \lambda))$$

composed with the bijection

$$\mathrm{Uch}_{\mathbb{G}}(\mathbb{L}, \lambda) \xrightarrow{\sim} \mathrm{Irr}\,(\mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda))$$

is the partition of $\mathrm{Irr}\,(\mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda))$ *into Rouquier blocks.*

4.2.6. Ennola transform. —

For $z \in \boldsymbol{\mu}(K)$, we define

$$\mathbb{G}_z := (V, Wz\varphi)\,.$$

Axiom 4.34. —

Let $\xi \in \boldsymbol{\mu}$ *such that* $z := \xi^{|ZW|} \in \boldsymbol{\mu}(K)$*. There is a bijection*

$$E_\xi \;:\; \mathrm{Uch}(\mathbb{G}) \xrightarrow{\sim} \mathrm{Uch}(\mathbb{G}_z)$$

with the following properties.

1. E_ξ *is stable under the action of* $N_{\mathrm{GL}(V)}(W\varphi)/W$.
2. *For all* $\rho \in \mathrm{Uch}(\mathbb{G})$*, we have*

$$\mathrm{Deg}(E_\xi(\rho))(x) = \pm \mathrm{Deg}(\rho)(z^{-1}x)\,.$$

Axiom 4.35. —

1. *Let* $\zeta \in \boldsymbol{\mu}(K)$*, a root of the* K*-cyclotomic polynomial* $\Phi(x)$*. Let* $(\mathbb{L}, \lambda)$ *be a* Φ*-cuspidal pair.*
   (a) $(\mathbb{L}_z, E_\xi(\lambda))$ *is a* $\Phi(z^{-1}x)$*-cuspidal pair of* $\mathbb{G}_z$.

(b) E_ξ *induces a bijection*

$$\mathrm{Uch}_{\mathbb{G}}(\mathbb{L},\lambda) \xrightarrow{\sim} \mathrm{Uch}_{\mathbb{G}_z}(\mathbb{L}_z, E_z(\lambda))\,.$$

(c) *The parameters of the* $\Phi(z^{-1}x)$*-cyclotomic Hecke algebra* $\mathcal{H}_{\mathbb{G}_z}(\mathbb{L}_z, E_z(\lambda))$ *are obtained from those of* $\mathcal{H}_W(\mathbb{L},\lambda)$ *by changing* x *into* $z^{-1}x$*.*

2. *The bijection* E_ξ *induces a bijection* $\mathrm{Fam}(\mathbb{G}) \xrightarrow{\sim} \mathrm{Fam}(\mathbb{G}_z)$.

CHAPTER 5

DETERMINATION OF $\mathrm{Uch}(\mathbb{G})$: THE ALGORITHM

In this section, we consider reflection cosets $\mathbb{G} = (V, W\varphi)$ which have a *split semi-simple part*, *i.e.*, V has a $W\varphi$-stable decomposition

$$V = V_1 \oplus V_2 \ \text{ with } \ W|_{V_2} = 1 \ \text{ and } \ \varphi|_{V_1} = 1\,.$$

In addition, we assume that W is a spetsial group (see 3.44 above).

We show with the help of computations done with the `CHEVIE` package of `GAP3`, that for all primitive special reflection groups there is a unique solution which satisfies the axioms given in §5. Actually, a subset of the axioms is sufficient to ensure unicity. More specifically, we finish the determination of unipotent degrees and Frobenius eigenvalues except for a few cases in G_{26} and G_{32} in 6.5, and at this stage we only use 4.16 for the pair $(\mathbb{T}, \mathrm{Id})$. Also we only use 4.20(3) to determine the families of characters.

The tables in the appendix describe this solution.

5.1. Determination of $\mathrm{Uch}(\mathbb{G})$

The construction of $\mathrm{Uch}(\mathbb{G})$ proceeds as follows:

1. First stage.
   - We start by constructing the principal series $\mathrm{Uch}(\mathbb{G}, 1)$ using 4.16(2)(c) for the pair $(\mathbb{T}, \mathrm{Id})$.
   - We extend it by Ennola transform using 4.13 to construct the union of the series $\mathrm{Uch}(\mathbb{G}, \xi)$ for ξ central in W.

     Let us denote by U_1 the subset of the set of unipotent characters that we have constructed at this stage.
2. Second stage.

   - Let $w_1 \in W$ be a regular element of largest order in W, with regular eigenvalue ζ_1. We have an algorithm allowing us to determine the parameters of the ζ_1-spetsial cyclotomic Hecke algebra $\mathcal{H}_W(w_1\varphi)$, which in turn determines $\mathrm{Uch}(\mathbb{G}, \zeta_1)$.
   - We again use Ennola transform to determine $\mathrm{Uch}(\mathbb{G}, \zeta_1\xi)$ for ξ central in W. Thus we know the series $\mathrm{Uch}(\mathbb{G}, \xi\zeta_1)$, which can be added to our set U_1.

     Let us denote by U_2 the subset of the set of unipotent characters that we have constructed at this stage.
3. Third stage.
   We iterate the previous steps (proceeding in decreasing orders of w, finding each time at least one reachable ζ) until no $\mathrm{Uch}(\mathbb{G}, \zeta)$ can be determined for any new ζ. At each iteration we can use 4.20(2) (whose right-hand side we know in advance) to check if we have finished the determination of $\mathrm{Uch}(\mathbb{G})$.

This will succeed for every spetsial irreducible exceptional group, except for G_{26}, G_{32}, where we will find *a posteriori* that 1 (resp. 14) unipotent characters are missing at this point.

For these remaining cases we have to consider some series corresponding to 1-cuspidal pairs $(\mathbb{L}, \lambda)$.

Since we also want to label unipotent characters according to the 1-Harish-Chandra series in which they lie, we shall actually determine (using a variation of the previous algorithm) the parameters of all these algebras $\mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda)$. We detail the steps (1)—(3) outlined above in sections 5.7 to 5.11.

5.2. The principal series $\mathrm{Uch}(\mathbb{G}, 1)$

By 4.16(2)(c), the principal series $\mathrm{Uch}(\mathbb{G}, 1)$ is given by the 1-spetsial algebra $\mathcal{H}_W$.

For $\chi \in \mathrm{Irr}(\mathcal{H}_W)$ we have $\mathrm{Fr}(\chi) = 1$ by 4.16(2)(d) and 4.1(2), and $\mathrm{Deg}(\rho_\chi) = \mathrm{Deg}(\chi)$ by 4.6(2).

5.2.1. Example: the cyclic Spets.—

Let $\mathbb{G}_e := (\mathbb{C}, \boldsymbol{\mu}_e)$ be the untwisted spets associated with the cyclic group $W = \boldsymbol{\mu}_e$ acting on $\mathbb{C}$ by multiplication.

We set $\mathbb{Z}_e := \mathbb{Z}[\boldsymbol{\mu}_e]$ and $\zeta := \exp(2\pi i/e)$.

The spetsial Hecke algebra $\mathcal{H}_W$ attached to $\mathbb{G}_e$ is by 3.44 the $\mathbb{Z}_e[x^{\pm 1}]$-algebra $\mathcal{H}_e$ defined by

$$\mathcal{H}_e := \mathbb{Z}_e[T]/(T - x)(T - \zeta)\cdots(T - \zeta^{e-1}) .$$

We denote by $\chi_0, \chi_1, \ldots, \chi_{e-1} : \mathcal{H}_e \to \mathbb{Z}_e[x, x^{-1}]$ the irreducible characters of $\mathcal{H}_e$, defined by

$$\begin{cases} \chi_0 : T \mapsto x\,, \\ \chi_i : T \mapsto \zeta^i \quad \text{for } 1 \leq i \leq e-1\,. \end{cases}$$

We denote by $S_0, S_1, \ldots, S_{e-1}$ the corresponding family of Schur elements, and by $\rho_0, \ldots, \rho_{e-1}$ the corresponding (by 4.6(2) or 4.16(2)(c)) unipotent characters in $\mathrm{Uch}(\mathbb{G}_e, 1)$. We have $\rho_0 = \mathrm{Id}$.

By 4.16(2)(b) we have $\mathrm{Deg}(\rho_i) = \dfrac{S_0}{S_i}$ since by 1.68 we have

$$S_0 = (|\mathbb{G}|/|\mathbb{T}|)_{x'} = \frac{x^e - 1}{x - 1}\,.$$

So for $i \neq 0$ using 1.68 for the value of S_i we get

$$\mathrm{Deg}(\rho_i) = \frac{1 - \zeta^i}{e}\, x \prod_{j \neq 0, i} (x - \zeta^j)\,.$$

We will see below that we need to add $\frac{(e-1)(e-2)}{2}$ other 1-cuspidal unipotent characters to the principal series before formula 4.20(2) is satisfied.

5.3. The series $\mathrm{Uch}(\mathbb{G},\xi)$ for $\xi \in ZW$

From the principal series, we use (4.13) to determine the series $\mathrm{Uch}(\mathbb{G},\xi)$ for all $\xi \in ZW$.

Note that for $\rho \in \mathrm{Uch}(\mathbb{G},\xi)$, this gives $\mathrm{Deg}(\rho)$ only up to sign.

In practice, we assign a sign arbitrarily and go on. However, for any character which is not 1-cuspidal the sign will be determined by the sign chosen for the 1-cuspidal character when we will determine 1-Harish-Chandra series (see below).

We illustrate the process for the cyclic reflection coset, which in this case allows us to finish the determination of $\mathrm{Uch}(\mathbb{G}_e)$.

We go on with the example 5.2.1 of $W = \boldsymbol{\mu}_e = \langle \zeta \rangle$ where $\zeta = \exp(2i\pi/e)$.

We have $Z(W) = W = \{\zeta^k \mid k = 0, 1, \ldots, e-1\}$.

We determine $\mathrm{Uch}(\mathbb{G}_e, \zeta^k)$ by Ennola transform. Let $\mathbf{z}$ be the lift of ζ to $Z(\mathbf{B}_W)$. The Ennola transform of $\mathcal{H}_W$ by $\mathbf{z}^k$ (for $k = 0, 1, \ldots, e-1$) is

$$\mathcal{H}_W(\zeta^k) = \mathbb{Z}_e[T]/(T - \zeta^{-k}x)(T - \zeta)\cdots(T - \zeta^{e-1})\,,$$

and the corresponding family of generic degrees is

$$\mathrm{Deg}(\rho_0)(\zeta^{-k}x), \mathrm{Deg}(\rho_1)(\zeta^{-k}x), \ldots, \mathrm{Deg}(\rho_{e-1})(\zeta^{-k}x)\,.$$

It is easy to check that, for all $i = 1, 2, \ldots, e-1$,

$$\mathrm{Deg}(\rho_i)(\zeta^{-k}x) \begin{cases} = -\mathrm{Deg}(\rho_k)(x) & \text{if } i+k \equiv 0 \mod e, \\ \notin \{\pm\mathrm{Deg}(\rho) \mid \rho \in \mathrm{Uch}(\mathbb{G}_e, 1)\} & \text{if } i+k \not\equiv 0 \mod e. \end{cases}$$

Let us denote, for $0 \le k < i \le e-1$, by $\rho_{k,i}$ the element of $\mathrm{Uch}(\mathbb{G}_e, \zeta^k)$ corresponding to the $(i-k)$-th character of $\mathcal{H}_W(\zeta^k)$. Thus

$$\mathrm{Deg}(\rho_{k,i})(x) := \mathrm{Deg}(\rho_{i-k})(\zeta^{-k}x) = \frac{\zeta^k - \zeta^i}{e}\, x \frac{x^e - 1}{(x-\zeta^k)(x-\zeta^i)}\,,$$

and by 4.13,

$$\mathrm{Fr}(\rho_{k,i}) = \zeta^{ik}.$$

With this notation, our original characters ρ_i become $\rho_{i,0}$, except for $\rho_0 = \mathrm{Id}$ which is a special case. We see that if we extend the notation $\rho_{i,k}$ to be $-\rho_{k,i}$ when $k < i$ the above formulae for the degree and eigenvalue remain consistent. It can be checked that with this notation $E_{\mathbf{z}^j}(\rho_{i,k}) = \rho_{i+j,k+j}$, where the indices are taken $\pmod e$; thus we have taken in account all characters obtained by Ennola from the principal series. We claim we have obtained all the unipotent characters.

Theorem 5.1. —
$\mathrm{Uch}(\mathbb{G}_e)$ *consists of the* $1 + \binom{e}{2}$ *elements* $\{\mathrm{Id}\} \bigcup \{\rho_{i,k}\}_{0 \le k < i \le e-1}$ *with degrees and eigenvalues as given above.*

Proof. —
Using that $\mathrm{Feg}_{\mathbb{G}_e}(\chi_i) = x^i$, the reader can check (a non-trivial exercise) that formula 4.20(2) is satisfied. □

5.4. An algorithm to determine some $\mathrm{Uch}(\mathbb{G}, \zeta)$ for ζ regular

Assume that U is one of the sets $U_1, U_2, \ldots$ of unipotents characters of $\mathbb{G}$ mentioned in the introduction of section 5.1. In particular, for all $\rho \in U$, we know $\mathrm{Fr}(\rho)$ and $\pm\mathrm{Deg}(\rho)$.

We outline in this section an algorithm which allows us to determine the parameters of $\mathcal{H}_W(w\varphi)$ for some well-chosen ζ-regular elements $w\varphi$ (we call such well-chosen elements *reachable from* U). Knowing the ζ-spetsial algebra $\mathcal{H}_W(w\varphi)$, we then construct the series $\mathrm{Uch}(\mathbb{G}, \zeta)$.

First step: determine the complete list of degrees of parameters $m_{I,j}$ for the algebra $\mathcal{H}_W(w\varphi)$.—

- If $W(w\varphi)$ is cyclic this is easy since in this case (as explained in remark 4.23) formula 4.22 determines the list of parameters $m_{I,j}$.
- If $W(w\varphi)$ has only one conjugacy class of hyperplanes, for each $\rho_{\chi_\theta} \in U$ such that $\mathrm{Deg}(\rho_{\chi_\theta})(\zeta) = \pm 1$ we get (as explained in Remark 4.23) the number m_{χ_θ} .

  For the other m_χ, we can restrict the possibilities by using 4.22: they are equal to some $(N^{\mathrm{ref}} + N^{\mathrm{hyp}} - \delta_{\mathcal{B}})/e_{W(w\varphi)}$ where $\mathcal{B}$ runs over the set of Rouquier blocks of $\mathcal{H}_W$.

  Finally 3.7(CS3) is a good test to weed out possibilities.
- When $W(w\varphi)$ has more than one class of hyperplanes, one can do the same with the equations of 3.23 to restrict the possibilities; we are helped by the fact that, in this case, the e_I's are rather small.

  By the above, in all cases we are able to start with at most a few dozens of possibilities for the list of $m_{I,j}$, and we proceed with the following steps with each one of this lists.

Second step: for a given I, assign a specific j to each element of our collection of $m_{I,j}$.—

It turns out that when U has a "large enough" intersection with $\mathrm{Uch}(\mathbb{G}, \zeta)$, there is only one assignment such that $m_{I,1}$ is the largest of the $m_{I,j}$ and the resulting $\mathrm{Uch}(\mathbb{G}, \zeta)$ contains U as a subset.

However, trying all possible assignments for the above test is not feasible in general, since $W(w\varphi)$ can be for example the cyclic group of order 42 (and 42! is too big). It happens that the product of the $e_I!$ is small enough when there is more than one of them; so we can concentrate on the case where there is only one e_I, that we will denote e; the linear characters of $W(w\varphi)$ are the $\det^i$ for $i = 0, \ldots, e-1$, and we will denote u_i for u_{I,χ_θ} and ρ_i for ρ_{χ_θ} when $\theta = \det^i$.

If $\rho_i \in U$ we can reduce some of the arbitrariness for the assignment of m to j since 4.8 implies that the root of unity part of $\mathrm{Fr}(\rho_i)$ is given by $\zeta^{i+(a_{\rho_i}+A_{\rho_i})a/d}$, which gives $i \mod d$.

We can then reduce somewhat the remaining permutations by using the "rationality type property" $P_I(t, x) \in K(x)[t]$.

We say that ζ is *reachable from* U when we can determine a unique algebra $\mathcal{H}_W(w\varphi)$ in a reasonable time. Given U, we then define U' as the union of U with all the $\mathrm{Uch}(\mathbb{G}, \zeta\xi)$ where ζ is reachable from U, and $\xi \in ZW$ (see section 5.1).

5.5. An example of computational problems

The worst computation encountered during the process described above occurs in G_{27}, at the first step, *i.e.*, when starting from the initial $U_1 = \bigcup_{\xi\in ZW} \mathrm{Uch}(\mathbb{G},\xi)$, we try to determine $\mathcal{H}_W(w\varphi)$ for w of maximal order. In that case, $W(w\varphi)$ is a cyclic group of order $h := 30$. We know 18 out of 30 parameters as corresponding to elements of U. For the 12 remaining, we know the list of $m_{I,j}$, which is

$$[1, 1, 1, 1, 3/2, 3/2, 3/2, 3/2, 2, 2, 2, 2]$$

and there are 34650 arrangements of that list in the remaining 12 slots; in a few minutes of CPU we find that 420 of them provide $P_I(t, x) \in K(x)[t]$ and after a few more minutes that just one of them gives a $\mathrm{Uch}(\mathbb{G}, \zeta_h)$ containing the $U \cap \mathrm{Uch}(\mathbb{G}, \zeta_h)$ we started with.

As mentioned in section 5.4, at the end of this process, we discover by 4.20(2) that we have found all unipotent degrees, except in the cases of G_{26} and G_{32}, where we will find *a posteriori* that 1 (resp. 14) unipotent characters are missing.

5.6. Determination of 1-Harish-Chandra series

We next finish determining $\mathrm{Uch}(\mathbb{G})$ for G_{26} and G_{32}, by considering some series corresponding to 1-cuspidal pairs $(\mathbb{L}, \lambda)$. We will find that in G_{26} the missing character occurs in the series $\mathrm{Uch}(\mathbb{G}_3, \rho_{2,1})$ (where $\rho_{2,1}$, as seen in 5.1, is the only 1-cuspidal character for $\mathbb{G}_3$, and where $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ is $G(6, 1, 2)$), while in G_{32} the missing characters occur in $\mathrm{Uch}(\mathbb{G}_3, \rho_{2,1})$ and in $\mathrm{Uch}(\mathbb{G}_3 \times \mathbb{G}_3, \rho_{2,1} \otimes \rho_{2,1})$ where $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ is respectively G_{26} and $G(6, 1, 2)$. In these cases, $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ is not cyclic, so by the reduction to the cyclic case 4.19, the computation of the parameters of $\mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda)$ is reduced to the case of sub-Spets where all unipotent characters are known.

Following the practice of Lusztig and Carter for reductive groups, we will name unipotent characters by their 1-Harish-Chandra data, that is each character will be indexed by a 1-cuspidal pair $(\mathbb{L}, \lambda)$ and a character of $\mathcal{H}_{\mathbb{G}}(\mathbb{L}, \lambda)$; thus to do this indexing we want anyway to determine all the 1-series.

Let us examine now the computations involved (in a Spets where all unipotent characters are known).

Step1: determine cuspidal pairs $(\mathbb{L}, \lambda)$.—

First we must find the W-orbits of cuspidal pairs $(\mathbb{L}, \lambda)$ and the corresponding groups $W_{\mathbb{G}}(\mathbb{L}, \lambda)$, and for that we must know the action of an automorphism of $\mathbb{L}$ given by an element of $W_{\mathbb{G}}(\mathbb{L})$) on $\lambda \in \mathrm{Uch}(\mathbb{L})$.

Cases when this is determined by our axioms 4.3 and 4.16(2)(a) are

- when $\pm\mathrm{Deg}(\lambda)$ is unique,
- when the pair $(\pm\mathrm{Deg}(\lambda), \mathrm{Fr}(\lambda))$ is unique,

– when λ is in the principal 1-series; the automorphism then acts on λ as on $\mathrm{Irr}(W_{\mathbb{L}})$.

In the first two cases $W_{\mathbb{G}}(\mathbb{L},\lambda) = W_{\mathbb{G}}(\mathbb{L})$.

The above conditions are sufficient in the cases we need for G_{26} and G_{32}. They are not sufficient in some other cases.

Now, for 1-series where $W_{\mathbb{G}}(\mathbb{L},\lambda)$ is *not* cyclic, we know (by induction using 4.19) the parameters of $\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda)$. Since in the cases we need for G_{26} and G_{32}, the group $W_{\mathbb{G}}(\mathbb{L},\lambda)$ is not cyclic, we may assume from now on that we know all of $\mathrm{Uch}(\mathbb{G})$.

Step 2: find the elements of $\mathrm{Uch}_{\mathbb{G}}(\mathbb{L},\lambda)$.—

A candidate element $\rho \in \mathrm{Uch}_{\mathbb{G}}(\mathbb{L},\lambda)$ must satisfy the following properties.

– $\mathrm{Deg}(\rho)$ must be divisible by $\mathrm{Deg}(\lambda)$ by 4.16(2)(b).

– By specializing 4.16(2)(b) to $x = 1$, we get, if $\rho = \rho_\chi$,

$$|W_{\mathbb{G}}(\mathbb{L},\lambda)| \left(\frac{\mathrm{Deg}(\rho)|\mathbb{L}|_{x'}}{\mathrm{Deg}(\lambda)|\mathbb{G}|_{x'}}\right)(1) = \chi(1)\,.$$

Then $\chi(1)$ must be the degree of a character of $|W_{\mathbb{G}}(\mathbb{L},\lambda)|$.

– Formula 4.16(2)(b) yields a Schur element S_{χ_ρ} which must be a Laurent polynomial (indeed, we assume that the relative Hecke algebras are 1-cyclotomic in the sense of [**BMM99**, 6E], thus their Schur elements have these rationality properties).

If there are exactly $|\mathrm{Irr}(\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda))|$ candidates left at this stage, we are done. If there are too many candidates left, a useful test is to filter candidate $|\mathrm{Irr}(\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda))|$-tuples by the condition 4.18. In practice this always yields only one acceptable tuple(1).

Step 3: Parametrize elements of $\mathrm{Uch}_{\mathbb{G}}(\mathbb{L},\lambda)$ by characters of $\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda)$.—

This problem is equivalent to determining the Schur elements of $\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda)$, which in turn is equivalent to determining the parameters of this algebra (up to a common scalar). Thanks to 4.19, it is sufficient to consider the case when $W_{\mathbb{G}}(\mathbb{L},\lambda)$ is cyclic. Then one can use techniques analogous to that of section 5.1.

For example, let us consider the case where $\mathbb{G}$ is the split reflection coset associated with the exceptional reflection group G_{26}. Let us recall that $\mathbb{G}_e$ denotes the reflection coset associated with the cyclic group $\boldsymbol{\mu}_e$ (see 5.2.1).

We have $W_{\mathbb{G}}(\mathbb{G}_3,\rho_{2,1}) = G(6,2,2)$. To determine the parameters, for each hyperplane I of that group, we have to look at the same series $\mathrm{Uch}(\mathbb{G}_3,\rho_{2,1})$ in the group W_I which is respectively $G(3,1,2)$, G_4 and $\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_2$; in each case the relative group is $\mathbb{G}_{e_I}$ where e_I is respectively $3, 2, 2$.

(1) This is not always the case if one tries to determine ζ-Harish-Chandra series by similar techniques for $\zeta \neq 1$.

Remark 5.2. — In each case we can determine the parameters up to a constant, which must be a power of x times a root of unity for the algebra to be 1-cyclotomic; we have chosen them so that the lowest power of x is 0, in which case they are determined up to a cyclic permutation. We have chosen this permutation so that the polynomial $P_I(t,x)$ is as rational as possible, and amongst the remaining ones we chose the list $m_{I,0},\dots,m_{I,e_I-1}$ to be the lexicographically biggest possible.

In our case we get that the relative Hecke algebra $\mathcal{H}_{\mathbb{G}}(\mathbb{L},\lambda)$ is

$$\mathcal{H}_{G(6,2,2)}(1,\zeta_3x^2,\zeta_3^2x^2;x^3,-1;x,-1).$$

Here we put the group $W_{\mathbb{G}}(\mathbb{L},\lambda)$ as an index and each list separated by a semicolon is the list of parameters for one of the 3 orbits of hyperplanes. We list the parameters in each list $u_{I,0},\dots,u_{I,e_I-1}$ in an order such that $u_{I,j}$ specializes to $\zeta_{e_I}^j$.

Similarly, in G_{32}, for $\mathrm{Uch}(\mathbb{G}_3,\rho_{2,1})$ we get the Hecke algebra

$$\mathcal{H}_{G_{26}}(x^3,\zeta_3,\zeta_3^2;x,-1)$$

and for $\mathrm{Uch}(\mathbb{G}_3\times\mathbb{G}_3,\rho_{2,1}\otimes\rho_{2,1})$ we get

$$\mathcal{H}_{G(6,1,2)}(x^3,-\zeta_3^2x^3,\zeta_3x^2,-1,\zeta_3^2,-\zeta_3x^2;x^3,-1)\,.$$

5.7. Determination of families

The families can be completely determined from their intersection with the principal series, which are the Rouquier blocks which were determined in [**MR03**], and 4.20(3).

Indeed, the only cases in the tables of Malle and Rouquier where two blocks share the same pair (a,A), and none of them is a one-element block, are the pairs of blocks $(12,13)$, $(14,15)$, $(21,27)$ in G_{34} (the numbers refer to the order in which the families appear in the `CHEVIE` data; see the tables in the appendix to this paper).

For each of these pairs, we have a list L of unipotent characters that we must split into two families $\mathcal{F}_1$ and $\mathcal{F}_2$. To do this, we can use the axiom of stability of families by Ennola, since in each case all degrees in L are Ennola-transforms of those in the intersection of L with the principal series, and there are no degrees in common between the intersections of $\mathcal{F}_1$ and $\mathcal{F}_2$ with the principal series.

5.8. The main theorem

We can now summarize the main result of this paper as follows:

Theorem 5.3. —

Given a primitive irreducible spetsial reflection group W, and the associated split coset $\mathbb{G}$, there is a unique set $\mathrm{Uch}(\mathbb{G})$*, with a unique function* Fr *and a unique (up to an arbitrary choice of signs for 1-cuspidal characters) function* Deg *which satisfy the axioms 4.1, 4.3, 4.6, 4.13, 4.16, 4.19 and 4.20.*

Apart from the primitive exceptional type spetsial reflection groups, there exist two further doubly infinite series of irreducible complex reflection groups: the groups $G(e,1,n)$, $e,n \geq 1$, $(e,n) \neq (1,1)$, and the groups $G(e,e,n)$, $e,n \geq 2$, $(e,n) \neq (2,2)$. Unipotent characters and Frobenius eigenvalues for these reflection groups were introduced in [**Mal95**] in a combinatorial way. Note however that Frobenius eigenvalues in [**Mal95**] by definition are roots of unity, with no power of x attached. Here we comment on the present state of knowledge concerning the axioms set out for the exceptional type spets in Section 5, viz. the Axioms 4.1, 4.2, 4.3, 4.6, 4.13, 4.16, 4.19, 4.20. Note that [**Mal95**] does not contain any unicity statement about the data constructed there. While the methods presented here will certainly make it possible to prove unicity in given small examples, a general proof is not known at present.

We discuss the eight relevant axioms in turn.

Axiom 4.1 does not apply directly to this situation, since the corresponding spets is simple and not a torus. But it is used (implicitly) in [**Mal95**] for the description of unipotent characters of all reducible proper subspets.

Axiom 4.2 is satisfied by [**Mal95**, Folg. 3.18] (for $G(e,1,n)$) and [**Mal95**, Folg. 6.14] (for $G(e,e,n)$).

There is an action as in Axiom 4.3 which we now describe: for $W = G(e,1,n) \leq \mathrm{GL}(V)$ we have that $N := N_{\mathrm{GL}(V)}(W) = WZ(\mathrm{GL}(V))$ (see e.g. [**BMM99**, Prop. 3.13]), and the action of N is trivial on $\mathrm{Irr}(W)$ as well as on the unipotent characters. For $W = G(e,e,n)$, we have again by [**BMM99**, Prop. 3.13] that $N := N_{\mathrm{GL}(V)}(W)$ equals $G(e,1,n)Z(\mathrm{GL}(V))$, unless $(e,n) \in \{(3,3),(2,4)\}$. The second exceptional case corresponds to the generic finite reductive group of type D_4, and there an action of N on unipotent characters has been defined by Lusztig. In the first exceptional case, it is easy to define an action of N on the unipotent characters which has the desired properties. In the general case when $N = G(e,1,n)Z(\mathrm{GL}(V))$, let s be the standard generating reflection of $G(e,1,n)$ of order e. It acts on unipotent characters of $G(e,e,n)$ by permuting cyclically the unipotent characters belonging to a fixed degenerate symbol (i.e., any symbol with non-trivial symmetry group), see [**Mal95**, Def. 6.3]. The definitions given for unipotent characters and Frobenius eigenvalues shows that Axiom 4.3 is satisfied by this action.

The construction of a bijection as in Axiom 4.6 is a particular case of [**Mal95**, Sätze 3.14 and 6.10], which also yields (2) (as a consequence of the main result of [**GIM00**]). Property (3) is shown in [**Mal95**, Satz 4.21] for $G(e,1,n)$, but it was not considered for $G(e,e,n)$.

The fact that Ennola transforms preserve unipotent degrees up to sign as in Axiom 4.13 is shown in [**Mal95**, Folg. 3.11 und 6.7], but the behaviour of Frobenius eigenvalues under Ennola transform has not been determined.

Axiom 4.16 on Harish-Chandra series in general was shown in [**Mal95**, Sätze 3.14 and 6.10], which also shows that the parameters are determined locally, as required by Axiom 4.19.

The families for types $G(e, 1, n)$ and $G(e, e, n)$ were introduced and studied in [**Mal95**, §4C, 6D]. Axiom 4.20(2) is proved in [**Mal95**, Sätze 4.17 and 6.26], and Axiom 4.20(1) is implicit for example in the hook formulas in [**Mal95**, Bem. 3.12 and 6.8]. The final part of Axiom 4.20(3) is not considered in [**Mal95**].

APPENDIX A

TABLES

In this appendix, we give tables of unipotent degrees for split Spetses of primitive finite complex reflexion groups, as well as for the imprimitive groups $\mathbb{Z}/3$, $\mathbb{Z}/4$, $G(3,1,2)$, $G(3,3,3)$ and $G(4,4,3)$ which are involved in their construction or in the labeling of their unipotent characters.

The characters are named by the pair of the 1-cuspidal unipotent above which they lie and the corresponding character of the relative Weyl group. As pointed out in 5.2, this last character is for the moment somewhat arbitrary since it depends on an ordering (defined up to a cyclic permutation) of the parameters of the relative Hecke algebra, thus may have to be changed in the future if the theory comes to prescribe a different ordering than the one we have chosen. Also, as pointed out in 5.3, the sign of the degree of 1-cuspidal characters (and consequently of the corresponding 1-Harish-Chandra series) is arbitrary, though we fixed it such that the leading term is positive when it is real. For the imprimitive groups we give the correspondence between our labels and symbols as in [**Mal95**].

For primitive groups, the labeling of characters of W is as in [**Mal00**].

The unipotent characters are listed family by family. In a Rouquier family, there is a unique character θ such that $a_\theta = b_\theta$, the special character (see 3.49), and a unique character such that $A_\theta = B_\theta$, called the *cospecial character*, which may or may not coincide with the special character. The special character in a family is indicated by a $*$ sign in the first column. If it is different from the special character, the cospecial character is indicated by a # sign in the first column.

In the third column we give Fr, as a root of unity times a power of x in $\mathbb{Q}/\mathbb{Z}$ (this power is most of the time equal to 0).

We denote the cuspidal unipotent characters by the name of the group if there is only one, otherwise the name of the group followed by the Fr, with an additional exponent if needed to resolve ambiguities. For instance $G_6[\zeta_8^3]$ is the cuspidal unipotent character of G_6 with $\mathrm{Fr} = \zeta_8^3$, while $G_6^2[-1]$ is the second one with $\mathrm{Fr} = -1$.

For each group we list the Hecke algebras used in the construction: we give the parameters of the spetsial ζ-cyclotomic Hecke algebras of compact type for representatives of the regular ζ under the action of the centre; we omit the central ζ for which the parameters are always given by 3.43 and its Ennola transforms.

We also list the parameters of the 1-cyclotomic Hecke algebras attached to cuspidal pairs, chosen as in 5.2.

For each of these Hecke algebras, the parameters are displayed as explained in remark 5.2.

A.1. Irreducible K-cyclotomic polynomials

$\mathbb{Q}(i)$-cyclotomic polynomials. — $\Phi'_4 = x - i$, $\Phi''_4 = x + i$, $\Phi'_8 = x^2 - i$, $\Phi''_8 = x^2 + i$, $\Phi'_{12} = x^2 - ix - 1$, $\Phi''_{12} = x^2 + ix - 1$, $\Phi'''_{20} = x^4 + ix^3 - x^2 - ix + 1$, $\Phi''''_{20} = x^4 - ix^3 - x^2 + ix + 1$

$\mathbb{Q}(\zeta_3)$-cyclotomic polynomials. — $\Phi'_3 = x - \zeta_3$, $\Phi''_3 = x - \zeta_3^2$, $\Phi'_6 = x + \zeta_3^2$, $\Phi''_6 = x + \zeta_3$, $\Phi'_9 = x^3 - \zeta_3$, $\Phi''_9 = x^3 - \zeta_3^2$, $\Phi'''_{12} = x^2 + \zeta_3^2$, $\Phi''''_{12} = x^2 + \zeta_3$, $\Phi'''_{15} = x^4 + \zeta_3^2 x^3 + \zeta_3 x^2 + x + \zeta_3^2$, $\Phi''''_{15} = x^4 + \zeta_3 x^3 + \zeta_3^2 x^2 + x + \zeta_3$, $\Phi'_{18} = x^3 + \zeta_3^2$, $\Phi''_{18} = x^3 + \zeta_3$, $\Phi'_{21} = x^6 + \zeta_3 x^5 + \zeta_3^2 x^4 + x^3 + \zeta_3 x^2 + \zeta_3^2 x + 1$, $\Phi''_{21} = x^6 + \zeta_3^2 x^5 + \zeta_3 x^4 + x^3 + \zeta_3^2 x^2 + \zeta_3 x + 1$, $\Phi'_{24} = x^4 + \zeta_3^2$, $\Phi''_{24} = x^4 + \zeta_3$, $\Phi'''_{30} = x^4 - \zeta_3 x^3 + \zeta_3^2 x^2 - x + \zeta_3$, $\Phi''''_{30} = x^4 - \zeta_3^2 x^3 + \zeta_3 x^2 - x + \zeta_3^2$, $\Phi'_{42} = x^6 - \zeta_3^2 x^5 + \zeta_3 x^4 - x^3 + \zeta_3^2 x^2 - \zeta_3 x + 1$, $\Phi''_{42} = x^6 - \zeta_3 x^5 + \zeta_3^2 x^4 - x^3 + \zeta_3 x^2 - \zeta_3^2 x + 1$

$\mathbb{Q}(\sqrt{3})$-cyclotomic polynomials. — $\Phi^{(5)}_{12} = x^2 - \sqrt{3}x + 1$, $\Phi^{(6)}_{12} = x^2 + \sqrt{3}x + 1$

$\mathbb{Q}(\sqrt{5})$-cyclotomic polynomials. — $\Phi'_5 = x^2 + \frac{1-\sqrt{5}}{2}x + 1$, $\Phi''_5 = x^2 + \frac{1+\sqrt{5}}{2}x + 1$, $\Phi'_{10} = x^2 + \frac{-1-\sqrt{5}}{2}x + 1$, $\Phi''_{10} = x^2 + \frac{-1+\sqrt{5}}{2}x + 1$, $\Phi'_{15} = x^4 + \frac{-1-\sqrt{5}}{2}x^3 + \frac{1+\sqrt{5}}{2}x^2 + \frac{-1-\sqrt{5}}{2}x + 1$, $\Phi''_{15} = x^4 + \frac{-1+\sqrt{5}}{2}x^3 + \frac{1-\sqrt{5}}{2}x^2 + \frac{-1+\sqrt{5}}{2}x + 1$, $\Phi'_{30} = x^4 + \frac{1-\sqrt{5}}{2}x^3 + \frac{1-\sqrt{5}}{2}x^2 + \frac{1-\sqrt{5}}{2}x + 1$, $\Phi''_{30} = x^4 + \frac{1+\sqrt{5}}{2}x^3 + \frac{1+\sqrt{5}}{2}x^2 + \frac{1+\sqrt{5}}{2}x + 1$

$\mathbb{Q}(\sqrt{-2})$-cyclotomic polynomials. — $\Phi^{(5)}_8 = x^2 - \sqrt{-2}x - 1$, $\Phi^{(6)}_8 = x^2 + \sqrt{-2}x - 1$, $\Phi^{(7)}_{24} = x^4 + \sqrt{-2}x^3 - x^2 - \sqrt{-2}x + 1$, $\Phi^{(8)}_{24} = x^4 - \sqrt{-2}x^3 - x^2 + \sqrt{-2}x + 1$

$\mathbb{Q}(\sqrt{-7})$-cyclotomic polynomials. — $\Phi'_7 = x^3 + \frac{1-\sqrt{-7}}{2}x^2 + \frac{-1-\sqrt{-7}}{2}x - 1$, $\Phi''_7 = x^3 + \frac{1+\sqrt{-7}}{2}x^2 + \frac{-1+\sqrt{-7}}{2}x - 1$, $\Phi'_{14} = x^3 + \frac{-1+\sqrt{-7}}{2}x^2 + \frac{-1-\sqrt{-7}}{2}x + 1$, $\Phi''_{14} = x^3 + \frac{-1-\sqrt{-7}}{2}x^2 + \frac{-1+\sqrt{-7}}{2}x + 1$

$\mathbb{Q}(\sqrt{6})$-cyclotomic polynomials. — $\Phi^{(5)}_{24} = x^4 - \sqrt{6}x^3 + 3x^2 - \sqrt{6}x + 1$, $\Phi^{(6)}_{24} = x^4 + \sqrt{6}x^3 + 3x^2 + \sqrt{6}x + 1$

$\mathbb{Q}(\zeta_{12})$-cyclotomic polynomials. — $\Phi^{(7)}_{12} = x + \zeta_{12}^7$, $\Phi^{(8)}_{12} = x + \zeta_{12}^{11}$, $\Phi^{(9)}_{12} = x + \zeta_{12}$, $\Phi^{(10)}_{12} = x + \zeta_{12}^5$

$\mathbb{Q}(\sqrt{5}, \zeta_3)$-cyclotomic polynomials. — $\Phi^{(5)}_{15} = x^2 + \frac{(1+\sqrt{5})\zeta_3^2}{2}x + \zeta_3$, $\Phi^{(6)}_{15} = x^2 + \frac{(1-\sqrt{5})\zeta_3^2}{2}x + \zeta_3$, $\Phi^{(7)}_{15} = x^2 + \frac{(1+\sqrt{5})\zeta_3}{2}x + \zeta_3^2$, $\Phi^{(8)}_{15} = x^2 + \frac{(1-\sqrt{5})\zeta_3}{2}x + \zeta_3^2$, $\Phi^{(5)}_{30} = x^2 + \frac{(-1+\sqrt{5})\zeta_3^2}{2}x + \zeta_3$, $\Phi^{(6)}_{30} = x^2 + \frac{(-1-\sqrt{5})\zeta_3^2}{2}x + \zeta_3$, $\Phi^{(7)}_{30} = x^2 + \frac{(-1+\sqrt{5})\zeta_3}{2}x + \zeta_3^2$, $\Phi^{(8)}_{30} = x^2 + \frac{(-1-\sqrt{5})\zeta_3}{2}x + \zeta_3^2$

$\mathbb{Q}(\sqrt{-2}, \zeta_3)$-cyclotomic polynomials. — $\Phi^{(9)}_{24} = x^2 + \sqrt{-2}\zeta_3^2 x - \zeta_3$, $\Phi^{(10)}_{24} = x^2 - \sqrt{-2}\zeta_3^2 x - \zeta_3$, $\Phi^{(11)}_{24} = x^2 + \sqrt{-2}\zeta_3 x - \zeta_3^2$, $\Phi^{(12)}_{24} = x^2 - \sqrt{-2}\zeta_3 x - \zeta_3^2$

A.2. Unipotent characters for Z_3

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$	Symbol
$*\ 1$	1	1	$(1,,)$
Z_3	$\frac{-\sqrt{-3}}{3}x\Phi_1$	ζ_3^2	$(,01,01)$
$\#\zeta_3^2$	$\frac{3+\sqrt{-3}}{6}x\Phi'_3$	1	$(01,0,1)$
$*\ \zeta_3$	$\frac{3-\sqrt{-3}}{6}x\Phi''_3$	1	$(01,1,0)$

To simplify, we used an obvious notation for the characters of the principal series, that is ζ_3 denotes the reflection character, and denoted by Z_3 the unique cuspidal unipotent character. The corresponding symbols are given in the last column.

A.3. Unipotent characters for Z_4

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$	Symbol
$*\ 1$	1	1	$(1,,,)$
-1	$\frac{1}{2}x\Phi_4$	1	$(01,0,1,0)$
$*\ i$	$\frac{-i+1}{4}x\Phi_2\Phi''_4$	1	$(01,1,0,0)$
Z_4^{1022}	$\frac{-i+1}{4}x\Phi_1\Phi'_4$	-1	$(0,,01,01)$
Z_4^{0212}	$\frac{-i}{2}x\Phi_1\Phi_2$	$-i$	$(,01,0,01)$
$\#-i$	$\frac{i+1}{4}x\Phi_2\Phi'_4$	1	$(01,0,0,1)$
Z_4^{1220}	$\frac{-i-1}{4}x\Phi_1\Phi''_4$	-1	$(0,01,01,)$

We used an obvious notation for the characters of the principal series, and the shape of the symbols for the cuspidal characters.

A.4. Unipotent characters for G_4

Some principal ζ-series

$\zeta_4 : \mathcal{H}_{Z_4}(ix^3, i, ix, -i)$
$\zeta_3^2 : \mathcal{H}_{Z_6}(\zeta_3^2 x^2, -\zeta_3^2, \zeta_3, -\zeta_3 x, \zeta_3^2, -\zeta_3^2 x)$
$\zeta_3 : \mathcal{H}_{Z_6}(\zeta_3 x^2, -\zeta_3 x, \zeta_3, -\zeta_3^2 x, \zeta_3^2, -\zeta_3)$

Non-principal 1-Harish-Chandra series

$\mathcal{H}_{G_4}(Z_3) = \mathcal{H}_{A_1}(x^3, -1)$

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$*\ \phi_{1,0}$	1	1
$*\ \phi_{2,1}$	$\frac{3-\sqrt{-3}}{6} x\Phi_3'\Phi_4\Phi_6''$	1
$\#\ \phi_{2,3}$	$\frac{3+\sqrt{-3}}{6} x\Phi_3''\Phi_4\Phi_6'$	1
$Z_3 : 2$	$\frac{-\sqrt{-3}}{3} x\Phi_1\Phi_2\Phi_4$	ζ_3^2
$*\ \phi_{3,2}$	$x^2\Phi_3\Phi_6$	1
$*\ \phi_{1,4}$	$\frac{-\sqrt{-3}}{6} x^4\Phi_3''\Phi_4\Phi_6''$	1
$\phi_{2,5}$	$\frac{1}{2} x^4\Phi_2^2\Phi_6$	1
G_4	$\frac{-1}{2} x^4\Phi_1^2\Phi_3$	-1
$Z_3 : 11$	$\frac{-\sqrt{-3}}{3} x^4\Phi_1\Phi_2\Phi_4$	ζ_3^2
$\#\ \phi_{1,8}$	$\frac{\sqrt{-3}}{6} x^4\Phi_3'\Phi_4\Phi_6'$	1

A.5. Unipotent characters for G_6

Some principal ζ-series

$\zeta_3 : \mathcal{H}_{Z_{12}}(\zeta_3 x^2, -ix, -\zeta_3 x, \zeta_{12}^7 x^{1/2}, x, \zeta_{12} x, -1, ix, \zeta_3 x, \zeta_{12} x^{1/2}, -\zeta_3, \zeta_{12}^7 x)$
$\zeta_3^2 : \mathcal{H}_{Z_{12}}(\zeta_3^2 x^2, \zeta_{12}^5 x, -\zeta_3^2, \zeta_{12}^{11} x^{1/2}, \zeta_3^2 x, -ix, -1, \zeta_{12}^{11} x, x, \zeta_{12}^5 x^{1/2}, -\zeta_3^2 x, ix)$

Non-principal 1-Harish-Chandra series

$\mathcal{H}_{G_6}(Z_3) = \mathcal{H}_{Z_4}(x^3, ix^3, -1, -ix^3)$

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$*\ \phi_{1,0}$	1	1
$*\ \phi_{2,1}$	$\frac{(3-\sqrt{3})(i+1)}{24} x\Phi_2^2\Phi_3'{\Phi_4'}^2\Phi_6\Phi_{12}''\Phi_{12}^{(8)}$	1
$\phi_{3,2}$	$\frac{1}{4} x\Phi_3\Phi_4^2\Phi_{12}$	1
$\phi_{2,3}'$	$\frac{(3+\sqrt{3})(-i+1)}{24} x\Phi_2^2\Phi_3'{\Phi_4''}^2\Phi_6\Phi_{12}'\Phi_{12}^{(10)}$	1
$\phi_{1,4}$	$\frac{-\sqrt{-3}}{12} x\Phi_3''\Phi_4^2\Phi_6''\Phi_{12}$	1
$G_6^2[-i]$	$\frac{\sqrt{3}}{12} x\Phi_1^2\Phi_2^2\Phi_3\Phi_6\Phi_{12}^{(5)}$	$-i$

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$G_6^2[-1]$	$\frac{(3+\sqrt{3})(-i+1)}{24} x\Phi_1^2\Phi_3{\Phi'_4}^2\Phi''_6\Phi''_{12}\Phi_{12}^{(8)}$	-1
$G_6[i]$	$\frac{-i}{4} x\Phi_1^2\Phi_2^2\Phi_3\Phi_6\Phi'_{12}$	i
$G_6[-\zeta_3^2]$	$\frac{-\sqrt{-3}(i+1)}{12} x\Phi_1^2\Phi_2\Phi_3\Phi_4\Phi'_4\Phi''_{12}$	$-\zeta_3^2$
$Z_3 : 1$	$\frac{-\sqrt{-3}}{6} x\Phi_1\Phi_2\Phi_4^2\Phi_{12}$	ζ_3^2
$G_6^2[-\zeta_3^2]$	$\frac{-\sqrt{3}(i+1)}{12} x\Phi_1^2\Phi_2\Phi_3\Phi_4\Phi''_4\Phi'_{12}$	$-\zeta_3^2$
# $\phi''_{2,3}$	$\frac{(3-\sqrt{3})(-i+1)}{24} x\Phi_2^2\Phi''_3{\Phi''_4}^2\Phi_6\Phi'_{12}\Phi_{12}^{(9)}$	1
$G_6^2[i]$	$\frac{-i}{4} x\Phi_1^2\Phi_2^2\Phi_3\Phi_6\Phi''_{12}$	i
$G_6^3[-1]$	$\frac{(-3-\sqrt{3})(i+1)}{24} x\Phi_1^2\Phi_3{\Phi''_4}^2\Phi'_6\Phi'_{12}\Phi_{12}^{(9)}$	-1
$\phi_{1,8}$	$\frac{\sqrt{-3}}{12} x\Phi'_3\Phi_4^2\Phi'_6\Phi_{12}$	1
$\phi'_{2,5}$	$\frac{(3+\sqrt{3})(i+1)}{24} x\Phi_2^2\Phi''_3{\Phi'_4}^2\Phi_6\Phi''_{12}\Phi_{12}^{(7)}$	1
$Z_3 : i$	$\frac{-\sqrt{3}(i+1)}{12} x\Phi_1\Phi_2^2\Phi_4\Phi'_4\Phi_6\Phi''_{12}$	ζ_3^2
$G_6[\zeta_{12}^5]$	$\frac{\sqrt{3}}{6} x\Phi_1^2\Phi_2^2\Phi_3\Phi_4\Phi_6$	ζ_{12}^5
$G_6[-1]$	$\frac{(-3+\sqrt{3})(i+1)}{24} x\Phi_1^2\Phi_3{\Phi''_4}^2\Phi''_6\Phi'_{12}\Phi_{12}^{(10)}$	-1
$\phi_{1,6}$	$\frac{1}{4} x\Phi_4^2\Phi_6\Phi_{12}$	1
$G_6[-i]$	$\frac{\sqrt{3}}{12} x\Phi_1^2\Phi_2^2\Phi_3\Phi_6\Phi_{12}^{(6)}$	$-i$
$Z_3 : -i$	$\frac{-\sqrt{-3}(i+1)}{12} x\Phi_1\Phi_2^2\Phi_4\Phi''_4\Phi_6\Phi'_{12}$	ζ_3^2
$G_6^4[-1]$	$\frac{(-3+\sqrt{3})(-i+1)}{24} x\Phi_1^2\Phi_3{\Phi'_4}^2\Phi'_6\Phi''_{12}\Phi_{12}^{(7)}$	-1
* $\phi_{3,4}$	$x^4\Phi_3\Phi_6\Phi_{12}$	1
$G_6[\zeta_8^3]$	$\frac{-i}{2} x^5\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	$\zeta_8^3 x^{1/2}$
* $\phi''_{2,5}$	$\frac{1}{2} x^5\Phi_4^2\Phi_{12}$	1
$G_6[\zeta_8^7]$	$\frac{-i}{2} x^5\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	$\zeta_8^7 x^{1/2}$
$\phi_{2,7}$	$\frac{1}{2} x^5\Phi_4^2\Phi_{12}$	1
* $\phi_{1,10}$	$\frac{3-\sqrt{-3}}{6} x^{10}\Phi''_3\Phi'_6\Phi''''_{12}$	1
# $\phi_{1,14}$	$\frac{3+\sqrt{-3}}{6} x^{10}\Phi'_3\Phi''_6\Phi'''_{12}$	1
$Z_3 : -1$	$\frac{-\sqrt{-3}}{3} x^{10}\Phi_1\Phi_2\Phi_4$	ζ_3^2

A.6. Unipotent characters for G_8

Some principal ζ-series

$\zeta_8 : \mathcal{H}_{Z_8}(\zeta_8^5 x^3, \zeta_8, \zeta_8 x, \zeta_8^3, \zeta_8^3 x, \zeta_8^5, \zeta_8^5 x, \zeta_8^7)$
$\zeta_3 : \mathcal{H}_{Z_{12}}(\zeta_3 x^2, \zeta_{12}, -\zeta_3^2, \zeta_{12}^7 x^{1/2}, \zeta_3, \zeta_{12} x, -1, \zeta_{12}^7, \zeta_3 x, \zeta_{12} x^{1/2}, -\zeta_3, \zeta_{12}^7 x)$
$\zeta_3^2 : \mathcal{H}_{Z_{12}}(\zeta_3^2 x^2, \zeta_{12}^5 x, -\zeta_3^2, \zeta_{12}^{11} x^{1/2}, \zeta_3^2 x, \zeta_{12}^5, -1, \zeta_{12}^{11} x, \zeta_3^2, \zeta_{12}^5 x^{1/2}, -\zeta_3, \zeta_{12}^{11})$

Non-principal 1-Harish-Chandra series

$\mathcal{H}_{G_8}(Z_4^{1220}) = \mathcal{H}_{Z_4}(x^3, i, -1, -ix^2)$
$\mathcal{H}_{G_8}(Z_4^{0212}) = \mathcal{H}_{Z_4}(x^3, i, -x^2, -i)$
$\mathcal{H}_{G_8}(Z_4^{1022}) = \mathcal{H}_{Z_4}(x^3, ix^2, -1, -i)$

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
* $\phi_{1,0}$	1	1
* $\phi_{2,1}$	$\frac{-i+1}{4}x\Phi_2\Phi'_4\Phi_6\Phi_8\Phi''_{12}$	1
$\phi_{2,4}$	$\frac{1}{2}x\Phi_4\Phi_8\Phi_{12}$	1
# $\phi'_{2,7}$	$\frac{i+1}{4}x\Phi_2\Phi''_4\Phi_6\Phi_8\Phi'_{12}$	1
$Z_4^{1220} : 1$	$\frac{-i-1}{4}x\Phi_1\Phi_3\Phi'_4\Phi_8\Phi''_{12}$	-1
$Z_4^{0212} : 1$	$\frac{-i}{2}x\Phi_1\Phi_2\Phi_3\Phi_6\Phi_8$	$-i$
$Z_4^{1022} : 1$	$\frac{-i+1}{4}x\Phi_1\Phi_3\Phi''_4\Phi_8\Phi'_{12}$	-1
* $\phi_{3,2}$	$\frac{-i+1}{4}x^2\Phi_3\Phi_4\Phi_6\Phi''_8\Phi_{12}$	1
$\phi_{3,4}$	$\frac{1}{2}x^2\Phi_3\Phi_6\Phi_8\Phi_{12}$	1
# $\phi_{3,6}$	$\frac{i+1}{4}x^2\Phi_3\Phi_4\Phi_6\Phi'_8\Phi_{12}$	1
$Z_4^{1022} : i$	$\frac{-i-1}{4}x^2\Phi_1\Phi_2\Phi_3\Phi_6\Phi''_8\Phi_{12}$	-1
$Z_4^{0212} : -1$	$\frac{-i}{2}x^2\Phi_1\Phi_2\Phi_3\Phi_4\Phi_6\Phi_{12}$	$-i$
$Z_4^{1220} : -i$	$\frac{-i+1}{4}x^2\Phi_1\Phi_2\Phi_3\Phi_6\Phi'_8\Phi_{12}$	-1
$G_8[\zeta_8^3]$	$\frac{-i}{2}x^3\Phi_1^2\Phi_2^2\Phi_3\Phi_6\Phi_8$	$\zeta_8^3 x^{1/2}$
* $\phi_{4,3}$	$\frac{1}{2}x^3\Phi_4^2\Phi_8\Phi_{12}$	1
$G_8[\zeta_8^7]$	$\frac{-i}{2}x^3\Phi_1^2\Phi_2^2\Phi_3\Phi_6\Phi_8$	$\zeta_8^7 x^{1/2}$
$\phi_{4,5}$	$\frac{1}{2}x^3\Phi_4^2\Phi_8\Phi_{12}$	1
* $\phi_{1,6}$	$\frac{-1}{12}x^6\Phi_3{\Phi''_4}^2\Phi_6\Phi_8\Phi''_{12}$	1
# $\phi_{1,18}$	$\frac{-1}{12}x^6\Phi_3{\Phi'_4}^2\Phi_6\Phi_8\Phi'_{12}$	1
$\phi_{1,12}$	$\frac{1}{4}x^6\Phi_4\Phi_6\Phi_8\Phi_{12}$	1
$\phi''_{2,7}$	$\frac{-i}{4}x^6\Phi_2\Phi_4\Phi''_4\Phi_6\Phi''_8\Phi_{12}$	1
$\phi_{2,13}$	$\frac{i}{4}x^6\Phi_2\Phi_4\Phi'_4\Phi_6\Phi'_8\Phi_{12}$	1
$\phi_{2,10}$	$\frac{1}{12}x^6\Phi_2^2\Phi_3\Phi_8\Phi_{12}$	1

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$\phi_{3,8}$	$\frac{1}{4}x^6\Phi_3\Phi_4\Phi_8\Phi_{12}$	1
$Z_4^{1220}:-1$	$\frac{-i}{4}x^6\Phi_1\Phi_3\Phi_4\Phi_4''\Phi_8'\Phi_{12}$	-1
$Z_4^{1022}:-1$	$\frac{-i}{4}x^6\Phi_1\Phi_3\Phi_4\Phi_4'\Phi_8''\Phi_{12}$	-1
$Z_4^{1220}:i$	$\frac{-1}{4}x^6\Phi_1\Phi_2\Phi_3\Phi_6\Phi_8\Phi_{12}''$	-1
$Z_4^{1022}:-i$	$\frac{1}{4}x^6\Phi_1\Phi_2\Phi_3\Phi_6\Phi_8\Phi_{12}'$	-1
$Z_4^{0212}:i$	$\frac{-i}{4}x^6\Phi_1\Phi_2^2\Phi_3\Phi_4'\Phi_6\Phi_8'\Phi_{12}''$	$-i$
$Z_4^{0212}:-i$	$\frac{-i}{4}x^6\Phi_1\Phi_2^2\Phi_3\Phi_4''\Phi_6\Phi_8''\Phi_{12}'$	$-i$
$G_8[1]$	$\frac{1}{12}x^6\Phi_1^2\Phi_6\Phi_8\Phi_{12}$	1
$G_8[i]$	$\frac{i}{4}x^6\Phi_1^2\Phi_2\Phi_3\Phi_4'\Phi_6\Phi_8''\Phi_{12}''$	i
$G_8^2[i]$	$\frac{-i}{4}x^6\Phi_1^2\Phi_2\Phi_3\Phi_4''\Phi_6\Phi_8'\Phi_{12}'$	i
$G_8[\zeta_3]$	$\frac{-1}{3}x^6\Phi_1^2\Phi_2^2\Phi_4^2\Phi_8$	ζ_3
$G_8[\zeta_3^2]$	$\frac{-1}{3}x^6\Phi_1^2\Phi_2^2\Phi_4^2\Phi_8$	ζ_3^2

A.7. Unipotent characters for G_{14}

Some principal ζ-series

ζ_4 : $\mathcal{H}_{Z_{24}}(-x^2, \zeta_{24}^{19}x, -\zeta_3 x, x^{1/2}, \zeta_3^2 x^{2/3}, \zeta_{24}^{23}x, x, \zeta_{24}x, \zeta_3, ix^{1/2}, -\zeta_3^2 x, \zeta_{24}^5 x, x^{2/3}, \zeta_{24}^7 x, \zeta_3 x, -x^{1/2}, \zeta_3^2, \zeta_{24}^{11}x, -x, \zeta_{24}^{13}x, \zeta_3 x^{2/3}, -ix^{1/2}, \zeta_3^2 x, \zeta_{24}^{17}x)$

ζ_8 : $\mathcal{H}_{Z_{24}}(-ix^2, \zeta_{12}^{11}x, \zeta_{12}, \zeta_{16}x^{1/2}, \zeta_{12}^5 x^{2/3}, \zeta_{12}x, \zeta_{16}^3 x^{1/2}, -\zeta_3^2 x, \zeta_{24}^5 x, ix, \zeta_{12}^5, \zeta_3 x, -ix^{2/3}, \zeta_{12}^5 x, \zeta_{24}^{11}x, \zeta_{16}^9 x^{1/2}, \zeta_{24}^{13}x, \zeta_{12}^7 x, \zeta_{16}^{11}x^{1/2}, \zeta_3^2 x, \zeta_{12}x^{2/3}, -ix, \zeta_{24}^{19}x, -\zeta_3 x)$

ζ_8^3 : $\mathcal{H}_{Z_{24}}(ix^2, \zeta_3^2 x, \zeta_{24}^{17}x, -ix, \zeta_{12}^{11}x^{2/3}, -\zeta_3 x, \zeta_{16}x^{1/2}, \zeta_{12}^{11}x, \zeta_{24}^{23}x, \zeta_{16}^3 x^{1/2}, \zeta_{24}x, \zeta_{12}x, ix^{2/3}, -\zeta_3^2 x, \zeta_{12}^7, ix, \zeta_{24}^7 x, \zeta_3 x, \zeta_{16}^9 x^{1/2}, \zeta_{12}^5 x, \zeta_{12}^7 x^{2/3}, \zeta_{16}^{11}x^{1/2}, \zeta_{12}^{11}, \zeta_{12}^7 x)$

Non-principal 1-Harish-Chandra series

$\mathcal{H}_{G_{14}}(Z_3) = \mathcal{H}_{Z_6}(x^3, -\zeta_3^2 x^4, \zeta_3 x^4, -1, \zeta_3^2 x^4, -\zeta_3 x^4)$

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$*\quad \phi_{1,0}$	1	1
$Z_3 : -\zeta_3^2$	$\frac{-\zeta_3}{6}x\Phi_1\Phi_2^2\Phi_3''\Phi_4{\Phi_6'}^2\Phi_8\Phi_{12}''''\Phi_{24}$	ζ_3^2
$G_{14}^2[\zeta_3^2]$	$\frac{\zeta_3}{6}x\Phi_1^2\Phi_2{\Phi_3''}^2\Phi_4\Phi_6'\Phi_8\Phi_{12}''''\Phi_{24}$	ζ_3^2
$Z_3 : \zeta_3^2$	$\frac{-\zeta_3}{6}x\Phi_1\Phi_2^2\Phi_3''\Phi_4\Phi_6^2\Phi_8\Phi_{12}\Phi_{24}'$	ζ_3^2
$G_{14}^2[-\zeta_3^2]$	$\frac{\zeta_3}{6}x\Phi_1^2\Phi_2\Phi_3^2\Phi_4\Phi_6'\Phi_8\Phi_{12}\Phi_{24}'$	$-\zeta_3^2$
$\phi_{2,4}$	$\frac{-\zeta_3}{12}x\Phi_2^2{\Phi_3'}^2{\Phi_6''}^2\Phi_8\Phi_{12}\Phi_{24}$	1
$\phi_{1,16}$	$\frac{\zeta_3}{12}x{\Phi_3'}^2\Phi_4\Phi_6^2\Phi_8\Phi_{12}'''\Phi_{24}$	1
$G_{14}[1]$	$\frac{-\zeta_3}{12}x\Phi_1^2{\Phi_3'}^2{\Phi_6''}^2\Phi_8\Phi_{12}\Phi_{24}$	1
$\phi_{3,4}$	$\frac{-\zeta_3}{12}x\Phi_3^2\Phi_4{\Phi_6''}^2\Phi_8\Phi_{12}'''\Phi_{24}$	1
$G_{14}[\zeta_8]$	$\frac{-\sqrt{6}\zeta_3}{12}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_4\Phi_6^2\Phi_{12}\Phi_{24}''$	ζ_8
$G_{14}[\zeta_8^3]$	$\frac{-\sqrt{6}\zeta_3}{12}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_4\Phi_6^2\Phi_{12}\Phi_{24}''$	ζ_8^3
$\phi_{2,7}$	$\frac{(-2-\sqrt{6})\zeta_3}{24}x\Phi_2^2{\Phi_3'}^2\Phi_4\Phi_6^2\Phi_8^{(6)}\Phi_{12}\Phi_{24}''\Phi_{24}^{(10)}$	1
$*\quad \phi_{2,1}$	$\frac{(-2+\sqrt{6})\zeta_3}{24}x\Phi_2^2{\Phi_3'}^2\Phi_4\Phi_6^2\Phi_8^{(5)}\Phi_{12}\Phi_{24}''\Phi_{24}^{(9)}$	1
$G_{14}^2[-1]$	$\frac{(-2-\sqrt{6})\zeta_3}{24}x\Phi_1^2\Phi_3^2\Phi_4{\Phi_6''}^2\Phi_8^{(5)}\Phi_{12}\Phi_{24}''\Phi_{24}^{(9)}$	-1
$G_{14}[-1]$	$\frac{(2-\sqrt{6})\zeta_3}{24}x\Phi_1^2\Phi_3^2\Phi_4{\Phi_6''}^2\Phi_8^{(6)}\Phi_{12}\Phi_{24}''\Phi_{24}^{(10)}$	-1
$G_{14}^2[i]$	$\frac{-\sqrt{6}\zeta_3}{24}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_8^{(6)}\Phi_{12}'''\Phi_{24}''\Phi_{24}^{(9)}$	i
$G_{14}^3[-i]$	$\frac{\sqrt{6}\zeta_3}{24}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_8^{(5)}\Phi_{12}'''\Phi_{24}''\Phi_{24}^{(10)}$	$-i$
$G_{14}^2[-i]$	$\frac{-\sqrt{6}\zeta_3}{24}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_8^{(6)}\Phi_{12}'''\Phi_{24}''\Phi_{24}^{(9)}$	$-i$
$G_{14}^3[i]$	$\frac{\sqrt{6}\zeta_3}{24}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_8^{(5)}\Phi_{12}'''\Phi_{24}''\Phi_{24}^{(10)}$	i
$Z_3 : -\zeta_3$	$\frac{\zeta_3^2}{6}x\Phi_1\Phi_2^2\Phi_3'\Phi_4{\Phi_6''}^2\Phi_8\Phi_{12}'''\Phi_{24}$	ζ_3^2
$G_{14}[\zeta_3^2]$	$\frac{\zeta_3^2}{6}x\Phi_1^2\Phi_2{\Phi_3'}^2\Phi_4\Phi_6''\Phi_8\Phi_{12}'''\Phi_{24}$	ζ_3^2
$Z_3 : \zeta_3$	$\frac{\zeta_3^2}{6}x\Phi_1\Phi_2^2\Phi_3'\Phi_4\Phi_6^2\Phi_8\Phi_{12}\Phi_{24}''$	ζ_3^2

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$G_{14}[-\zeta_3^2]$	$\frac{\zeta_3^2}{6}x\Phi_1^2\Phi_2\Phi_3^2\Phi_4\Phi''_6\Phi_8\Phi_{12}\Phi''_{24}$	$-\zeta_3^2$
$\phi_{1,8}$	$\frac{\zeta_3^2}{12}x{\Phi''_3}^2\Phi_4\Phi_6^2\Phi_8\Phi''''_{12}\Phi_{24}$	1
$\phi_{2,8}$	$\frac{-\zeta_3^2}{12}x\Phi_2^2{\Phi''_3}^2{\Phi'_6}^2\Phi_8\Phi_{12}\Phi_{24}$	1
$\phi_{3,2}$	$\frac{-\zeta_3^2}{12}x\Phi_3^2\Phi_4{\Phi'_6}^2\Phi_8\Phi''''_{12}\Phi_{24}$	1
$G_{14}^2[1]$	$\frac{-\zeta_3^2}{12}x\Phi_1^2{\Phi''_3}^2{\Phi'_6}^2\Phi_8\Phi_{12}\Phi_{24}$	1
$G_{14}^2[\zeta_8]$	$\frac{\sqrt{6}\zeta_3^2}{12}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_4\Phi_6^2\Phi_{12}\Phi'_{24}$	ζ_8
$G_{14}^2[\zeta_8^3]$	$\frac{\sqrt{6}\zeta_3^2}{12}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_4\Phi_6^2\Phi_{12}\Phi'_{24}$	ζ_8^3
$\phi_{2,11}$	$\frac{(-2-\sqrt{6})\zeta_3^2}{24}x\Phi_2^2{\Phi''_3}^2\Phi_4\Phi_6^2\Phi_8^{(5)}\Phi_{12}\Phi'_{24}\Phi_{24}^{(11)}$	1
# $\phi_{2,5}$	$\frac{(-2+\sqrt{6})\zeta_3^2}{24}x\Phi_2^2{\Phi''_3}^2\Phi_4\Phi_6^2\Phi_8^{(6)}\Phi_{12}\Phi'_{24}\Phi_{24}^{(12)}$	1
$G_{14}^4[-1]$	$\frac{(2+\sqrt{6})\zeta_3^2}{24}x\Phi_1^2\Phi_3^2\Phi_4{\Phi'_6}^2\Phi_8^{(6)}\Phi_{12}\Phi'_{24}\Phi_{24}^{(12)}$	-1
$G_{14}^3[-1]$	$\frac{(2-\sqrt{6})\zeta_3^2}{24}x\Phi_1^2\Phi_3^2\Phi_4{\Phi'_6}^2\Phi_8^{(5)}\Phi_{12}\Phi'_{24}\Phi_{24}^{(11)}$	-1
$G_{14}[i]$	$\frac{\sqrt{6}\zeta_3^2}{24}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_8^{(6)}\Phi''''_{12}\Phi'_{24}\Phi_{24}^{(11)}$	i
$G_{14}^4[-i]$	$\frac{\sqrt{6}\zeta_3^2}{24}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_8^{(5)}\Phi''''_{12}\Phi'_{24}\Phi_{24}^{(12)}$	$-i$
$G_{14}[-i]$	$\frac{\sqrt{6}\zeta_3^2}{24}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_8^{(6)}\Phi''''_{12}\Phi'_{24}\Phi_{24}^{(11)}$	$-i$
$G_{14}^4[i]$	$\frac{\sqrt{6}\zeta_3^2}{24}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_8^{(5)}\Phi''''_{12}\Phi'_{24}\Phi_{24}^{(12)}$	i
$\phi_{1,12}$	$\frac{1}{6}x\Phi_3\Phi_6^2\Phi_8\Phi_{12}\Phi_{24}$	1
$\phi'_{3,6}$	$\frac{1}{6}x\Phi_3^2\Phi_6\Phi_8\Phi_{12}\Phi_{24}$	1
$\phi_{4,3}$	$\frac{1}{6}x\Phi_2^2\Phi_3\Phi_4\Phi_6^2\Phi_{12}\Phi_{24}$	1
$G_{14}^5[-1]$	$\frac{1}{6}x\Phi_1^2\Phi_3^2\Phi_4\Phi_6\Phi_{12}\Phi_{24}$	-1
$G_{14}^2[\zeta_3]$	$\frac{1}{12}x\Phi_1^2\Phi_2^2\Phi_4{\Phi'_6}^2\Phi_8\Phi'''_{12}\Phi_{24}$	ζ_3
$G_{14}[\zeta_3]$	$\frac{1}{12}x\Phi_1^2\Phi_2^2\Phi_4{\Phi''_6}^2\Phi_8\Phi''''_{12}\Phi_{24}$	ζ_3
$G_{14}^4[\zeta_3]$	$\frac{-1}{12}x\Phi_1^2\Phi_2^2{\Phi''_3}^2\Phi_4\Phi_8\Phi'''_{12}\Phi_{24}$	ζ_3
$G_{14}^3[\zeta_3]$	$\frac{-1}{12}x\Phi_1^2\Phi_2^2{\Phi'_3}^2\Phi_4\Phi_8\Phi''''_{12}\Phi_{24}$	ζ_3
$G_{14}[\zeta_{24}^{11}]$	$\frac{\sqrt{6}}{12}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_4\Phi_6^2\Phi_8\Phi_{12}$	ζ_{24}^{11}
$G_{14}[\zeta_{24}^{17}]$	$\frac{\sqrt{6}}{12}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_4\Phi_6^2\Phi_8\Phi_{12}$	ζ_{24}^{17}
$G_{14}^6[\zeta_3]$	$\frac{2+\sqrt{6}}{24}x\Phi_1^2\Phi_2^2\Phi_4\Phi_6^2\Phi_8\Phi_{12}\Phi_{24}^{(6)}$	ζ_3
$G_{14}^5[\zeta_3]$	$\frac{2-\sqrt{6}}{24}x\Phi_1^2\Phi_2^2\Phi_4\Phi_6^2\Phi_8\Phi_{12}\Phi_{24}^{(5)}$	ζ_3
$G_{14}^2[-\zeta_3]$	$\frac{2+\sqrt{6}}{24}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_4\Phi_8\Phi_{12}\Phi_{24}^{(5)}$	$-\zeta_3$
$G_{14}[-\zeta_3]$	$\frac{-2+\sqrt{6}}{24}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_4\Phi_8\Phi_{12}\Phi_{24}^{(6)}$	$-\zeta_3$
$G_{14}[\zeta_{12}^7]$	$\frac{\sqrt{6}}{24}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_4\Phi_6^2\Phi_8\Phi_{24}^{(8)}$	ζ_{12}^7
$G_{14}^2[\zeta_{12}]$	$\frac{\sqrt{6}}{24}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_4\Phi_6^2\Phi_8\Phi_{24}^{(7)}$	ζ_{12}
$G_{14}[\zeta_{12}]$	$\frac{\sqrt{6}}{24}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_4\Phi_6^2\Phi_8\Phi_{24}^{(8)}$	ζ_{12}
$G_{14}^2[\zeta_{12}^7]$	$\frac{\sqrt{6}}{24}x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_4\Phi_6^2\Phi_8\Phi_{24}^{(7)}$	ζ_{12}^7
* $\phi_{4,5}$	$\frac{3-\sqrt{-3}}{6}x^5\Phi'_3\Phi_4\Phi''_6\Phi_8\Phi_{12}\Phi_{24}$	1
# $\phi_{4,7}$	$\frac{3+\sqrt{-3}}{6}x^5\Phi''_3\Phi_4\Phi'_6\Phi_8\Phi_{12}\Phi_{24}$	1
$Z_3 : 1$	$\frac{-\sqrt{-3}}{3}x^5\Phi_1\Phi_2\Phi_4\Phi_8\Phi_{12}\Phi_{24}$	ζ_3^2

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
* $\phi''_{3,6}$	$\frac{1}{3}x^6\Phi_3^2\Phi_6^2\Phi_{12}\Phi_{24}$	1
$G_{14}^2[\zeta_9^5]$	$\frac{\zeta_3^2}{3}x^6\Phi_1^2\Phi_2^2{\Phi'_3}^2\Phi_4{\Phi''_6}^2\Phi_8\Phi'''_{12}\Phi''_{24}$	$\zeta_9^5x^{2/3}$
$G_{14}[\zeta_9^2]$	$\frac{\zeta_3}{3}x^6\Phi_1^2\Phi_2^2{\Phi''_3}^2\Phi_4{\Phi'_6}^2\Phi_8\Phi''''_{12}\Phi'_{24}$	$\zeta_9^2x^{1/3}$
# $\phi_{3,8}$	$\frac{1}{3}x^6\Phi_3^2\Phi_6^2\Phi_{12}\Phi_{24}$	1
$G_{14}^2[\zeta_9^8]$	$\frac{\zeta_3^2}{3}x^6\Phi_1^2\Phi_2^2{\Phi'_3}^2\Phi_4{\Phi''_6}^2\Phi_8\Phi'''_{12}\Phi''_{24}$	$\zeta_9^8x^{2/3}$
$G_{14}[\zeta_9^8]$	$\frac{\zeta_3}{3}x^6\Phi_1^2\Phi_2^2{\Phi''_3}^2\Phi_4{\Phi'_6}^2\Phi_8\Phi''''_{12}\Phi'_{24}$	$\zeta_9^8x^{1/3}$
$\phi_{3,10}$	$\frac{1}{3}x^6\Phi_3^2\Phi_6^2\Phi_{12}\Phi_{24}$	1
$G_{14}^2[\zeta_9^2]$	$\frac{\zeta_3^2}{3}x^6\Phi_1^2\Phi_2^2{\Phi'_3}^2\Phi_4{\Phi''_6}^2\Phi_8\Phi'''_{12}\Phi''_{24}$	$\zeta_9^2x^{2/3}$
$G_{14}[\zeta_9^5]$	$\frac{\zeta_3}{3}x^6\Phi_1^2\Phi_2^2{\Phi''_3}^2\Phi_4{\Phi'_6}^2\Phi_8\Phi''''_{12}\Phi'_{24}$	$\zeta_9^5x^{1/3}$
$\phi_{2,15}$	$\frac{1}{4}x^9\Phi_2^2\Phi_6^2\Phi_8\Phi_{24}$	1
* $\phi_{2,9}$	$\frac{1}{4}x^9\Phi_2^2\Phi_6^2\Phi_8\Phi_{24}$	1
$\phi_{2,12}$	$\frac{1}{2}x^9\Phi_4\Phi_8\Phi_{12}\Phi_{24}$	1
$G_{14}^6[-1]$	$\frac{1}{4}x^9\Phi_1^2\Phi_3^2\Phi_8\Phi_{24}$	-1
$G_{14}^7[-1]$	$\frac{1}{4}x^9\Phi_1^2\Phi_3^2\Phi_8\Phi_{24}$	-1
$G_{14}[\zeta_{16}^5]$	$\frac{-\sqrt{-2}}{4}x^9\Phi_1^2\Phi_2^2\Phi_3^2\Phi_4\Phi_6^2\Phi_{12}$	$\zeta_{16}^5x^{1/2}$
$G_{14}[\zeta_{16}^7]$	$\frac{-\sqrt{-2}}{4}x^9\Phi_1^2\Phi_2^2\Phi_3^2\Phi_4\Phi_6^2\Phi_{12}$	$\zeta_{16}^7x^{1/2}$
$G_{14}[\zeta_{16}^{13}]$	$\frac{-\sqrt{-2}}{4}x^9\Phi_1^2\Phi_2^2\Phi_3^2\Phi_4\Phi_6^2\Phi_{12}$	$\zeta_{16}^{13}x^{1/2}$
$G_{14}[\zeta_{16}^{15}]$	$\frac{-\sqrt{-2}}{4}x^9\Phi_1^2\Phi_2^2\Phi_3^2\Phi_4\Phi_6^2\Phi_{12}$	$\zeta_{16}^{15}x^{1/2}$
* $\phi_{1,20}$	$\frac{3-\sqrt{-3}}{6}x^{20}\Phi'_3\Phi''_6\Phi'''_{12}\Phi''_{24}$	1
# $\phi_{1,28}$	$\frac{3+\sqrt{-3}}{6}x^{20}\Phi''_3\Phi'_6\Phi''''_{12}\Phi'_{24}$	1
$Z_3:-1$	$\frac{-\sqrt{-3}}{3}x^{20}\Phi_1\Phi_2\Phi_4\Phi_8$	ζ_3^2

A.8. Unipotent characters for $G_{3,1,2}$

Some principal ζ-series

$-1 : \mathcal{H}_{Z_6}(x^2, \zeta_3^2 x, \zeta_3, x, \zeta_3^2, \zeta_3 x)$

Non-principal 1-Harish-Chandra series

$\mathcal{H}_{G_{3,1,2}}(Z_3) = \mathcal{H}_{Z_3}(1, \zeta_3 x^2, \zeta_3^2 x^2)$

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$	Symbol
11..	$\frac{1}{3}x\Phi_3\Phi_6$	1	$(12, 0, 0)$
$Z_3 : \zeta_3^2$	$\frac{\zeta_3^2}{3}x\Phi_1\Phi_2\Phi_3'\Phi_6''$	ζ_3^2	$(, 01, 02)$
$Z_3 : \zeta_3$	$\frac{-\zeta_3}{3}x\Phi_1\Phi_2\Phi_3''\Phi_6'$	ζ_3^2	$(, 02, 01)$
..2	$\frac{-\zeta_3^2}{3}x{\Phi_3'}^2\Phi_6$	1	$(01, 0, 2)$
$G_{3,1,2}^{130}$	$\frac{-\zeta_3}{3}x\Phi_1^2\Phi_2\Phi_6''$	ζ_3	$(0, 012,)$
# 1..1	$\frac{1}{3}x\Phi_2{\Phi_3''}^2\Phi_6'$	1	$(02, 0, 1)$
.2.	$\frac{-\zeta_3}{3}x{\Phi_3''}^2\Phi_6$	1	$(01, 2, 0)$
$*$ 1.1.	$\frac{1}{3}x\Phi_2{\Phi_3'}^2\Phi_6''$	1	$(02, 1, 0)$
$G_{3,1,2}^{103}$	$\frac{-\zeta_3^2}{3}x\Phi_1^2\Phi_2\Phi_6'$	ζ_3	$(0, , 012)$
$*$ .1.1	$x^3\Phi_2\Phi_6$	1	$(01, 1, 1)$
$*$ 2..	1	1	$(2, ,)$
$Z_3 : 1$	$\frac{-\sqrt{-3}}{3}x^5\Phi_1\Phi_2$	ζ_3^2	$(1, 012, 012)$
# ..11	$\frac{3+\sqrt{-3}}{6}x^5\Phi_3''\Phi_6'$	1	$(012, 01, 12)$
$*$ .11.	$\frac{3-\sqrt{-3}}{6}x^5\Phi_3'\Phi_6''$	1	$(012, 12, 01)$

We used partition tuples for the principal series, the shape of the symbol for cuspidals and notation coming the relative group Z_3 for the characters Harish-Chandra induced from Z_3.

A.9. Unipotent characters for G_{24}

Some principal ζ-series

$\zeta_3^2 : \mathcal{H}_{Z_6}(\zeta_3 x^7, -\zeta_3 x^{7/2}, \zeta_3 x^3, -\zeta_3 x^4, \zeta_3 x^{7/2}, -\zeta_3)$
$\zeta_3 : \mathcal{H}_{Z_6}(\zeta_3^2 x^7, -\zeta_3^2, \zeta_3^2 x^{7/2}, -\zeta_3^2 x^4, \zeta_3^2 x^3, -\zeta_3^2 x^{7/2})$
$\zeta_7^4 : \mathcal{H}_{Z_{14}}(\zeta_7^2 x^3, -\zeta_7^3 x^2, \zeta_7^2 x^{3/2}, -\zeta_7^4 x^2, \zeta_7^5 x, -\zeta_7^2 x, \zeta_7^2 x^2, -\zeta_7^6 x^2, x, -\zeta_7^2 x^{3/2}, \zeta_7 x, -\zeta_7^2, \zeta_7^2 x, -\zeta_7^2 x^2)$
$\zeta_7 : \mathcal{H}_{Z_{14}}(\zeta_7^4 x^3, -\zeta_7^4, x, -\zeta_7^4 x, \zeta_7^4 x^{3/2}, -\zeta_7^4 x^2, \zeta_7^2 x, -\zeta_7^5 x^2, \zeta_7^3 x, -\zeta_7^6 x^2, \zeta_7^4 x, -\zeta_7^4 x^{3/2}, \zeta_7^4 x^2, -\zeta_7 x^2)$

Non-principal 1-Harish-Chandra series

$\mathcal{H}_{G_{24}}(B_2) = \mathcal{H}_{A_1}(x^7, -1)$

γ	Deg(γ)	Fr(γ)
$*\ \phi_{1,0}$	1	1
$*\ \phi_{3,1}$	$\frac{\sqrt{-7}}{14} x\Phi_3\Phi_4\Phi_6\Phi'_7\Phi''_{14}$	1
$\#\ \phi_{3,3}$	$\frac{-\sqrt{-7}}{14} x\Phi_3\Phi_4\Phi_6\Phi''_7\Phi'_{14}$	1
$\phi_{6,2}$	$\frac{1}{2} x\Phi_2^2\Phi_3\Phi_6\Phi_{14}$	1
$B_2 : 2$	$\frac{1}{2} x\Phi_1^2\Phi_3\Phi_6\Phi_7$	-1
$G_{24}[\zeta_7^4]$	$\frac{\sqrt{-7}}{7} x\Phi_1^3\Phi_2^3\Phi_3\Phi_4\Phi_6$	ζ_7^4
$G_{24}[\zeta_7^2]$	$\frac{\sqrt{-7}}{7} x\Phi_1^3\Phi_2^3\Phi_3\Phi_4\Phi_6$	ζ_7^2
$G_{24}[\zeta_7]$	$\frac{\sqrt{-7}}{7} x\Phi_1^3\Phi_2^3\Phi_3\Phi_4\Phi_6$	ζ_7
$*\ \phi_{7,3}$	$x^3\Phi_7\Phi_{14}$	1
$*\ \phi_{8,4}$	$\frac{1}{2} x^4\Phi_2^3\Phi_4\Phi_6\Phi_{14}$	1
$\#\ \phi_{8,5}$	$\frac{1}{2} x^4\Phi_2^3\Phi_4\Phi_6\Phi_{14}$	1
$G_{24}[i]$	$\frac{1}{2} x^4\Phi_1^3\Phi_3\Phi_4\Phi_7$	$ix^{1/2}$
$G_{24}[-i]$	$\frac{1}{2} x^4\Phi_1^3\Phi_3\Phi_4\Phi_7$	$-ix^{1/2}$
$*\ \phi_{7,6}$	$x^6\Phi_7\Phi_{14}$	1
$*\ \phi_{3,8}$	$\frac{-\sqrt{-7}}{14} x^8\Phi_3\Phi_4\Phi_6\Phi''_7\Phi'_{14}$	1
$\#\ \phi_{3,10}$	$\frac{\sqrt{-7}}{14} x^8\Phi_3\Phi_4\Phi_6\Phi'_7\Phi''_{14}$	1
$\phi_{6,9}$	$\frac{1}{2} x^8\Phi_2^2\Phi_3\Phi_6\Phi_{14}$	1
$B_2 : 11$	$\frac{1}{2} x^8\Phi_1^2\Phi_3\Phi_6\Phi_7$	-1
$G_{24}[\zeta_7^3]$	$\frac{-\sqrt{-7}}{7} x^8\Phi_1^3\Phi_2^3\Phi_3\Phi_4\Phi_6$	ζ_7^3
$G_{24}[\zeta_7^5]$	$\frac{-\sqrt{-7}}{7} x^8\Phi_1^3\Phi_2^3\Phi_3\Phi_4\Phi_6$	ζ_7^5
$G_{24}[\zeta_7^6]$	$\frac{-\sqrt{-7}}{7} x^8\Phi_1^3\Phi_2^3\Phi_3\Phi_4\Phi_6$	ζ_7^6
$*\ \phi_{1,21}$	x^{21}	1

A.10. Unipotent characters for G_{25}

Some principal ζ-series

$\zeta_9 : \mathcal{H}_{Z_9}(\zeta_9^5 x^4, \zeta_9^8 x^2, \zeta_9^2, \zeta_9^2 x, \zeta_9^2 x^2, \zeta_9^5, \zeta_9^5 x, \zeta_9^5 x^2, \zeta_9^8)$
$\zeta_4^3 : \mathcal{H}_{Z_{12}}(-ix^3, \zeta_{12}, \zeta_{12}^5 x, i, \zeta_{12}^7 x, \zeta_{12}^5, -ix, \zeta_{12} x^2, \zeta_{12}^{11} x, -i, \zeta_{12} x, \zeta_{12}^5 x^2)$
$\zeta_4 : \mathcal{H}_{Z_{12}}(ix^3, \zeta_{12}^7 x^2, \zeta_{12}^{11} x, i, \zeta_{12} x, \zeta_{12}^{11} x^2, ix, \zeta_{12}^7, \zeta_{12}^5 x, -i, \zeta_{12}^7 x, \zeta_{12}^{11})$
$-1 : \mathcal{H}_{G_5}(x^2, \zeta_3, \zeta_3^2; -x, \zeta_3, \zeta_3^2)$

Non-principal 1-Harish-Chandra series

$\mathcal{H}_{G_{25}}(Z_3) = \mathcal{H}_{G_{3,1,2}}(x, \zeta_3, \zeta_3^2; x^3, -1)$
$\mathcal{H}_{G_{25}}(Z_3 \otimes Z_3) = \mathcal{H}_{Z_6}(x^3, -\zeta_3^2 x^3, \zeta_3 x^2, -1, \zeta_3^2, -\zeta_3 x^2)$
$\mathcal{H}_{G_{25}}(G_4) = \mathcal{H}_{Z_3}(1, \zeta_3 x^4, \zeta_3^2 x^4)$

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
*	$\phi_{1,0}$	1	1
*	$\phi_{3,1}$	$\frac{3-\sqrt{-3}}{6} x \Phi_3'' \Phi_6' \Phi_9 \Phi_{12}''''$	1
#	$\phi_{3,5}'$	$\frac{3+\sqrt{-3}}{6} x \Phi_3' \Phi_6'' \Phi_9 \Phi_{12}'''$	1
	$Z_3 : 2..$	$\frac{-\sqrt{-3}}{3} x \Phi_1 \Phi_2 \Phi_4 \Phi_9$	ζ_3^2
	$\phi_{8,3}$	$\frac{1}{3} x^2 \Phi_2^2 \Phi_4 \Phi_9 \Phi_{12}$	1
*	$\phi_{6,2}$	$\frac{1}{3} x^2 \Phi_3 \Phi_3' \Phi_4 \Phi_6 \Phi_9'' \Phi_{12}$	1
#	$\phi_{6,4}'$	$\frac{1}{3} x^2 \Phi_3 \Phi_3'' \Phi_4 \Phi_6 \Phi_9' \Phi_{12}$	1
	$\phi_{2,9}$	$\frac{\zeta_3}{3} x^2 \Phi_4 \Phi_6'^2 \Phi_9 \Phi_{12}$	1
	$Z_3 \otimes Z_3 : -\zeta_3^2$	$\frac{\zeta_3}{3} x^2 \Phi_1^2 \Phi_2 \Phi_3' \Phi_4 \Phi_6'' \Phi_9'' \Phi_{12}$	ζ_3
	$Z_3 : ..2$	$\frac{-\zeta_3}{3} x^2 \Phi_1 \Phi_2 \Phi_3'^2 \Phi_4 \Phi_6'' \Phi_9' \Phi_{12}$	ζ_3^2
	$\phi_{2,3}$	$\frac{\zeta_3^2}{3} x^2 \Phi_4 \Phi_6''^2 \Phi_9 \Phi_{12}$	1
	$Z_3 : .2.$	$\frac{\zeta_3^2}{3} x^2 \Phi_1 \Phi_2 \Phi_3''^2 \Phi_4 \Phi_6' \Phi_9'' \Phi_{12}$	ζ_3^2
	$Z_3 \otimes Z_3 : 1$	$\frac{\zeta_3^2}{3} x^2 \Phi_1^2 \Phi_2 \Phi_3'' \Phi_4 \Phi_6' \Phi_9' \Phi_{12}$	ζ_3
*	$\phi_{6,4}''$	$\frac{-\zeta_3}{6} x^4 \Phi_2^2 \Phi_3'^2 \Phi_6''^2 \Phi_9 \Phi_{12}$	1
	$\phi_{9,5}$	$\frac{3-\sqrt{-3}}{12} x^4 \Phi_3''^3 \Phi_4 \Phi_6' \Phi_9 \Phi_{12}$	1
	$\phi_{3,6}$	$\frac{1}{3} x^4 \Phi_3 \Phi_6^2 \Phi_9 \Phi_{12}$	1
	$\phi_{9,7}$	$\frac{3+\sqrt{-3}}{12} x^4 \Phi_3'^3 \Phi_4 \Phi_6'' \Phi_9 \Phi_{12}$	1
#	$\phi_{6,8}''$	$\frac{-\zeta_3^2}{6} x^4 \Phi_2^2 \Phi_3''^2 \Phi_6'^2 \Phi_9 \Phi_{12}$	1
	$Z_3 \otimes Z_3 : \zeta_3$	$\frac{1}{6} x^4 \Phi_1^2 \Phi_2^2 \Phi_4 \Phi_6''^2 \Phi_9 \Phi_{12}''''$	ζ_3
	$G_4 : \zeta_3$	$\frac{-3-\sqrt{-3}}{12} x^4 \Phi_1^2 \Phi_3^2 \Phi_3' \Phi_6'' \Phi_9 \Phi_{12}'''$	-1
	$Z_3 : 1..1$	$\frac{\zeta_3^2}{3} x^4 \Phi_1 \Phi_2^2 \Phi_3' \Phi_4 \Phi_6''^2 \Phi_9 \Phi_{12}'''$	ζ_3^2
	$G_{25}[-\zeta_3]$	$\frac{\sqrt{-3}}{6} x^4 \Phi_1^3 \Phi_2 \Phi_3^2 \Phi_4 \Phi_9$	$-\zeta_3$

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
	$\phi''_{3,5}$	$\frac{\zeta_3^2}{6}x^4{\Phi''_3}^2\Phi_4\Phi_6^2\Phi_9\Phi''''_{12}$	1
	$G_{25}[\zeta_3]$	$\frac{\sqrt{-3}}{6}x^4\Phi_1^3\Phi_2\Phi_4\Phi_9\Phi_{12}$	ζ_3
	$Z_3 : 1.1.$	$\frac{-\zeta_3}{3}x^4\Phi_1\Phi_2^2\Phi''_3\Phi_4{\Phi'_6}^2\Phi_9\Phi''''_{12}$	ζ_3^2
	$\phi''_{3,13}$	$\frac{\zeta_3}{6}x^4{\Phi'_3}^2\Phi_4\Phi_6^2\Phi_9\Phi'''_{12}$	1
	$G_4 : \zeta_3^2$	$\frac{-3+\sqrt{-3}}{12}x^4\Phi_1^2\Phi_3^2\Phi''_3\Phi'_6\Phi_9\Phi''''_{12}$	-1
	$Z_3 \otimes Z_3 : -\zeta_3$	$\frac{1}{6}x^4\Phi_1^2\Phi_2^2\Phi_4{\Phi'_6}^2\Phi_9\Phi'''_{12}$	ζ_3
$*$	$\phi_{8,6}$	$\frac{3-\sqrt{-3}}{6}x^6\Phi_2^2\Phi_4\Phi_6^2\Phi''_9\Phi_{12}$	1
#	$\phi_{8,9}$	$\frac{3+\sqrt{-3}}{6}x^6\Phi_2^2\Phi_4\Phi_6^2\Phi'_9\Phi_{12}$	1
	$Z_3 : .1.1$	$\frac{-\sqrt{-3}}{3}x^6\Phi_1\Phi_2^2\Phi_3\Phi_4\Phi_6^2\Phi_{12}$	ζ_3^2
$*$	$\phi'_{6,8}$	$\frac{3-\sqrt{-3}}{6}x^8\Phi'_3\Phi_4\Phi''_6\Phi_9\Phi_{12}$	1
#	$\phi_{6,10}$	$\frac{3+\sqrt{-3}}{6}x^8\Phi''_3\Phi_4\Phi'_6\Phi_9\Phi_{12}$	1
	$Z_3 : 11..$	$\frac{-\sqrt{-3}}{3}x^8\Phi_1\Phi_2\Phi_4\Phi_9\Phi_{12}$	ζ_3^2
$*$	$\phi_{1,12}$	$\frac{-1}{6}x^{12}\Phi_4{\Phi''_6}^2\Phi_9\Phi''''_{12}$	1
	$\phi'_{3,13}$	$\frac{-\zeta_3}{3}x^{12}\Phi_3\Phi''_3\Phi_6\Phi''_9\Phi_{12}$	1
	$Z_3 : ..11$	$\frac{-\zeta_3}{3}x^{12}\Phi_1\Phi_2{\Phi'_3}^2\Phi_4\Phi''_6\Phi''_9\Phi'''_{12}$	ζ_3^2
	$\phi_{3,17}$	$\frac{-\zeta_3^2}{3}x^{12}\Phi_3\Phi'_3\Phi_6\Phi'_9\Phi_{12}$	1
	$Z_3 : .11.$	$\frac{\zeta_3^2}{3}x^{12}\Phi_1\Phi_2{\Phi''_3}^2\Phi_4\Phi'_6\Phi'_9\Phi''''_{12}$	ζ_3^2
#	$\phi_{1,24}$	$\frac{-1}{6}x^{12}\Phi_4{\Phi'_6}^2\Phi_9\Phi'''_{12}$	1
	$Z_3 \otimes Z_3 : \zeta_3^2$	$\frac{\zeta_3^2}{3}x^{12}\Phi_1^2\Phi_2\Phi'_3\Phi_4\Phi''_6\Phi'_9\Phi'''_{12}$	ζ_3
	$G_4 : 1$	$\frac{-1}{2}x^{12}\Phi_1^2\Phi_3^2\Phi_9$	-1
	$\phi_{2,15}$	$\frac{1}{6}x^{12}\Phi_2^2\Phi_9\Phi_{12}$	1
	$Z_3 \otimes Z_3 : -1$	$\frac{\zeta_3}{3}x^{12}\Phi_1^2\Phi_2\Phi''_3\Phi_4\Phi'_6\Phi''_9\Phi''''_{12}$	ζ_3

A.11. Unipotent characters for G_{26}

Some principal ζ-series

ζ_9^2 : $\mathcal{H}_{Z_{18}}(\zeta_3 x^3, -\zeta_3 x, \zeta_9^8 x, -\zeta_3 x^{3/2}, x, -\zeta_3 x^2, \zeta_3, -\zeta_3^2 x, \zeta_9^2 x, -1, \zeta_3 x, -\zeta_3^2 x^2, \zeta_3 x^{3/2}, -x, \zeta_9^5 x, -\zeta_3, \zeta_3^2 x, -x^2)$

ζ_9 : $\mathcal{H}_{Z_{18}}(\zeta_3^2 x^3, -\zeta_3 x^2, x, -\zeta_3^2, \zeta_9 x, -\zeta_3^2 x, \zeta_3^2 x^{3/2}, -\zeta_3^2 x^2, \zeta_3 x, -1, \zeta_9^4 x, -x, \zeta_3^2, -x^2, \zeta_3^2 x, -\zeta_3^2 x^{3/2}, \zeta_9^7 x, -\zeta_3 x)$

Non-principal 1-Harish-Chandra series

$\mathcal{H}_{G_{26}}(Z_3) = \mathcal{H}_{G_{6,2,2}}(1, \zeta_3 x^2, \zeta_3^2 x^2; x^3, -1; x, -1)$
$\mathcal{H}_{G_{26}}(G_4) = \mathcal{H}_{Z_6}(x^3, -\zeta_3^2 x^4, \zeta_3 x^3, -1, \zeta_3^2 x^3, -\zeta_3 x^4)$
$\mathcal{H}_{G_{26}}(G_{3,1,2}^{103}) = \mathcal{H}_{Z_6}(x^4, -\zeta_3^2 x^3, \zeta_3 x, -x, \zeta_3^2, -\zeta_3 x)$
$\mathcal{H}_{G_{26}}(G_{3,1,2}^{130}) = \mathcal{H}_{Z_6}(x^4, -\zeta_3^2 x, \zeta_3, -x, \zeta_3^2 x, -\zeta_3 x^3)$

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
*	$\phi_{1,0}$	1	1
	$\phi_{1,9}$	$\frac{1}{3}x\Phi_9\Phi_{12}\Phi_{18}$	1
#	$\phi'_{3,5}$	$\frac{1}{3}x\Phi_3\Phi'_3\Phi_6\Phi''_6\Phi''_9\Phi_{12}\Phi'_{18}$	1
*	$\phi_{3,1}$	$\frac{1}{3}x\Phi_3\Phi''_3\Phi_6\Phi'_6\Phi'_9\Phi_{12}\Phi''_{18}$	1
	$\phi_{2,9}$	$\frac{-\zeta_3^2}{3}x\Phi_4\Phi_9\Phi'''_{12}\Phi_{18}$	1
	$G_{3,1,2}^{103}:1$	$\frac{-\zeta_3^2}{3}x\Phi_1^2\Phi_2^2\Phi'_3\Phi_4\Phi''_6\Phi''_9\Phi'''_{12}\Phi'_{18}$	ζ_3
	$Z_3:.....2$	$\frac{\zeta_3^2}{3}x\Phi_1\Phi_2{\Phi'_3}^2\Phi_4{\Phi''_6}^2\Phi'_9\Phi'''_{12}\Phi''_{18}$	ζ_3^2
	$\phi_{2,3}$	$\frac{-\zeta_3}{3}x\Phi_4\Phi_9\Phi''''_{12}\Phi_{18}$	1
	$Z_3:....2.$	$\frac{-\zeta_3}{3}x\Phi_1\Phi_2{\Phi''_3}^2\Phi_4{\Phi'_6}^2\Phi''_9\Phi''''_{12}\Phi'_{18}$	ζ_3^2
	$G_{3,1,2}^{130}:1$	$\frac{-\zeta_3}{3}x\Phi_1^2\Phi_2^2\Phi''_3\Phi_4\Phi'_6\Phi'_9\Phi''''_{12}\Phi''_{18}$	ζ_3
	$\phi_{3,6}$	$\frac{1}{3}x^2\Phi_3\Phi_6\Phi_9\Phi_{12}\Phi_{18}$	1
	$\phi'_{3,8}$	$\frac{-\zeta_3^2}{3}x^2{\Phi''_3}^2{\Phi'_6}^2\Phi_9\Phi_{12}\Phi_{18}$	1
	$\phi_{3,4}$	$\frac{-\zeta_3}{3}x^2{\Phi'_3}^2{\Phi''_6}^2\Phi_9\Phi_{12}\Phi_{18}$	1
#	$\phi'_{6,4}$	$\frac{1}{3}x^2{\Phi'_3}^2\Phi_4{\Phi''_6}^2\Phi_9\Phi'''_{12}\Phi_{18}$	1
	$G_{3,1,2}^{130}:-\zeta_3$	$\frac{-\zeta_3^2}{3}x^2\Phi_1^2\Phi_2^2\Phi_4\Phi_9\Phi'''_{12}\Phi_{18}$	ζ_3
	$Z_3:.1.\cdot-$	$\frac{-\zeta_3}{3}x^2\Phi_1\Phi_2\Phi'_3\Phi_4\Phi''_6\Phi_9\Phi'''_{12}\Phi_{18}$	ζ_3^2
*	$\phi_{6,2}$	$\frac{1}{3}x^2{\Phi''_3}^2\Phi_4{\Phi'_6}^2\Phi_9\Phi''''_{12}\Phi_{18}$	1
	$Z_3:..1\cdot-$	$\frac{\zeta_3^2}{3}x^2\Phi_1\Phi_2\Phi''_3\Phi_4\Phi'_6\Phi_9\Phi''''_{12}\Phi_{18}$	ζ_3^2
	$G_{3,1,2}^{103}:-\zeta_3^2$	$\frac{-\zeta_3}{3}x^2\Phi_1^2\Phi_2^2\Phi_4\Phi_9\Phi''''_{12}\Phi_{18}$	ζ_3
*	$\phi_{8,3}$	$\frac{1}{2}x^3\Phi_2^3\Phi_4\Phi_6^3\Phi_{12}\Phi_{18}$	1
#	$\phi'_{8,6}$	$\frac{1}{2}x^3\Phi_2^3\Phi_4\Phi_6^3\Phi_{12}\Phi_{18}$	1

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$G_{26}[i]$	$\frac{1}{2}x^3\Phi_1^3\Phi_3^3\Phi_4\Phi_9\Phi_{12}$	$ix^{1/2}$
$G_{26}[-i]$	$\frac{1}{2}x^3\Phi_1^3\Phi_3^3\Phi_4\Phi_9\Phi_{12}$	$-ix^{1/2}$
$*$ $\phi''_{6,4}$	$\frac{3-\sqrt{-3}}{12}x^4\Phi_2^2\Phi''_3\Phi_6^2\Phi'_6\Phi_9\Phi''''_{12}\Phi_{18}$	1
$\#$ $\phi''_{6,8}$	$\frac{3+\sqrt{-3}}{12}x^4\Phi_2^2\Phi'_3\Phi_6^2\Phi''_6\Phi_9\Phi'''_{12}\Phi_{18}$	1
$Z_3 : ....1.1$	$\frac{-\sqrt{-3}}{6}x^4\Phi_1\Phi_2^3\Phi_4\Phi_6^2\Phi_9\Phi_{18}$	ζ_3^2
$\phi''_{3,13}$	$\frac{-3+\sqrt{-3}}{12}x^4\Phi''_3\Phi_4{\Phi'_6}^3\Phi_9\Phi_{12}\Phi_{18}$	1
$\phi''_{3,5}$	$\frac{-3-\sqrt{-3}}{12}x^4\Phi'_3\Phi_4{\Phi''_6}^3\Phi_9\Phi_{12}\Phi_{18}$	1
$Z_3 : ..1..1.$	$\frac{-\sqrt{-3}}{6}x^4\Phi_1\Phi_2^3\Phi_4\Phi_9\Phi_{12}\Phi_{18}$	ζ_3^2
$\phi_{9,7}$	$\frac{3-\sqrt{-3}}{12}x^4{\Phi''_3}^3\Phi_4\Phi'_6\Phi_9\Phi_{12}\Phi_{18}$	1
$\phi_{9,5}$	$\frac{3+\sqrt{-3}}{12}x^4{\Phi'_3}^3\Phi_4\Phi''_6\Phi_9\Phi_{12}\Phi_{18}$	1
$G_{26}[\zeta_3^2]$	$\frac{\sqrt{-3}}{6}x^4\Phi_1^3\Phi_2\Phi_4\Phi_9\Phi_{12}\Phi_{18}$	ζ_3^2
$G_4 : -\zeta_3^2$	$\frac{-3+\sqrt{-3}}{12}x^4\Phi_1^2\Phi_3^2\Phi''_3\Phi'_6\Phi_9\Phi''''_{12}\Phi_{18}$	-1
$G_4 : -\zeta_3$	$\frac{-3-\sqrt{-3}}{12}x^4\Phi_1^2\Phi_3^2\Phi'_3\Phi''_6\Phi_9\Phi'''_{12}\Phi_{18}$	-1
$G_{26}[-\zeta_3^2]$	$\frac{\sqrt{-3}}{6}x^4\Phi_1^3\Phi_2\Phi_3^2\Phi_4\Phi_9\Phi_{18}$	$-\zeta_3^2$
$*$ $\phi_{6,5}$	$\frac{3-\sqrt{-3}}{6}x^5\Phi'_3\Phi_4\Phi''_6\Phi_9\Phi_{12}\Phi_{18}$	1
$\#$ $\phi'_{6,7}$	$\frac{3+\sqrt{-3}}{6}x^5\Phi''_3\Phi_4\Phi'_6\Phi_9\Phi_{12}\Phi_{18}$	1
$Z_3 : ...2..$	$\frac{-\sqrt{-3}}{3}x^5\Phi_1\Phi_2\Phi_4\Phi_9\Phi_{12}\Phi_{18}$	ζ_3^2
$*$ $\phi''_{8,6}$	$\frac{-\sqrt{-3}}{54}x^6\Phi_2^3{\Phi''_3}^3\Phi_4\Phi_9\Phi_{12}\Phi_{18}$	1
$\#$ $\phi_{8,12}$	$\frac{\sqrt{-3}}{54}x^6\Phi_2^3{\Phi'_3}^3\Phi_4\Phi_9\Phi_{12}\Phi_{18}$	1
$\phi''_{6,11}$	$\frac{-\zeta_3^2}{6}x^6\Phi_2^2\Phi_3\Phi'_3\Phi_6^3\Phi'_9\Phi_{12}\Phi_{18}$	1
$\phi''_{6,7}$	$\frac{-\zeta_3}{6}x^6\Phi_2^2\Phi_3\Phi''_3\Phi_6^3\Phi''_9\Phi_{12}\Phi_{18}$	1
$\phi_{2,15}$	$\frac{1}{6}x^6\Phi_2^2\Phi_6^3\Phi_9\Phi_{12}\Phi_{18}$	1
$\phi''_{8,9}$	$\frac{1}{6}x^6\Phi_2^3\Phi_4\Phi_6^2\Phi''_6\Phi_9\Phi'''_{12}\Phi_{18}$	1
$\phi'_{8,9}$	$\frac{1}{6}x^6\Phi_2^3\Phi_4\Phi_6^2\Phi'_6\Phi_9\Phi''''_{12}\Phi_{18}$	1
$\phi'_{6,8}$	$\frac{3-\sqrt{-3}}{18}x^6\Phi_3\Phi'_3\Phi_4\Phi_6\Phi''_6\Phi_9\Phi_{12}\Phi_{18}$	1
$\phi_{6,10}$	$\frac{3+\sqrt{-3}}{18}x^6\Phi_3\Phi''_3\Phi_4\Phi_6\Phi'_6\Phi_9\Phi_{12}\Phi_{18}$	1
$\phi''_{3,8}$	$\frac{3-\sqrt{-3}}{36}x^6\Phi_3\Phi'_3\Phi_4\Phi_6^3\Phi_9\Phi_{12}\Phi''_{18}$	1
$\phi''_{3,16}$	$\frac{3+\sqrt{-3}}{36}x^6\Phi_3\Phi''_3\Phi_4\Phi_6^3\Phi_9\Phi_{12}\Phi'_{18}$	1
$\phi_{9,8}$	$\frac{3-\sqrt{-3}}{36}x^6\Phi_3^3\Phi_4\Phi_6\Phi''_6\Phi'_9\Phi_{12}\Phi_{18}$	1
$\phi_{9,10}$	$\frac{3+\sqrt{-3}}{36}x^6\Phi_3^3\Phi_4\Phi_6\Phi'_6\Phi''_9\Phi_{12}\Phi_{18}$	1
$\phi_{1,12}$	$\frac{\sqrt{-3}}{54}x^6{\Phi''_3}^3\Phi_4{\Phi''_6}^3\Phi_9\Phi_{12}\Phi_{18}$	1
$\phi_{1,24}$	$\frac{-\sqrt{-3}}{54}x^6{\Phi'_3}^3\Phi_4{\Phi'_6}^3\Phi_9\Phi_{12}\Phi_{18}$	1
$\phi_{2,18}$	$\frac{\sqrt{-3}}{27}x^6{\Phi''_3}^3\Phi_4{\Phi'_6}^3\Phi_9\Phi_{12}\Phi_{18}$	1
$\phi_{2,12}$	$\frac{-\sqrt{-3}}{27}x^6{\Phi'_3}^3\Phi_4{\Phi''_6}^3\Phi_9\Phi_{12}\Phi_{18}$	1
$Z_3 : .1.\cdot+$	$\frac{3-\sqrt{-3}}{18}x^6\Phi_1\Phi_2{\Phi''_3}^2\Phi_4{\Phi'_6}^2\Phi_9\Phi_{12}\Phi_{18}$	ζ_3^2
$Z_3 : ..1\cdot+$	$\frac{-3-\sqrt{-3}}{18}x^6\Phi_1\Phi_2{\Phi'_3}^2\Phi_4{\Phi''_6}^2\Phi_9\Phi_{12}\Phi_{18}$	ζ_3^2

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$Z_3 : 1..\cdot-$	$\frac{-\sqrt{-3}}{9}x^6\Phi_1\Phi_2\Phi_3\Phi_4\Phi_6\Phi_9\Phi_{12}\Phi_{18}$	ζ_3^2
$Z_3 : ...1.1.$	$\frac{-3-\sqrt{-3}}{36}x^6\Phi_1\Phi_2^3{\Phi_3'}^2\Phi_4{\Phi_6''}^3\Phi_9\Phi_{12}\Phi_{18}'$	ζ_3^2
$Z_3 : ..1.1..$	$\frac{3-\sqrt{-3}}{36}x^6\Phi_1\Phi_2^3{\Phi_3''}^2\Phi_4{\Phi_6'}^3\Phi_9\Phi_{12}\Phi_{18}''$	ζ_3^2
$Z_3 : .1..1..$	$\frac{-\zeta_3}{6}x^6\Phi_1\Phi_2^3{\Phi_3'}^2\Phi_4\Phi_6^2\Phi_6''\Phi_9''\Phi_{12}'''\Phi_{18}$	ζ_3^2
$Z_3 : ...1..1$	$\frac{\zeta_3^2}{6}x^6\Phi_1\Phi_2^3{\Phi_3''}^2\Phi_4\Phi_6^2\Phi_6'\Phi_9'\Phi_{12}''''\Phi_{18}$	ζ_3^2
$G_4 : 1$	$\frac{-1}{6}x^6\Phi_1^2\Phi_3^3\Phi_9\Phi_{12}\Phi_{18}$	-1
$G_4 : \zeta_3$	$\frac{-\zeta_3^2}{6}x^6\Phi_1^2\Phi_3^3\Phi_6\Phi_6''\Phi_9\Phi_{12}\Phi_{18}''$	-1
$G_4 : \zeta_3^2$	$\frac{-\zeta_3}{6}x^6\Phi_1^2\Phi_3^3\Phi_6\Phi_6'\Phi_9\Phi_{12}\Phi_{18}'$	-1
$G_{3,1,2}^{103} : \zeta_3$	$\frac{-\zeta_3}{6}x^6\Phi_1^2\Phi_2^3\Phi_3''\Phi_4\Phi_6^2\Phi_6'\Phi_9''\Phi_{12}''''\Phi_{18}$	ζ_3
$G_{3,1,2}^{130} : \zeta_3^2$	$\frac{-\zeta_3^2}{6}x^6\Phi_1^2\Phi_2^3\Phi_3'\Phi_4\Phi_6^2\Phi_6''\Phi_9'\Phi_{12}'''\Phi_{18}$	ζ_3
$G_{3,1,2}^{103} : -\zeta_3$	$\frac{-3-\sqrt{-3}}{36}x^6\Phi_1^2\Phi_2^3\Phi_3''\Phi_4{\Phi_6'}^3\Phi_9\Phi_{12}\Phi_{18}'$	ζ_3
$G_{3,1,2}^{130} : -\zeta_3^2$	$\frac{-3+\sqrt{-3}}{36}x^6\Phi_1^2\Phi_2^3\Phi_3'\Phi_4{\Phi_6''}^3\Phi_9\Phi_{12}\Phi_{18}''$	ζ_3
$G_{3,1,2}^{130} : -1$	$\frac{3+\sqrt{-3}}{18}x^6\Phi_1^2\Phi_2^2\Phi_3''\Phi_4\Phi_6'\Phi_9\Phi_{12}\Phi_{18}$	ζ_3
$G_{3,1,2}^{103} : -1$	$\frac{3-\sqrt{-3}}{18}x^6\Phi_1^2\Phi_2^2\Phi_3'\Phi_4\Phi_6''\Phi_9\Phi_{12}\Phi_{18}$	ζ_3
$G_{26}[-1]$	$\frac{-1}{6}x^6\Phi_1^3\Phi_3^2\Phi_3'\Phi_4\Phi_9\Phi_{12}'''\Phi_{18}$	-1
$G_{26}^2[-1]$	$\frac{-1}{6}x^6\Phi_1^3\Phi_3^2\Phi_3''\Phi_4\Phi_9\Phi_{12}''''\Phi_{18}$	-1
$G_{26}^2[\zeta_3^2]$	$\frac{3+\sqrt{-3}}{36}x^6\Phi_1^3\Phi_2{\Phi_3'}^3\Phi_4{\Phi_6''}^2\Phi_9''\Phi_{12}\Phi_{18}$	ζ_3^2
$G_{26}^3[\zeta_3^2]$	$\frac{-3+\sqrt{-3}}{36}x^6\Phi_1^3\Phi_2{\Phi_3''}^3\Phi_4{\Phi_6'}^2\Phi_9'\Phi_{12}\Phi_{18}$	ζ_3^2
$G_{26}^2[-\zeta_3^2]$	$\frac{\zeta_3}{6}x^6\Phi_1^3\Phi_2\Phi_3^2\Phi_3'\Phi_4{\Phi_6''}^2\Phi_9\Phi_{12}'''\Phi_{18}'$	$-\zeta_3^2$
$G_{26}^3[-\zeta_3^2]$	$\frac{\zeta_3^2}{6}x^6\Phi_1^3\Phi_2\Phi_3^2\Phi_3''\Phi_4{\Phi_6'}^2\Phi_9\Phi_{12}''''\Phi_{18}''$	$-\zeta_3^2$
$G_{26}[\zeta_3]$	$\frac{3+\sqrt{-3}}{36}x^6\Phi_1^3\Phi_2^2{\Phi_3''}^3\Phi_4\Phi_6'\Phi_9''\Phi_{12}\Phi_{18}$	ζ_3
$G_{26}^2[\zeta_3]$	$\frac{-3+\sqrt{-3}}{36}x^6\Phi_1^3\Phi_2^2{\Phi_3'}^3\Phi_4\Phi_6''\Phi_9'\Phi_{12}\Phi_{18}$	ζ_3
$G_{26}[-\zeta_3]$	$\frac{\zeta_3}{6}x^6\Phi_1^3\Phi_2^2\Phi_3^2\Phi_3''\Phi_4\Phi_6'\Phi_9\Phi_{12}''''\Phi_{18}'$	$-\zeta_3$
$G_{26}^2[-\zeta_3]$	$\frac{\zeta_3^2}{6}x^6\Phi_1^3\Phi_2^2\Phi_3^2\Phi_3'\Phi_4\Phi_6''\Phi_9\Phi_{12}'''\Phi_{18}''$	$-\zeta_3$
$G_{26}[1]$	$\frac{\sqrt{-3}}{27}x^6\Phi_1^3\Phi_2^3\Phi_4\Phi_9\Phi_{12}\Phi_{18}$	1
$G_{26}^2[1]$	$\frac{\sqrt{-3}}{54}x^6\Phi_1^3\Phi_4{\Phi_6''}^3\Phi_9\Phi_{12}\Phi_{18}$	1
$G_{26}^3[1]$	$\frac{\sqrt{-3}}{54}x^6\Phi_1^3\Phi_4{\Phi_6'}^3\Phi_9\Phi_{12}\Phi_{18}$	1
$G_{26}[\zeta_9^8]$	$\frac{-\sqrt{-3}}{9}x^6\Phi_1^3\Phi_2^3\Phi_3^3\Phi_4\Phi_6^3\Phi_{12}$	ζ_9^8
$G_{26}[\zeta_9^5]$	$\frac{-\sqrt{-3}}{9}x^6\Phi_1^3\Phi_2^3\Phi_3^3\Phi_4\Phi_6^3\Phi_{12}$	ζ_9^5
$G_{26}[\zeta_9^2]$	$\frac{-\sqrt{-3}}{9}x^6\Phi_1^3\Phi_2^3\Phi_3^3\Phi_4\Phi_6^3\Phi_{12}$	ζ_9^2
$\phi_{3,15}$	$\frac{1}{3}x^{11}\Phi_3\Phi_6\Phi_9\Phi_{12}\Phi_{18}$	1
# $\phi_{6,13}$	$\frac{1}{3}x^{11}{\Phi_3'}^2\Phi_4{\Phi_6''}^2\Phi_9\Phi_{12}'''\Phi_{18}$	1
* $\phi_{6,11}'$	$\frac{1}{3}x^{11}{\Phi_3''}^2\Phi_4{\Phi_6'}^2\Phi_9\Phi_{12}''''\Phi_{18}$	1
$\phi_{3,17}$	$\frac{-\zeta_3^2}{3}x^{11}{\Phi_3''}^2{\Phi_6'}^2\Phi_9\Phi_{12}\Phi_{18}$	1
$G_{3,1,2}^{103} : \zeta_3^2$	$\frac{-\zeta_3^2}{3}x^{11}\Phi_1^2\Phi_2^2\Phi_4\Phi_9\Phi_{12}'''\Phi_{18}$	ζ_3
$Z_3 : .....11$	$\frac{\zeta_3^2}{3}x^{11}\Phi_1\Phi_2\Phi_3''\Phi_4\Phi_6'\Phi_9\Phi_{12}''''\Phi_{18}$	ζ_3^2

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$\phi'_{3,13}$	$\frac{-\zeta_3}{3}x^{11}\Phi_3'^{\,2}\Phi_6''^{2}\Phi_9\Phi_{12}\Phi_{18}$	1
$Z_3 : ....11.$	$\frac{-\zeta_3}{3}x^{11}\Phi_1\Phi_2\Phi'_3\Phi_4\Phi''_6\Phi_9\Phi'''_{12}\Phi_{18}$	ζ_3^2
$G_{3,1,2}^{130} : \zeta_3$	$\frac{-\zeta_3}{3}x^{11}\Phi_1^2\Phi_2^2\Phi_4\Phi_9\Phi''''_{12}\Phi_{18}$	ζ_3
$*$ $\phi'_{3,16}$	$\frac{3-\sqrt{-3}}{6}x^{16}\Phi''_3\Phi'_6\Phi_9\Phi''''_{12}\Phi_{18}$	1
# $\phi_{3,20}$	$\frac{3+\sqrt{-3}}{6}x^{16}\Phi'_3\Phi''_6\Phi_9\Phi'''_{12}\Phi_{18}$	1
$Z_3 : 1.. \cdot +$	$\frac{-\sqrt{-3}}{3}x^{16}\Phi_1\Phi_2\Phi_4\Phi_9\Phi_{18}$	ζ_3^2
$*$ $\phi_{1,21}$	$\frac{-\sqrt{-3}}{6}x^{21}\Phi_4\Phi''_9\Phi_{12}\Phi''_{18}$	1
$\phi_{2,24}$	$\frac{1}{2}x^{21}\Phi_2^2\Phi_6^2\Phi_{18}$	1
$G_4 : -1$	$\frac{-1}{2}x^{21}\Phi_1^2\Phi_3^2\Phi_9$	-1
$Z_3 : ...11..$	$\frac{-\sqrt{-3}}{3}x^{21}\Phi_1\Phi_2\Phi_3\Phi_4\Phi_6\Phi_{12}$	ζ_3^2
# $\phi_{1,33}$	$\frac{\sqrt{-3}}{6}x^{21}\Phi_4\Phi'_9\Phi_{12}\Phi'_{18}$	1

A.12. Unipotent characters for G_{27}

Some principal ζ-series

ζ_5^3 : $\mathcal{H}_{Z_{30}}(\zeta_5 x^3, -\zeta_3 x^2, \zeta_{15}^{13} x^2, -\zeta_5 x^{3/2}, \zeta_{15}^8 x, -\zeta_{15}^{13} x^{4/3}, \zeta_5 x^{5/3}, -\zeta_{15}^8 x^2, \zeta_3^2 x, -\zeta_5^2 x^{3/2}, \zeta_{15}^{11} x, -\zeta_3^2 x^2, x^{3/2}, -\zeta_{15}^{11} x^2, \zeta_{15}^{13} x, -\zeta_5 x^{4/3}, \zeta_{15}^8 x^{5/3}, -\zeta_{15}^{13} x^2, \zeta_5 x^{3/2}, -\zeta_{15}^8 x, \zeta_{15} x, -\zeta_5, \zeta_{15}^8 x^2, -\zeta_{15} x^2, \zeta_5^2 x^{3/2}, -\zeta_{15}^8 x^{4/3}, \zeta_{15}^{13} x^{5/3}, -x^{3/2}, \zeta_3 x, -\zeta_{15}^{13} x)$

ζ_5 : $\mathcal{H}_{Z_{30}}(\zeta_5^2 x^3, -\zeta_{15}^2 x^2, \zeta_{15} x^{5/3}, -\zeta_5^4 x^{3/2}, \zeta_{15}^{11} x^2, -\zeta_{15} x^{4/3}, \zeta_5^2 x^{3/2}, -\zeta_3 x^2, \zeta_{15} x, -x^{3/2}, \zeta_{15}^2 x, -\zeta_{15}^7 x^2, \zeta_5^2 x^{5/3}, -\zeta_{15}^{11} x, \zeta_{15} x^2, -\zeta_5^2 x^{4/3}, \zeta_3 x, -\zeta_3^2 x^2, \zeta_5^4 x^{3/2}, -\zeta_{15}^{11} x^2, \zeta_{15}^7 x, -\zeta_5^2 x^{3/2}, \zeta_{15}^{11} x^{5/3}, -\zeta_{15} x, x^{3/2}, -\zeta_{15}^{11} x^{4/3}, \zeta_3^2 x, -\zeta_5^2, \zeta_{15}^{11} x, -\zeta_{15} x^2)$

ζ_5^2 : $\mathcal{H}_{Z_{30}}(\zeta_5^4 x^3, -\zeta_{15}^2 x, \zeta_3^2 x, -x^{3/2}, \zeta_{15}^2 x^{5/3}, -\zeta_{15}^7 x^{4/3}, \zeta_5^3 x^{3/2}, -\zeta_{15}^{14} x^2, \zeta_{15}^7 x^2, -\zeta_5^4, \zeta_{15}^{14} x, -\zeta_{15}^7 x, \zeta_5^4 x^{3/2}, -\zeta_{15}^2 x^2, \zeta_{15}^7 x^{5/3}, -\zeta_5^4 x^{4/3}, \zeta_{15}^2 x, -\zeta_{15}^4 x^2, x^{3/2}, -\zeta_3 x^2, \zeta_{15}^4 x, -\zeta_5^3 x^{3/2}, \zeta_3 x, -\zeta_{15}^7 x^2, \zeta_5^4 x^{5/3}, -\zeta_{15}^2 x^{4/3}, \zeta_{15}^7 x, -\zeta_5^4 x^{3/2}, \zeta_{15}^2 x^2, -\zeta_3^2 x^2)$

Non-principal 1-Harish-Chandra series

$\mathcal{H}_{G_{27}}(I_2(5)[1,3]) = \mathcal{H}_{Z_6}(x^{5/2}, -\zeta_3^2 x^5, \zeta_3, -x^{5/2}, \zeta_3^2, -\zeta_3 x^5)$
$\mathcal{H}_{G_{27}}(I_2(5)[1,2]) = \mathcal{H}_{Z_6}(x^{5/2}, -\zeta_3^2 x^5, \zeta_3, -x^{5/2}, \zeta_3^2, -\zeta_3 x^5)$
$\mathcal{H}_{G_{27}}(B_2) = \mathcal{H}_{Z_6}(x^4, -\zeta_3^2 x^5, \zeta_3, -x, \zeta_3^2, -\zeta_3 x^5)$

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
*	$\phi_{1,0}$	1	1
*	$\phi_{3,1}$	$\frac{-\sqrt{-15}\zeta_3^2}{30} x\Phi_3''^3\Phi_4\Phi_5'\Phi_6'^3\Phi_{10}'\Phi_{12}\Phi_{15}''\Phi_{15}^{(5)}\Phi_{30}''\Phi_{30}^{(5)}$	1
	$\phi_{3,7}$	$\frac{\sqrt{-15}\zeta_3^2}{30} x\Phi_3''^3\Phi_4\Phi_5''\Phi_6'^3\Phi_{10}''\Phi_{12}\Phi_{15}'\Phi_{15}^{(6)}\Phi_{30}'\Phi_{30}^{(6)}$	1
	$I_2(5)[1,2]:-\zeta_3^2$	$\frac{\sqrt{-15}\zeta_3^2}{15} x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_3''\Phi_4\Phi_6^2\Phi_6'\Phi_{12}\Phi_{15}'''\Phi_{30}''''$	ζ_5^3
	$I_2(5)[1,3]:-\zeta_3^2$	$\frac{\sqrt{-15}\zeta_3^2}{15} x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_3''\Phi_4\Phi_6^2\Phi_6'\Phi_{12}\Phi_{15}'''\Phi_{30}''''$	ζ_5^2
	$B_2:-\zeta_3^2$	$\frac{3-\sqrt{-3}}{12} x\Phi_1^2\Phi_3^2\Phi_3''\Phi_5\Phi_6'^3\Phi_{12}''''\Phi_{15}\Phi_{30}''''$	-1
	$\phi_{6,4}$	$\frac{3-\sqrt{-3}}{12} x\Phi_2^2\Phi_3''^3\Phi_6^2\Phi_6'\Phi_{10}\Phi_{12}''''\Phi_{15}'''\Phi_{30}$	1
#	$\phi_{3,5}''$	$\frac{\sqrt{-15}\zeta_3}{30} x\Phi_3'^3\Phi_4\Phi_5'\Phi_6''^3\Phi_{10}'\Phi_{12}\Phi_{15}''\Phi_{15}^{(7)}\Phi_{30}''\Phi_{30}^{(7)}$	1
	$\phi_{3,5}'$	$\frac{-\sqrt{-15}\zeta_3}{30} x\Phi_3'^3\Phi_4\Phi_5''\Phi_6''^3\Phi_{10}''\Phi_{12}\Phi_{15}'\Phi_{15}^{(8)}\Phi_{30}'\Phi_{30}^{(8)}$	1
	$I_2(5)[1,2]:-\zeta_3$	$\frac{-\sqrt{-15}\zeta_3}{15} x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_3'\Phi_4\Phi_6^2\Phi_6''\Phi_{12}\Phi_{15}''''\Phi_{30}'''$	ζ_5^3
	$I_2(5)[1,3]:-\zeta_3$	$\frac{-\sqrt{-15}\zeta_3}{15} x\Phi_1^2\Phi_2^2\Phi_3^2\Phi_3'\Phi_4\Phi_6^2\Phi_6''\Phi_{12}\Phi_{15}''''\Phi_{30}'''$	ζ_5^2
	$B_2:-\zeta_3$	$\frac{3+\sqrt{-3}}{12} x\Phi_1^2\Phi_3^2\Phi_3'\Phi_5\Phi_6''^3\Phi_{12}'''\Phi_{15}\Phi_{30}'''$	-1
	$\phi_{6,2}$	$\frac{3+\sqrt{-3}}{12} x\Phi_2^2\Phi_3'^3\Phi_6^2\Phi_6''\Phi_{10}\Phi_{12}'''\Phi_{15}''''\Phi_{30}$	1
	$G_{27}^3[\zeta_3^2]$	$\frac{\sqrt{-15}}{30} x\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_{10}\Phi_{12}\Phi_{15}''\Phi_{30}''$	ζ_3^2
	$G_{27}^2[\zeta_3^2]$	$\frac{-\sqrt{-15}}{30} x\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_{10}\Phi_{12}\Phi_{15}'\Phi_{30}'$	ζ_3^2
	$G_{27}[\zeta_{15}^4]$	$\frac{\sqrt{-15}}{15} x\Phi_1^3\Phi_2^3\Phi_3^2\Phi_4\Phi_5\Phi_6^2\Phi_{10}\Phi_{12}$	ζ_{15}^4
	$G_{27}[\zeta_{15}]$	$\frac{\sqrt{-15}}{15} x\Phi_1^3\Phi_2^3\Phi_3^2\Phi_4\Phi_5\Phi_6^2\Phi_{10}\Phi_{12}$	ζ_{15}
	$G_{27}^2[-\zeta_3]$	$\frac{\sqrt{-3}}{6} x\Phi_1^3\Phi_2^3\Phi_3^2\Phi_4\Phi_5\Phi_{10}\Phi_{15}$	$-\zeta_3^2$

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
	$G_{27}^4[\zeta_3^2]$	$\frac{\sqrt{-3}}{6}x\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_6^2\Phi_{10}\Phi_{30}$	ζ_3^2
*	$\phi_{10,3}$	$\frac{1}{2}x^3\Phi_2^2\Phi_5\Phi_6^2\Phi_{10}\Phi_{15}\Phi_{30}$	1
	$\phi''_{5,6}$	$\frac{1}{2}x^3\Phi_4\Phi_5\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{30}$	1
	$\phi'_{5,6}$	$\frac{1}{2}x^3\Phi_4\Phi_5\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{30}$	1
	$B_2:1$	$\frac{1}{2}x^3\Phi_1^2\Phi_3^2\Phi_5\Phi_{10}\Phi_{15}\Phi_{30}$	-1
#	$\phi_{9,6}$	$\frac{1}{3}x^4\Phi_3^3\Phi_6^3\Phi_{12}\Phi_{15}\Phi_{30}$	1
	$G_{27}^2[\zeta_9^4]$	$\frac{1}{3}x^4\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4\Phi_5{\Phi'_6}^3\Phi_{10}\Phi''''_{12}\Phi'''_{15}\Phi''''_{30}$	$\zeta_9^4x^{2/3}$
	$G_{27}^2[\zeta_9^7]$	$\frac{1}{3}x^4\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4\Phi_5{\Phi''_6}^3\Phi_{10}\Phi'''_{12}\Phi''''_{15}\Phi'''_{30}$	$\zeta_9^7x^{1/3}$
	$\phi_{9,8}$	$\frac{1}{3}x^4\Phi_3^3\Phi_6^3\Phi_{12}\Phi_{15}\Phi_{30}$	1
	$G_{27}^2[\zeta_9]$	$\frac{1}{3}x^4\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4\Phi_5{\Phi'_6}^3\Phi_{10}\Phi''''_{12}\Phi'''_{15}\Phi''''_{30}$	$\zeta_9x^{2/3}$
	$G_{27}[\zeta_9]$	$\frac{1}{3}x^4\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4\Phi_5{\Phi''_6}^3\Phi_{10}\Phi'''_{12}\Phi''''_{15}\Phi'''_{30}$	$\zeta_9x^{1/3}$
*	$\phi_{9,4}$	$\frac{1}{3}x^4\Phi_3^3\Phi_6^3\Phi_{12}\Phi_{15}\Phi_{30}$	1
	$G_{27}[\zeta_9^7]$	$\frac{1}{3}x^4\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4\Phi_5{\Phi'_6}^3\Phi_{10}\Phi''''_{12}\Phi'''_{15}\Phi''''_{30}$	$\zeta_9^7x^{2/3}$
	$G_{27}[\zeta_9^4]$	$\frac{1}{3}x^4\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4\Phi_5{\Phi''_6}^3\Phi_{10}\Phi'''_{12}\Phi''''_{15}\Phi'''_{30}$	$\zeta_9^4x^{1/3}$
*	$\phi_{15,5}$	$\frac{3+\sqrt{-3}}{6}x^5{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_{10}\Phi''''_{12}\Phi_{15}\Phi_{30}$	1
#	$\phi_{15,7}$	$\frac{3-\sqrt{-3}}{6}x^5{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_{10}\Phi'''_{12}\Phi_{15}\Phi_{30}$	1
	$G_{27}^4[\zeta_3]$	$\frac{-\sqrt{-3}}{3}x^5\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_{10}\Phi_{15}\Phi_{30}$	ζ_3
*	$\phi_{8,6}$	$\frac{5-\sqrt{5}}{20}x^6\Phi_2^3\Phi_4\Phi'_5\Phi_6^3\Phi_{10}\Phi_{12}\Phi''_{15}\Phi_{30}$	1
	$\phi_{8,12}$	$\frac{5+\sqrt{5}}{20}x^6\Phi_2^3\Phi_4\Phi''_5\Phi_6^3\Phi_{10}\Phi_{12}\Phi'_{15}\Phi_{30}$	1
	$I_2(5)[1,2]:1$	$\frac{\sqrt{5}}{10}x^6\Phi_1^2\Phi_2^3\Phi_3^2\Phi_4\Phi_6^3\Phi_{10}\Phi_{12}\Phi_{30}$	ζ_5^3
	$I_2(5)[1,3]:1$	$\frac{\sqrt{5}}{10}x^6\Phi_1^2\Phi_2^3\Phi_3^2\Phi_4\Phi_6^3\Phi_{10}\Phi_{12}\Phi_{30}$	ζ_5^2
#	$\phi'_{8,9}$	$\frac{5-\sqrt{5}}{20}x^6\Phi_2^3\Phi_4\Phi'_5\Phi_6^3\Phi_{10}\Phi_{12}\Phi''_{15}\Phi_{30}$	1
	$\phi''_{8,9}$	$\frac{5+\sqrt{5}}{20}x^6\Phi_2^3\Phi_4\Phi''_5\Phi_6^3\Phi_{10}\Phi_{12}\Phi'_{15}\Phi_{30}$	1
	$I_2(5)[1,2]:-1$	$\frac{\sqrt{5}}{10}x^6\Phi_1^2\Phi_2^3\Phi_3^2\Phi_4\Phi_6^3\Phi_{10}\Phi_{12}\Phi_{30}$	ζ_5^3
	$I_2(5)[1,3]:-1$	$\frac{\sqrt{5}}{10}x^6\Phi_1^2\Phi_2^3\Phi_3^2\Phi_4\Phi_6^3\Phi_{10}\Phi_{12}\Phi_{30}$	ζ_5^2
	$G_{27}[i]$	$\frac{5-\sqrt{5}}{20}x^6\Phi_1^3\Phi_3^3\Phi_4\Phi_5\Phi''_{10}\Phi_{12}\Phi_{15}\Phi'_{30}$	$ix^{1/2}$
	$G_{27}^2[i]$	$\frac{5+\sqrt{5}}{20}x^6\Phi_1^3\Phi_3^3\Phi_4\Phi_5\Phi'_{10}\Phi_{12}\Phi_{15}\Phi''_{30}$	$ix^{1/2}$
	$G_{27}[\zeta_{20}^{17}]$	$\frac{\sqrt{5}}{10}x^6\Phi_1^3\Phi_2^2\Phi_3^3\Phi_4\Phi_5\Phi_6^2\Phi_{12}\Phi_{15}$	$\zeta_{20}^{17}x^{1/2}$
	$G_{27}[\zeta_{20}^{13}]$	$\frac{\sqrt{5}}{10}x^6\Phi_1^3\Phi_2^2\Phi_3^3\Phi_4\Phi_5\Phi_6^2\Phi_{12}\Phi_{15}$	$\zeta_{20}^{13}x^{1/2}$
	$G_{27}^2[-i]$	$\frac{5-\sqrt{5}}{20}x^6\Phi_1^3\Phi_3^3\Phi_4\Phi_5\Phi''_{10}\Phi_{12}\Phi_{15}\Phi'_{30}$	$-ix^{1/2}$
	$G_{27}[-1]$	$\frac{5+\sqrt{5}}{20}x^6\Phi_1^3\Phi_3^3\Phi_4\Phi_5\Phi'_{10}\Phi_{12}\Phi_{15}\Phi''_{30}$	$-ix^{1/2}$
	$G_{27}[\zeta_{20}^7]$	$\frac{\sqrt{5}}{10}x^6\Phi_1^3\Phi_2^2\Phi_3^3\Phi_4\Phi_5\Phi_6^2\Phi_{12}\Phi_{15}$	$\zeta_{20}^7x^{1/2}$
	$G_{27}[\zeta_{20}^3]$	$\frac{\sqrt{5}}{10}x^6\Phi_1^3\Phi_2^2\Phi_3^3\Phi_4\Phi_5\Phi_6^2\Phi_{12}\Phi_{15}$	$\zeta_{20}^3x^{1/2}$
*	$\phi_{15,8}$	$\frac{3-\sqrt{-3}}{6}x^8{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_{10}\Phi'''_{12}\Phi_{15}\Phi_{30}$	1
#	$\phi_{15,10}$	$\frac{3+\sqrt{-3}}{6}x^8{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_{10}\Phi''''_{12}\Phi_{15}\Phi_{30}$	1
	$G_{27}[\zeta_3^2]$	$\frac{\sqrt{-3}}{3}x^8\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_{10}\Phi_{15}\Phi_{30}$	ζ_3^2

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
*	$\phi_{9,9}$	$\frac{1}{3}x^9\Phi_3^3\Phi_6^3\Phi_{12}\Phi_{15}\Phi_{30}$	1
	$G_{27}^2[\zeta_9^5]$	$\frac{\zeta_3^2}{3}x^9\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4\Phi_5{\Phi''_6}^3\Phi_{10}\Phi'''_{12}\Phi''''_{15}\Phi'''_{30}$	$\zeta_9^5x^{2/3}$
	$G_{27}^5[1]$	$\frac{-\zeta_3}{3}x^9\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4\Phi_5{\Phi'_6}^3\Phi_{10}\Phi''''_{12}\Phi'''_{15}\Phi''''_{30}$	$\zeta_9^2x^{1/3}$
#	$\phi_{9,11}$	$\frac{1}{3}x^9\Phi_3^3\Phi_6^3\Phi_{12}\Phi_{15}\Phi_{30}$	1
	$G_{27}[\zeta_9^8]$	$\frac{\zeta_3^2}{3}x^9\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4\Phi_5{\Phi''_6}^3\Phi_{10}\Phi'''_{12}\Phi''''_{15}\Phi'''_{30}$	$\zeta_9^8x^{2/3}$
	$G_{27}^2[\zeta_9^8]$	$\frac{-\zeta_3}{3}x^9\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4\Phi_5{\Phi'_6}^3\Phi_{10}\Phi''''_{12}\Phi'''_{15}\Phi''''_{30}$	$\zeta_9^8x^{1/3}$
	$\phi_{9,13}$	$\frac{1}{3}x^9\Phi_3^3\Phi_6^3\Phi_{12}\Phi_{15}\Phi_{30}$	1
	$G_{27}^7[\zeta_3]$	$\frac{\zeta_3^2}{3}x^9\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4\Phi_5{\Phi''_6}^3\Phi_{10}\Phi'''_{12}\Phi''''_{15}\Phi'''_{30}$	$\zeta_9^2x^{2/3}$
	$G_{27}[\zeta_9^5]$	$\frac{-\zeta_3}{3}x^9\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4\Phi_5{\Phi'_6}^3\Phi_{10}\Phi''''_{12}\Phi'''_{15}\Phi''''_{30}$	$\zeta_9^5x^{1/3}$
*	$\phi_{10,12}$	$\frac{1}{2}x^{12}\Phi_2^2\Phi_5\Phi_6^2\Phi_{10}\Phi_{15}\Phi_{30}$	1
	$\phi''_{5,15}$	$\frac{1}{2}x^{12}\Phi_4\Phi_5\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{30}$	1
	$\phi'_{5,15}$	$\frac{1}{2}x^{12}\Phi_4\Phi_5\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{30}$	1
	$B_2:-1$	$\frac{1}{2}x^{12}\Phi_1^2\Phi_3^2\Phi_5\Phi_{10}\Phi_{15}\Phi_{30}$	-1
*	$\phi_{3,16}$	$\frac{\sqrt{-15}\zeta_3}{30}x^{16}{\Phi'_3}^3\Phi_4\Phi'_5{\Phi''_6}^3\Phi'_{10}\Phi_{12}\Phi''_{15}\Phi_{15}^{(7)}\Phi''_{30}\Phi_{30}^{(7)}$	1
	$\phi_{3,22}$	$\frac{-\sqrt{-15}\zeta_3}{30}x^{16}{\Phi'_3}^3\Phi_4\Phi''_5{\Phi''_6}^3\Phi''_{10}\Phi_{12}\Phi'_{15}\Phi_{15}^{(8)}\Phi'_{30}\Phi_{30}^{(8)}$	1
	$I_2(5)[1,3]:\zeta_3^2$	$\frac{-\sqrt{-15}\zeta_3}{15}x^{16}\Phi_1^2\Phi_2^2\Phi_3^2\Phi'_3\Phi_4\Phi_6^2\Phi''_6\Phi_{12}\Phi''''_{15}\Phi'''_{30}$	ζ_5^2
	$I_2(5)[1,2]:\zeta_3^2$	$\frac{-\sqrt{-15}\zeta_3}{15}x^{16}\Phi_1^2\Phi_2^2\Phi_3^2\Phi'_3\Phi_4\Phi_6^2\Phi''_6\Phi_{12}\Phi''''_{15}\Phi'''_{30}$	ζ_5^3
	$B_2:\zeta_3^2$	$\frac{3+\sqrt{-3}}{12}x^{16}\Phi_1^2\Phi_3^2\Phi'_3\Phi_5{\Phi''_6}^3\Phi'''_{12}\Phi_{15}\Phi'''_{30}$	-1
	$\phi_{6,19}$	$\frac{3+\sqrt{-3}}{12}x^{16}\Phi_2^2{\Phi'_3}^3\Phi_6^2\Phi''_6\Phi_{10}\Phi'''_{12}\Phi''''_{15}\Phi_{30}$	1
#	$\phi''_{3,20}$	$\frac{-\sqrt{-15}\zeta_3^2}{30}x^{16}{\Phi''_3}^3\Phi_4\Phi'_5{\Phi'_6}^3\Phi'_{10}\Phi_{12}\Phi''_{15}\Phi_{15}^{(5)}\Phi''_{30}\Phi_{30}^{(5)}$	1
	$\phi'_{3,20}$	$\frac{\sqrt{-15}\zeta_3^2}{30}x^{16}{\Phi''_3}^3\Phi_4\Phi''_5{\Phi'_6}^3\Phi''_{10}\Phi_{12}\Phi'_{15}\Phi_{15}^{(6)}\Phi'_{30}\Phi_{30}^{(6)}$	1
	$I_2(5)[1,3]:\zeta_3$	$\frac{\sqrt{-15}\zeta_3^2}{15}x^{16}\Phi_1^2\Phi_2^2\Phi_3^2\Phi''_3\Phi_4\Phi_6^2\Phi'_6\Phi_{12}\Phi'''_{15}\Phi''''_{30}$	ζ_5^2
	$I_2(5)[1,2]:\zeta_3$	$\frac{\sqrt{-15}\zeta_3^2}{15}x^{16}\Phi_1^2\Phi_2^2\Phi_3^2\Phi''_3\Phi_4\Phi_6^2\Phi'_6\Phi_{12}\Phi'''_{15}\Phi''''_{30}$	ζ_5^3
	$B_2:\zeta_3$	$\frac{3-\sqrt{-3}}{12}x^{16}\Phi_1^2\Phi_3^2\Phi''_3\Phi_5{\Phi'_6}^3\Phi''''_{12}\Phi_{15}\Phi''''_{30}$	-1
	$\phi_{6,17}$	$\frac{3-\sqrt{-3}}{12}x^{16}\Phi_2^2{\Phi''_3}^3\Phi_6^2\Phi'_6\Phi_{10}\Phi''''_{12}\Phi'''_{15}\Phi_{30}$	1
	$G_{27}^2[\zeta_3]$	$\frac{-\sqrt{-15}}{30}x^{16}\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_{10}\Phi_{12}\Phi''_{15}\Phi''_{30}$	ζ_3
	$G_{27}[\zeta_3]$	$\frac{\sqrt{-15}}{30}x^{16}\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_{10}\Phi_{12}\Phi'_{15}\Phi'_{30}$	ζ_3
	$G_{27}[\zeta_{15}^{11}]$	$\frac{-\sqrt{-15}}{15}x^{16}\Phi_1^3\Phi_2^3\Phi_3^2\Phi_4\Phi_5\Phi_6^2\Phi_{10}\Phi_{12}$	ζ_{15}^{11}
	$G_{27}[\zeta_{15}^{14}]$	$\frac{-\sqrt{-15}}{15}x^{16}\Phi_1^3\Phi_2^3\Phi_3^2\Phi_4\Phi_5\Phi_6^2\Phi_{10}\Phi_{12}$	ζ_{15}^{14}
	$G_{27}[-\zeta_3]$	$\frac{-\sqrt{-3}}{6}x^{16}\Phi_1^3\Phi_2^3\Phi_3^2\Phi_4\Phi_5\Phi_{10}\Phi_{15}$	$-\zeta_3$
	$G_{27}^3[\zeta_3]$	$\frac{-\sqrt{-3}}{6}x^{16}\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_6^2\Phi_{10}\Phi_{30}$	ζ_3
*	$\phi_{1,45}$	x^{45}	1

A.13. Unipotent characters for $G_{3,3,3}$

Some principal ζ-series

$-1 : \mathcal{H}_{Z_6}(-x^3, -\zeta_3^2 x^2, -\zeta_3 x, -1, -\zeta_3^2 x, -\zeta_3 x^2)$

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$	Symbol
$*\quad 1\cdot+$	$x^3\Phi_2\Phi_6$	1	$(1+)$
$*\quad 1\cdot\zeta_3$	$x^3\Phi_2\Phi_6$	1	$(1E3)$
$*\quad 1\cdot\zeta_3^2$	$x^3\Phi_2\Phi_6$	1	$(1E3^2)$
$*\quad .11.1$	$\frac{3-\sqrt{-3}}{6}x^4\Phi_3''^3\Phi_6'$	1	$(01,12,02)$
$\#\quad .1.11$	$\frac{3+\sqrt{-3}}{6}x^4\Phi_3'^3\Phi_6''$	1	$(01,02,12)$
$G_{3,3,3}[\zeta_3^2]$	$\frac{-\sqrt{-3}}{3}x^4\Phi_1^3\Phi_2$	ζ_3^2	$(012,012,)$
$*\quad ..111$	x^9	1	$(012,012,123)$
$*\quad .2.1$	$\frac{3+\sqrt{-3}}{6}x\Phi_3'^3\Phi_6''$	1	$(0,2,1)$
$\#\quad .1.2$	$\frac{3-\sqrt{-3}}{6}x\Phi_3''^3\Phi_6'$	1	$(0,1,2)$
$G_{3,3,3}[\zeta_3]$	$\frac{\sqrt{-3}}{3}x\Phi_1^3\Phi_2$	ζ_3	$(012,,)$
$*\quad ..21$	$x^3\Phi_2\Phi_6$	1	$(01,01,13)$
$*\quad ..3$	1	1	$(0,0,3)$

We used partition tuples for the principal series. The partition with repeated parts 1.1.1 gives rise to 3 characters denoted by $1.+$, $1.\zeta_3$ and $1.\zeta_3^2$. The cuspidals are labeled by Fr.

A.14. Unipotent characters for $G_{4,4,3}$

Some principal ζ-series

$\zeta_3 : \mathcal{H}_{Z_3}(\zeta_3 x^8, \zeta_3, \zeta_3 x^4)$
$\zeta_8 : \mathcal{H}_{Z_8}(\zeta_8^5 x^3, \zeta_8^7 x^2, \zeta_8 x, \zeta_8 x^2, \zeta_8^3 x, \zeta_8^5, \zeta_8^5 x, \zeta_8^5 x^2)$
$-1 : \mathcal{H}_{G_{4,1,2}(\frac{-i-1}{2})}(x^2, -ix, -1, ix; -x, -1)$

Non-principal 1-Harish-Chandra series

$\mathcal{H}_{G_{4,4,3}}(B_2) = \mathcal{H}_{Z_4}(x^2, ix^2, -1, -i)$

γ	Deg(γ)	Fr(γ)	Symbol
$*$ .1.1.1	$x^3\Phi_3\Phi_8$	1	$(0,1,1,1)$
$*$ ..11.1	$\frac{-i+1}{4}x^5\Phi_3\Phi_4''^2\Phi_8'$	1	$(01,01,12,02)$
.1..11	$\frac{1}{2}x^5\Phi_3\Phi_8$	1	$(01,02,01,12)$
# ..1.11	$\frac{i+1}{4}x^5\Phi_3\Phi_4'^2\Phi_8''$	1	$(01,01,02,12)$
$B_2 : -i$	$\frac{i+1}{4}x^5\Phi_1^2\Phi_3\Phi_8'$	-1	$(012,012,0,1)$
$G_{4,4,3}[-i]$	$\frac{-i}{2}x^5\Phi_1^3\Phi_2\Phi_3$	$-i$	$(012,01,012,)$
$B_2 : -1$	$\frac{-i+1}{4}x^5\Phi_1^2\Phi_3\Phi_8''$	-1	$(012,012,1,0)$
$*$ ...111	x^{12}	1	$(012,012,012,123)$
$*$ ..2.1	$\frac{i+1}{4}x\Phi_3\Phi_4'^2\Phi_8''$	1	$(0,0,2,1)$
.1..2	$\frac{1}{2}x\Phi_3\Phi_8$	1	$(0,1,0,2)$
# ..1.2	$\frac{-i+1}{4}x\Phi_3\Phi_4''^2\Phi_8'$	1	$(0,0,1,2)$
$B_2 : i$	$\frac{-i+1}{4}x\Phi_1^2\Phi_3\Phi_8''$	-1	$(01,02,,)$
$G_{4,4,3}[i]$	$\frac{i}{2}x\Phi_1^3\Phi_2\Phi_3$	i	$(012,,0,)$
$B_2 : 1$	$\frac{i+1}{4}x\Phi_1^2\Phi_3\Phi_8'$	-1	$(02,01,,)$
$*$ ...21	$x^4\Phi_8$	1	$(01,01,01,13)$
$*$ ...3	1	1	$(0,0,0,3)$

We used partition tuples for the principal series. The cuspidals are labeled by Fr, and the characters 1-Harish-Chandra induced from B_2 by the corresponding labels.

A.15. Unipotent characters for G_{29}

Some principal ζ-series

ζ_5^4 : $\mathcal{H}_{Z_{20}}(\zeta_5^4 x^4, \zeta_{20} x^{5/2}, -x^2, \zeta_{20}^{11} x^2, \zeta_5^4 x^3, \zeta_{20} x^{3/2}, -\zeta_5 x^2, \zeta_{20}^{11} x, \zeta_5^4 x^2, \zeta_{20} x^3, -\zeta_5^2 x^2, \zeta_{20}^{11} x^{5/2}, \zeta_5^4 x, \zeta_{20} x^2, -\zeta_5^3 x^2, \zeta_{20}^{11} x^{3/2}, \zeta_5^4, \zeta_{20} x, -\zeta_5^4 x^2, \zeta_{20}^{11} x^3)$

ζ_5^3 : $\mathcal{H}_{Z_{20}}(\zeta_5^3 x^4, \zeta_{20}^{17} x^2, -\zeta_5^2 x^2, \zeta_{20}^{7} x^3, \zeta_5^3 x, \zeta_{20}^{17} x^{3/2}, -\zeta_5^3 x^2, \zeta_{20}^{7} x^{5/2}, \zeta_5^3 x^3, \zeta_{20}^{17} x, -\zeta_5^4 x^2, \zeta_{20}^{7} x^2, \zeta_5^3, \zeta_{20}^{17} x^3, -x^2, \zeta_{20}^{7} x^{3/2}, \zeta_5^3 x^2, \zeta_{20}^{17} x^{5/2}, -\zeta_5 x^2, \zeta_{20}^{7} x)$

ζ_5 : $\mathcal{H}_{Z_{20}}(\zeta_5 x^4, \zeta_{20}^{9} x^3, -\zeta_5 x^2, \zeta_{20}^{19} x, \zeta_5, \zeta_{20}^{9} x^{3/2}, -\zeta_5^2 x^2, \zeta_{20}^{19} x^2, \zeta_5 x, \zeta_{20}^{9} x^{5/2}, -\zeta_5^3 x^2, \zeta_{20}^{19} x^3, \zeta_5 x^2, \zeta_{20}^{9} x, -\zeta_5^4 x^2, \zeta_{20}^{19} x^{3/2}, \zeta_5 x^3, \zeta_{20}^{9} x^2, -x^2, \zeta_{20}^{19} x^{5/2})$

Non-principal 1-Harish-Chandra series

$\mathcal{H}_{G_{29}}(B_2) = \mathcal{H}_{G_{4,1,2}}(x^2, ix^2, -1, -i; x^3, -1)$
$\mathcal{H}_{G_{29}}(G_{4,4,3}[i]) = \mathcal{H}_{Z_4}(x^6, ix, -1, -ix)$
$\mathcal{H}_{G_{29}}(G_{4,4,3}[-i]) = \mathcal{H}_{Z_4}(x^6, ix^5, -1, -ix^5)$

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
*	$\phi_{1,0}$	1	1
*	$\phi_{4,1}$	$\frac{i+1}{4}x\Phi_2^2\Phi_4''^2\Phi_6\Phi_8'\Phi_{10}\Phi_{12}'\Phi_{20}''''$	1
	$\phi_{4,4}$	$\frac{1}{2}x\Phi_4^3\Phi_{12}\Phi_{20}$	1
#	$\phi_{4,3}$	$\frac{-i+1}{4}x\Phi_2^2\Phi_4'^2\Phi_6\Phi_8''\Phi_{10}\Phi_{12}''\Phi_{20}'''$	1
	$B_2 : .2..$	$\frac{-i+1}{4}x\Phi_1^2\Phi_3\Phi_4''^2\Phi_5\Phi_8''\Phi_{12}'\Phi_{20}''''$	-1
	$G_{4,4,3}[i] : 1$	$\frac{i}{2}x\Phi_1^3\Phi_2^3\Phi_3\Phi_5\Phi_6\Phi_{10}$	i
	$B_2 : 2...$	$\frac{i+1}{4}x\Phi_1^2\Phi_3\Phi_4'^2\Phi_5\Phi_8'\Phi_{12}''\Phi_{20}'''$	-1
*	$\phi_{10,2}$	$x^2\Phi_5\Phi_8\Phi_{10}\Phi_{20}$	1
	$G_{29}[\zeta_8^5]$	$\frac{i}{2}x^3\Phi_1^4\Phi_2^4\Phi_3\Phi_5\Phi_6\Phi_8\Phi_{10}$	$\zeta_8^5 x^{1/2}$
*	$\phi_{16,3}$	$\frac{1}{2}x^3\Phi_4^4\Phi_8\Phi_{12}\Phi_{20}$	1
	$G_{29}[\zeta_8]$	$\frac{i}{2}x^3\Phi_1^4\Phi_2^4\Phi_3\Phi_5\Phi_6\Phi_8\Phi_{10}$	$\zeta_8 x^{1/2}$
	$\phi_{16,5}$	$\frac{1}{2}x^3\Phi_4^4\Phi_8\Phi_{12}\Phi_{20}$	1
*	$\phi_{15,4}''$	$\frac{1}{2}x^4\Phi_3\Phi_5\Phi_6\Phi_8\Phi_{10}\Phi_{20}$	1
	$\phi_{5,8}$	$\frac{1}{2}x^4\Phi_5\Phi_8\Phi_{10}\Phi_{12}\Phi_{20}$	1
	$\phi_{10,6}$	$\frac{1}{2}x^4\Phi_4^2\Phi_5\Phi_{10}\Phi_{12}\Phi_{20}$	1
	$B_2 : 1.1..$	$\frac{1}{2}x^4\Phi_1^2\Phi_2^2\Phi_3\Phi_5\Phi_6\Phi_{10}\Phi_{20}$	-1
*	$\phi_{15,4}'$	$x^4\Phi_3\Phi_5\Phi_6\Phi_{10}\Phi_{12}\Phi_{20}$	1
*	$\phi_{20,5}$	$\frac{-i+1}{4}x^5\Phi_2^2\Phi_4''^2\Phi_5\Phi_6\Phi_8''\Phi_{10}\Phi_{12}'\Phi_{20}$	1
	$\phi_{20,6}$	$\frac{1}{2}x^5\Phi_4^3\Phi_5\Phi_{10}\Phi_{12}\Phi_{20}$	1
#	$\phi_{20,7}$	$\frac{i+1}{4}x^5\Phi_2^2\Phi_4'^2\Phi_5\Phi_6\Phi_8'\Phi_{10}\Phi_{12}''\Phi_{20}$	1
	$B_2 : ...2$	$\frac{i+1}{4}x^5\Phi_1^2\Phi_3\Phi_4''^2\Phi_5\Phi_8'\Phi_{10}\Phi_{12}'\Phi_{20}$	-1
	$G_{4,4,3}[-i] : 1$	$\frac{-i}{2}x^5\Phi_1^3\Phi_2^3\Phi_3\Phi_5\Phi_6\Phi_{10}\Phi_{20}$	$-i$

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
	$B_2 : ..2.$	$\frac{-i+1}{4}x^5\Phi_1^2\Phi_3{\Phi'_4}^2\Phi_5\Phi''_8\Phi_{10}\Phi''_{12}\Phi_{20}$	-1
$*$	$\phi_{24,6}$	$\frac{1}{20}x^6\Phi_2^4\Phi_3\Phi_5\Phi_6\Phi_8\Phi_{12}\Phi_{20}$	1
	$G_{29}[1]$	$\frac{1}{20}x^6\Phi_1^4\Phi_3\Phi_6\Phi_8\Phi_{10}\Phi_{12}\Phi_{20}$	1
	$\phi''''_{6,10}$	$\frac{-1}{20}x^6\Phi_3{\Phi''_4}^4\Phi_5\Phi_6\Phi_8\Phi_{10}\Phi_{12}\Phi'''_{20}$	1
	$\phi'''_{6,10}$	$\frac{-1}{20}x^6\Phi_3{\Phi'_4}^4\Phi_5\Phi_6\Phi_8\Phi_{10}\Phi_{12}\Phi''''_{20}$	1
	$\phi''_{6,10}$	$\frac{1}{5}x^6\Phi_3\Phi_5\Phi_6\Phi_8\Phi_{10}\Phi_{12}\Phi_{20}$	1
	$\phi_{24,9}$	$\frac{1}{4}x^6\Phi_2^2\Phi_3\Phi_4^3\Phi_6\Phi_{10}\Phi_{12}\Phi_{20}$	1
	$B_2 : .1..1$	$\frac{1}{4}x^6\Phi_1^2\Phi_3\Phi_4^3\Phi_5\Phi_6\Phi_{12}\Phi_{20}$	-1
	$G_{4,4,3}[i] : -i$	$\frac{-1}{4}x^6\Phi_1^3\Phi_2^3\Phi_3{\Phi'_4}^2\Phi_5\Phi_6\Phi_{10}\Phi_{12}\Phi'''_{20}$	i
	$G_{4,4,3}[-i] : i$	$\frac{-1}{4}x^6\Phi_1^3\Phi_2^3\Phi_3{\Phi''_4}^2\Phi_5\Phi_6\Phi_{10}\Phi_{12}\Phi''''_{20}$	$-i$
	$\phi_{24,7}$	$\frac{1}{4}x^6\Phi_2^2\Phi_3\Phi_4^3\Phi_6\Phi_{10}\Phi_{12}\Phi_{20}$	1
	$B_2 : 1..1.$	$\frac{1}{4}x^6\Phi_1^2\Phi_3\Phi_4^3\Phi_5\Phi_6\Phi_{12}\Phi_{20}$	-1
	$G_{4,4,3}[-i] : -i$	$\frac{1}{4}x^6\Phi_1^3\Phi_2^3\Phi_3{\Phi'_4}^2\Phi_5\Phi_6\Phi_{10}\Phi_{12}\Phi'''_{20}$	$-i$
	$G_{4,4,3}[i] : i$	$\frac{1}{4}x^6\Phi_1^3\Phi_2^3\Phi_3{\Phi''_4}^2\Phi_5\Phi_6\Phi_{10}\Phi_{12}\Phi''''_{20}$	i
	$\phi_{30,8}$	$\frac{1}{4}x^6\Phi_3\Phi_4^2\Phi_5\Phi_6\Phi_8\Phi_{12}\Phi_{20}$	1
	$\phi_{6,12}$	$\frac{1}{4}x^6\Phi_3\Phi_4^2\Phi_6\Phi_8\Phi_{10}\Phi_{12}\Phi_{20}$	1
	$B_2 : .1.1.$	$\frac{1}{4}x^6\Phi_1^2\Phi_2^2\Phi_3\Phi_5\Phi_6\Phi_8\Phi_{10}\Phi_{12}\Phi''''_{20}$	-1
	$B_2 : 1...1$	$\frac{1}{4}x^6\Phi_1^2\Phi_2^2\Phi_3\Phi_5\Phi_6\Phi_8\Phi_{10}\Phi_{12}\Phi'''_{20}$	-1
	$\phi'_{6,10}$	$\frac{1}{5}x^6\Phi_3\Phi_5\Phi_6\Phi_8\Phi_{10}\Phi_{12}\Phi_{20}$	1
	$G_{29}[\zeta_5]$	$\frac{1}{5}x^6\Phi_1^4\Phi_2^4\Phi_3\Phi_4^4\Phi_6\Phi_8\Phi_{12}$	ζ_5
	$G_{29}[\zeta_5^2]$	$\frac{1}{5}x^6\Phi_1^4\Phi_2^4\Phi_3\Phi_4^4\Phi_6\Phi_8\Phi_{12}$	ζ_5^2
	$G_{29}[\zeta_5^3]$	$\frac{1}{5}x^6\Phi_1^4\Phi_2^4\Phi_3\Phi_4^4\Phi_6\Phi_8\Phi_{12}$	ζ_5^3
	$G_{29}[\zeta_5^4]$	$\frac{1}{5}x^6\Phi_1^4\Phi_2^4\Phi_3\Phi_4^4\Phi_6\Phi_8\Phi_{12}$	ζ_5^4
$*$	$\phi_{20,9}$	$\frac{i+1}{4}x^9\Phi_2^2{\Phi'_4}^2\Phi_5\Phi_6\Phi'_8\Phi_{10}\Phi''_{12}\Phi_{20}$	1
	$\phi_{20,10}$	$\frac{1}{2}x^9\Phi_4^3\Phi_5\Phi_{10}\Phi_{12}\Phi_{20}$	1
#	$\phi_{20,11}$	$\frac{-i+1}{4}x^9\Phi_2^2{\Phi''_4}^2\Phi_5\Phi_6\Phi''_8\Phi_{10}\Phi'_{12}\Phi_{20}$	1
	$B_2 : .11..$	$\frac{-i+1}{4}x^9\Phi_1^2\Phi_3{\Phi'_4}^2\Phi_5\Phi''_8\Phi_{10}\Phi''_{12}\Phi_{20}$	-1
	$G_{4,4,3}[i] : -1$	$\frac{i}{2}x^9\Phi_1^3\Phi_2^3\Phi_3\Phi_5\Phi_6\Phi_{10}\Phi_{20}$	i
	$B_2 : 11...$	$\frac{i+1}{4}x^9\Phi_1^2\Phi_3{\Phi''_4}^2\Phi_5\Phi'_8\Phi_{10}\Phi'_{12}\Phi_{20}$	-1
$*$	$\phi'_{15,12}$	$x^{12}\Phi_3\Phi_5\Phi_6\Phi_{10}\Phi_{12}\Phi_{20}$	1
$*$	$\phi''_{15,12}$	$\frac{1}{2}x^{12}\Phi_3\Phi_5\Phi_6\Phi_8\Phi_{10}\Phi_{20}$	1
	$\phi_{5,16}$	$\frac{1}{2}x^{12}\Phi_5\Phi_8\Phi_{10}\Phi_{12}\Phi_{20}$	1
	$\phi_{10,14}$	$\frac{1}{2}x^{12}\Phi_4^2\Phi_5\Phi_{10}\Phi_{12}\Phi_{20}$	1
	$B_2 : ..1.1$	$\frac{1}{2}x^{12}\Phi_1^2\Phi_2^2\Phi_3\Phi_5\Phi_6\Phi_{10}\Phi_{20}$	-1
	$G_{29}[\zeta_8^3]$	$\frac{-i}{2}x^{13}\Phi_1^4\Phi_2^4\Phi_3\Phi_5\Phi_6\Phi_8\Phi_{10}$	$\zeta_8^3x^{1/2}$
$*$	$\phi_{16,13}$	$\frac{1}{2}x^{13}\Phi_4^4\Phi_8\Phi_{12}\Phi_{20}$	1
	$G_{29}[\zeta_8^7]$	$\frac{-i}{2}x^{13}\Phi_1^4\Phi_2^4\Phi_3\Phi_5\Phi_6\Phi_8\Phi_{10}$	$\zeta_8^7x^{1/2}$
	$\phi_{16,15}$	$\frac{1}{2}x^{13}\Phi_4^4\Phi_8\Phi_{12}\Phi_{20}$	1
$*$	$\phi_{10,18}$	$x^{18}\Phi_5\Phi_8\Phi_{10}\Phi_{20}$	1

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
*	$\phi_{4,21}$	$\frac{-i+1}{4}x^{21}\Phi_2^2\Phi_4'^2\Phi_6\Phi_8''\Phi_{10}\Phi_{12}''\Phi_{20}'''$	1
	$\phi_{4,24}$	$\frac{1}{2}x^{21}\Phi_4^3\Phi_{12}\Phi_{20}$	1
#	$\phi_{4,23}$	$\frac{i+1}{4}x^{21}\Phi_2^2\Phi_4''^2\Phi_6\Phi_8'\Phi_{10}\Phi_{12}'\Phi_{20}''''$	1
	$B_2 : ...11$	$\frac{i+1}{4}x^{21}\Phi_1^2\Phi_3\Phi_4'^2\Phi_5\Phi_8'\Phi_{12}''\Phi_{20}'''$	-1
	$G_{4,4,3}[-i] : -1$	$\frac{-i}{2}x^{21}\Phi_1^3\Phi_2^3\Phi_3\Phi_5\Phi_6\Phi_{10}$	$-i$
	$B_2 : ..11.$	$\frac{-i+1}{4}x^{21}\Phi_1^2\Phi_3\Phi_4''^2\Phi_5\Phi_8''\Phi_{12}'\Phi_{20}''''$	-1
*	$\phi_{1,40}$	x^{40}	1

A.16. Unipotent characters for G_{32}

Some principal ζ-series

$\zeta_8^3 : \mathcal{H}_{Z_{24}}(\zeta_8 x^5, \zeta_{24}^7 x^2, \zeta_{24}^{11} x^{5/3}, \zeta_8^3 x^2, \zeta_{24}^{19} x, \zeta_{24}^{11} x^2, \zeta_8 x^3, \zeta_{24}^7, \zeta_{24}^{23} x, \zeta_8^5 x^2, \zeta_{24}^{19} x^{5/3}, \zeta_{24}^{11}, \zeta_8 x, \zeta_{24}^{19} x^2, \zeta_{24}^{11} x^3, \zeta_8^7 x^2, \zeta_{24}^7 x, \zeta_{24}^{23} x^2, \zeta_8 x^{5/3}, \zeta_{24}^{19}, \zeta_{24}^{11} x, \zeta_8 x^2, \zeta_{24}^{19} x^3, \zeta_{24}^{23})$

$\zeta_8 : \mathcal{H}_{Z_{24}}(\zeta_8^3 x^5, \zeta_{24}, \zeta_{24}^{17} x^3, \zeta_8^7 x^2, \zeta_{24} x, \zeta_{24}^5, \zeta_8^3 x^{5/3}, \zeta_{24} x^2, \zeta_{24}^5 x, \zeta_8 x^2, \zeta_{24} x^3, \zeta_{24}^5 x^2, \zeta_8^3 x, \zeta_{24}^{13}, \zeta_{24}^{17} x^{5/3}, \zeta_8^3 x^2, \zeta_{24}^{13} x, \zeta_{24}^{17}, \zeta_8^3 x^3, \zeta_{24}^{13} x^2, \zeta_{24}^{17} x, \zeta_8^5 x^2, \zeta_{24} x^{5/3}, \zeta_{24}^{17} x^2)$

$\zeta_5^3 : \mathcal{H}_{Z_{30}}(\zeta_5^3 x^4, -\zeta_{15}^{14} x, \zeta_{15}^4 x^{4/3}, -x, \zeta_{15}^{14} x^2, -\zeta_{15}^4 x^{3/2}, \zeta_5^3 x, -\zeta_{15}^{14} x^3, \zeta_{15}^4, -\zeta_5 x, \zeta_{15}^{14} x^{3/2}, -\zeta_{15}^4 x, \zeta_5^3 x^{4/3}, -\zeta_{15}^{14}, \zeta_{15}^4 x^2, -\zeta_5^2 x, \zeta_{15}^{14} x, -\zeta_{15}^4 x^3, \zeta_5^3, -\zeta_{15}^{14} x^2, \zeta_{15}^4 x^{3/2}, -\zeta_5^3 x, \zeta_{15}^{14} x^{4/3}, -\zeta_{15}^4, \zeta_5^3 x^2, -\zeta_{15}^{14} x^{3/2}, \zeta_{15}^4 x, -\zeta_5^4 x, \zeta_{15}^{14}, -\zeta_{15}^4 x^2)$

$\zeta_5^2 : \mathcal{H}_{Z_{30}}(\zeta_5^2 x^4, -\zeta_{15}^{11} x^2, \zeta_{15}, -\zeta_5 x, \zeta_{15}^{11} x, -\zeta_{15} x^{3/2}, \zeta_5^2 x^2, -\zeta_{15}^{11}, \zeta_{15} x^{4/3}, -\zeta_5^2 x, \zeta_{15}^{11} x^{3/2}, -\zeta_{15} x^2, \zeta_5^2, -\zeta_{15}^{11} x^3, \zeta_{15} x, -\zeta_5^3 x, \zeta_{15}^{11} x^2, -\zeta_{15}, \zeta_5^2 x^{4/3}, -\zeta_{15}^{11} x, \zeta_{15} x^{3/2}, -\zeta_5^4 x, \zeta_{15}^{11}, -\zeta_{15} x^3, \zeta_5^2 x, -\zeta_{15}^{11} x^{3/2}, \zeta_{15} x^2, -x, \zeta_{15}^{11} x^{4/3}, -\zeta_{15} x)$

$\zeta_5 : \mathcal{H}_{Z_{30}}(\zeta_5 x^4, -\zeta_{15}^8, \zeta_{15}^{13} x, -\zeta_5^2 x, \zeta_{15}^8 x^{4/3}, -\zeta_{15}^{13} x^{3/2}, \zeta_5, -\zeta_{15}^8 x, \zeta_{15}^{13} x^2, -\zeta_5^3 x, \zeta_{15}^8 x^{3/2}, -\zeta_{15}^{13}, \zeta_5 x, -\zeta_{15}^8 x^2, \zeta_{15}^{13} x^{4/3}, -\zeta_5^4 x, \zeta_{15}^8, -\zeta_{15}^{13} x, \zeta_5 x^2, -\zeta_{15}^8 x^3, \zeta_{15}^{13} x^{3/2}, -x, \zeta_{15}^8 x, -\zeta_{15}^{13} x^2, \zeta_5 x^{4/3}, -\zeta_{15}^8 x^{3/2}, \zeta_{15}^{13}, -\zeta_5 x, \zeta_{15}^8 x^2, -\zeta_{15}^{13} x^3)$

$\zeta_4 : \mathcal{H}_{G_{10}}(-x^2, \zeta_3, \zeta_3^2; ix^3, i, ix, -i)$

Non-principal 1-Harish-Chandra series

$\mathcal{H}_{G_{32}}(Z_3) = \mathcal{H}_{G_{26}}(x^3, -1; x, \zeta_3, \zeta_3^2; x, \zeta_3, \zeta_3^2)$

$\mathcal{H}_{G_{32}}(G_4) = \mathcal{H}_{G_5}(x, \zeta_3, \zeta_3^2; 1, \zeta_3 x^4, \zeta_3^2 x^4)$

$\mathcal{H}_{G_{32}}(Z_3 \otimes Z_3) = \mathcal{H}_{G_{6,1,2}}(x^3, -\zeta_3^2 x^3, \zeta_3 x^2, -1, \zeta_3^2, -\zeta_3 x^2; x^3, -1)$

$\mathcal{H}_{G_{32}}(G_{25}[\zeta_3]) = \mathcal{H}_{Z_6}(x^6, -\zeta_3^2 x^4, \zeta_3 x, -1, \zeta_3^2 x, -\zeta_3 x^4)$

$\mathcal{H}_{G_{32}}(G_{25}[-\zeta_3]) = \mathcal{H}_{Z_6}(x^6, -\zeta_3^2 x, \zeta_3 x^4, -1, \zeta_3^2 x^4, -\zeta_3 x)$

$\mathcal{H}_{G_{32}}(G_4 \otimes Z_3) = \mathcal{H}_{Z_6}(x^9, -\zeta_3^2 x^8, \zeta_3 x^5, -1, \zeta_3^2 x^5, -\zeta_3 x^8)$

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
*	$\phi_{1,0}$	1	1
*	$\phi_{4,1}$	$\frac{3-\sqrt{-3}}{6} x\Phi_3''\Phi_4\Phi_6'\Phi_8\Phi_{12}\Phi_{15}'''\Phi_{24}\Phi_{30}''''$	1
#	$\phi_{4,11}$	$\frac{3+\sqrt{-3}}{6} x\Phi_3'\Phi_4\Phi_6''\Phi_8\Phi_{12}\Phi_{15}''''\Phi_{24}\Phi_{30}'''$	1
	$Z_3 : \phi_{1,0}$	$\frac{-\sqrt{-3}}{3} x\Phi_1\Phi_2\Phi_4\Phi_5\Phi_8\Phi_{10}\Phi_{12}\Phi_{24}$	ζ_3^2
	$\phi_{15,6}$	$\frac{1}{3} x^2\Phi_3^2\Phi_5\Phi_6^2\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{24}\Phi_{30}$	1
*	$\phi_{10,2}$	$\frac{1}{3} x^2\Phi_3'\Phi_5\Phi_6''\Phi_8\Phi_9'\Phi_{10}{\Phi_{12}'''}^2\Phi_{15}\Phi_{18}''\Phi_{24}''\Phi_{30}$	1
#	$\phi_{10,10}$	$\frac{1}{3} x^2\Phi_3''\Phi_5\Phi_6'\Phi_8\Phi_9''\Phi_{10}{\Phi_{12}''''}^2\Phi_{15}\Phi_{18}'\Phi_{24}'\Phi_{30}$	1
	$\phi_{5,20}$	$\frac{\zeta_3}{3} x^2\Phi_3'\Phi_5\Phi_6''\Phi_9''\Phi_{10}{\Phi_{12}'''}^2\Phi_{15}\Phi_{18}'\Phi_{24}\Phi_{30}$	1
	$Z_3 \otimes Z_3 : .2....$	$\frac{\zeta_3}{3} x^2\Phi_1^2\Phi_2^2{\Phi_3'}^2\Phi_4\Phi_5{\Phi_6''}^2\Phi_8\Phi_{10}\Phi_{12}'''\Phi_{15}\Phi_{24}''\Phi_{30}$	ζ_3
	$Z_3 : \phi_{2,9}$	$\frac{-\zeta_3}{3} x^2\Phi_1\Phi_2\Phi_4^2\Phi_5\Phi_8\Phi_9'\Phi_{10}\Phi_{15}\Phi_{18}''\Phi_{24}'\Phi_{30}$	ζ_3^2
	$\phi_{5,4}$	$\frac{\zeta_3^2}{3} x^2\Phi_3''\Phi_5\Phi_6'\Phi_9'\Phi_{10}{\Phi_{12}''''}^2\Phi_{15}\Phi_{18}''\Phi_{24}\Phi_{30}$	1
	$Z_3 : \phi_{2,3}$	$\frac{\zeta_3^2}{3} x^2\Phi_1\Phi_2\Phi_4^2\Phi_5\Phi_8\Phi_9''\Phi_{10}\Phi_{15}\Phi_{18}'\Phi_{24}''\Phi_{30}$	ζ_3^2
	$Z_3 \otimes Z_3 : 2.....$	$\frac{\zeta_3^2}{3} x^2\Phi_1^2\Phi_2^2{\Phi_3''}^2\Phi_4\Phi_5{\Phi_6'}^2\Phi_8\Phi_{10}\Phi_{12}''''\Phi_{15}\Phi_{24}'\Phi_{30}$	ζ_3
*	$\phi_{20,3}$	$\frac{3-\sqrt{-3}}{6} x^3\Phi_4\Phi_5\Phi_8\Phi_9'\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{18}''\Phi_{24}\Phi_{30}$	1

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
#	$\phi'_{20,9}$	$\frac{3+\sqrt{-3}}{6}x^3\Phi_4\Phi_5\Phi_8\Phi''_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	1
	$Z_3:\phi_{3,6}$	$\frac{-\sqrt{-3}}{3}x^3\Phi_1\Phi_2\Phi_3\Phi_4\Phi_5\Phi_6\Phi_8\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{24}\Phi_{30}$	ζ_3^2
*	$\phi_{30,4}$	$\frac{-\zeta_3}{6}x^4{\Phi'_3}^2\Phi_4\Phi_5{\Phi''_6}^2\Phi_8\Phi_9\Phi_{12}\Phi'''_{12}\Phi''''_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$\phi_{36,5}$	$\frac{3-\sqrt{-3}}{12}x^4\Phi_2^2{\Phi''_3}^3\Phi_6^2\Phi'_6\Phi_8\Phi_9\Phi_{10}\Phi''''_{12}\Phi'''_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$\phi_{24,6}$	$\frac{1}{3}x^4\Phi_3\Phi_4^2\Phi_6\Phi_8\Phi_9\Phi_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$\phi_{36,7}$	$\frac{3+\sqrt{-3}}{12}x^4\Phi_2^2{\Phi'_3}^3\Phi_6^2\Phi''_6\Phi_8\Phi_9\Phi_{10}\Phi'''_{12}\Phi''''_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
#	$\phi_{30,8}$	$\frac{-\zeta_3^2}{6}x^4{\Phi''_3}^2\Phi_4\Phi_5{\Phi'_6}^2\Phi_8\Phi_9\Phi_{12}\Phi''''_{12}\Phi'''_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$Z_3\otimes Z_3:..2...$	$\frac{1}{6}x^4\Phi_1^2\Phi_2^2\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi''''_{15}\Phi_{18}\Phi_{24}\Phi''''_{30}$	ζ_3
	$G_4:\phi'_{1,4}$	$\frac{-3-\sqrt{-3}}{12}x^4\Phi_1^2\Phi_3^2\Phi'_3\Phi_5{\Phi''_6}^3\Phi_8\Phi_9\Phi'''_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi'''_{30}$	-1
	$Z_3:\phi'_{3,5}$	$\frac{\zeta_3^2}{3}x^4\Phi_1\Phi_2\Phi'_3\Phi_4\Phi_5\Phi''_6\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi''''_{15}\Phi_{18}\Phi_{24}\Phi'''_{30}$	ζ_3^2
	$G_{25}[-\zeta_3]:1$	$\frac{\sqrt{-3}}{6}x^4\Phi_1^3\Phi_2^3\Phi_3^2\Phi_4\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{15}\Phi_{18}\Phi_{24}$	$-\zeta_3$
	$\phi_{6,8}$	$\frac{\zeta_3^2}{6}x^4{\Phi''_3}^2\Phi_4{\Phi'_6}^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi''''_{30}$	1
	$G_{25}[\zeta_3]:1$	$\frac{\sqrt{-3}}{6}x^4\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{10}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3
	$Z_3:\phi_{3,1}$	$\frac{-\zeta_3}{3}x^4\Phi_1\Phi_2\Phi''_3\Phi_4\Phi_5\Phi'_6\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi'''_{15}\Phi_{18}\Phi_{24}\Phi''''_{30}$	ζ_3^2
	$\phi_{6,28}$	$\frac{\zeta_3}{6}x^4{\Phi'_3}^2\Phi_4{\Phi''_6}^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi'''_{30}$	1
	$G_4:\phi'_{1,8}$	$\frac{-3+\sqrt{-3}}{12}x^4\Phi_1^2\Phi_3^2\Phi''_3\Phi_5{\Phi'_6}^3\Phi_8\Phi_9\Phi''''_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi''''_{30}$	-1
	$Z_3\otimes Z_3:.....2$	$\frac{1}{6}x^4\Phi_1^2\Phi_2^2\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi'''_{15}\Phi_{18}\Phi_{24}\Phi'''_{30}$	ζ_3
*	$\phi_{20,5}$	$\frac{\zeta_3^2}{6}x^5\Phi'_3\Phi_4^2\Phi_5\Phi''_6\Phi_8\Phi''_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	1
	$\phi_{40,8}$	$\frac{3-\sqrt{-3}}{12}x^5\Phi_2^2\Phi'_3\Phi_4\Phi_5\Phi_6^2\Phi''_6\Phi_8\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$G_4:\phi'''_{2,5}$	$\frac{-3+\sqrt{-3}}{12}x^5\Phi_1^2\Phi_3^2\Phi'_3\Phi_4\Phi_5\Phi''_6\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{24}\Phi_{30}$	-1
	$Z_3:\phi_{6,2}$	$\frac{\zeta_3^2}{3}x^5\Phi_1\Phi_2\Phi_3\Phi'_3\Phi_4^2\Phi_5\Phi_6\Phi''_6\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{24}\Phi_{30}$	ζ_3^2
	$\phi_{20,17}$	$\frac{-\zeta_3^2}{6}x^5\Phi'_3\Phi_4^2\Phi_5\Phi''_6\Phi_8\Phi'_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	1
	$\phi_{20,7}$	$\frac{-\zeta_3}{6}x^5\Phi''_3\Phi_4^2\Phi_5\Phi'_6\Phi_8\Phi''_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	1
	$\phi_{40,10}$	$\frac{3+\sqrt{-3}}{12}x^5\Phi_2^2\Phi''_3\Phi_4\Phi_5\Phi_6^2\Phi'_6\Phi_8\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$G_4:\phi''_{2,7}$	$\frac{-3-\sqrt{-3}}{12}x^5\Phi_1^2\Phi_3^2\Phi''_3\Phi_4\Phi_5\Phi'_6\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{24}\Phi_{30}$	-1
	$Z_3:\phi'_{6,4}$	$\frac{-\zeta_3}{3}x^5\Phi_1\Phi_2\Phi_3\Phi''_3\Phi_4^2\Phi_5\Phi_6\Phi'_6\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{24}\Phi_{30}$	ζ_3^2
#	$\phi_{20,19}$	$\frac{\zeta_3}{6}x^5\Phi''_3\Phi_4^2\Phi_5\Phi'_6\Phi_8\Phi'_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	1
	$Z_3:\phi_{1,12}$	$\frac{-1}{6}x^5\Phi_1\Phi_2\Phi_4^2\Phi_5\Phi_8\Phi''_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
	$Z_3:\phi_{2,15}$	$\frac{-\sqrt{-3}}{6}x^5\Phi_1\Phi_2^3\Phi_4\Phi_5\Phi_6^2\Phi_8\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
	$G_4\otimes Z_3:1$	$\frac{\sqrt{-3}}{6}x^5\Phi_1^3\Phi_2\Phi_3^2\Phi_4\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{24}\Phi_{30}$	$-\zeta_3^2$
	$Z_3\otimes Z_3:1.1....$	$\frac{-1}{3}x^5\Phi_1^2\Phi_2^2\Phi_3\Phi_4^2\Phi_5\Phi_6\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{24}\Phi_{30}$	ζ_3
	$Z_3:\phi_{1,24}$	$\frac{1}{6}x^5\Phi_1\Phi_2\Phi_4^2\Phi_5\Phi_8\Phi'_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
	$\phi_{20,12}$	$\frac{1}{24}x^6\Phi_2^4\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$\phi_{5,12}$	$\frac{1}{24}x^6\Phi_4^2\Phi_5{\Phi''_6}^4\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$\phi_{5,36}$	$\frac{1}{24}x^6\Phi_4^2\Phi_5{\Phi'_6}^4\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$\phi_{80,9}$	$\frac{1}{12}x^6\Phi_2^4\Phi_4^2\Phi_5\Phi_6^2\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$\phi_{20,21}$	$\frac{1}{12}x^6\Phi_2^2\Phi_4\Phi_5\Phi_6^2{\Phi''_6}^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi_{15}\Phi_{18}\Phi'_{24}\Phi_{30}$	1
	$\phi''_{20,9}$	$\frac{1}{12}x^6\Phi_2^2\Phi_4\Phi_5\Phi_6^2{\Phi'_6}^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi_{15}\Phi_{18}\Phi''_{24}\Phi_{30}$	1
	$G_{32}^2[1]$	$\frac{1}{8}x^6\Phi_1^4\Phi_3^4\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$G_{32}[1]$	$\frac{1}{24}x^6\Phi_1^4\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
* $\phi_{45,6}$	$\frac{1}{24}x^6{\Phi'_3}^4\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
# $\phi_{45,18}$	$\frac{1}{24}x^6{\Phi''_3}^4\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$G_{32}[-1]$	$\frac{-1}{12}x^6\Phi_1^4\Phi_3^2\Phi_4^2\Phi_5\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	-1
$G_4:\phi''_{1,12}$	$\frac{1}{12}x^6\Phi_1^2\Phi_3^2{\Phi'_3}^2\Phi_4\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi_{15}\Phi_{18}\Phi'_{24}\Phi_{30}$	-1
$G_4:\phi'_{1,12}$	$\frac{1}{12}x^6\Phi_1^2\Phi_3^2{\Phi''_3}^2\Phi_4\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi_{15}\Phi_{18}\Phi''_{24}\Phi_{30}$	-1
$\phi_{60,12}$	$\frac{1}{8}x^6\Phi_2^4\Phi_5\Phi_6^4\Phi_8\Phi_9\Phi_{10}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$\phi''_{30,12}$	$\frac{1}{4}x^6\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$G_4:\phi_{2,9}$	$\frac{-1}{4}x^6\Phi_1^2\Phi_2^2\Phi_3^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{10}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	-1
$G_{32}[-i]$	$\frac{1}{4}x^6\Phi_1^4\Phi_2^4\Phi_3^4\Phi_5\Phi_6^4\Phi_9\Phi_{10}\Phi_{15}\Phi_{18}\Phi_{30}$	$-i$
$G_{32}[i]$	$\frac{1}{4}x^6\Phi_1^4\Phi_2^4\Phi_3^4\Phi_5\Phi_6^4\Phi_9\Phi_{10}\Phi_{15}\Phi_{18}\Phi_{30}$	i
$\phi_{45,14}$	$\frac{-\zeta_3^2}{6}x^6\Phi_3{\Phi'_3}^2\Phi_5\Phi_6\Phi''_6\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	1
$G_4:\phi''_{1,8}$	$\frac{\zeta_3^2}{6}x^6\Phi_1^2\Phi_3^3\Phi_4\Phi_5\Phi_6\Phi''_6\Phi_9\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	-1
$G_{25}[\zeta_3]:-\zeta_3$	$\frac{\zeta_3^2}{6}x^6\Phi_1^3\Phi_2^2\Phi'_3\Phi_4^2\Phi_5\Phi''_6\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	ζ_3
$G_{25}[-\zeta_3]:\zeta_3$	$\frac{-\zeta_3^2}{6}x^6\Phi_1^3\Phi_2^2\Phi_3^2\Phi'_3\Phi_4^2\Phi_5\Phi''_6\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi'_{18}\Phi''_{24}\Phi_{30}$	$-\zeta_3$
$Z_3:\phi_{9,5}$	$\frac{\zeta_3^2}{6}x^6\Phi_1\Phi_2{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^2\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$G_4\otimes Z_3:-\zeta_3$	$\frac{\zeta_3^2}{6}x^6\Phi_1^3\Phi_2\Phi_3^2\Phi''_3\Phi_4\Phi_5{\Phi'_6}^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi_{15}\Phi'_{18}\Phi'_{24}\Phi_{30}$	$-\zeta_3^2$
$\phi_{15,8}$	$\frac{\zeta_3^2}{6}x^6\Phi_3\Phi'_3\Phi_5\Phi_6{\Phi''_6}^2\Phi_8\Phi''_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$\phi'_{60,11}$	$\frac{-\zeta_3^2}{6}x^6\Phi_2^2\Phi_3\Phi'_3\Phi_4\Phi_5\Phi_6^3\Phi''_9\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$Z_3\otimes Z_3:.1....1$	$\frac{\zeta_3^2}{6}x^6\Phi_1^2\Phi_2^3\Phi'_3\Phi_4^2\Phi_5\Phi''_6\Phi_8\Phi''_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3
$Z_3\otimes Z_3:.1.1...$	$\frac{-\zeta_3^2}{6}x^6\Phi_1^2\Phi_2^3\Phi'_3\Phi_4^2\Phi_5\Phi_6^2\Phi''_6\Phi_8\Phi''_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{18}\Phi''_{24}\Phi_{30}$	ζ_3
$Z_3:\phi''_{3,5}$	$\frac{\zeta_3^2}{6}x^6\Phi_1\Phi_2{\Phi''_3}^2\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_8\Phi''_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$Z_3:\phi''_{6,8}$	$\frac{\zeta_3^2}{6}x^6\Phi_1\Phi_2^3{\Phi''_3}^2\Phi_4\Phi_5\Phi_6^2\Phi'_6\Phi_8\Phi''_9\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi_{15}\Phi_{18}\Phi'_{24}\Phi_{30}$	ζ_3^2
$\phi_{15,16}$	$\frac{\zeta_3}{6}x^6\Phi_3\Phi''_3\Phi_5\Phi_6{\Phi'_6}^2\Phi_8\Phi'_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{60,7}$	$\frac{-\zeta_3}{6}x^6\Phi_2^2\Phi_3\Phi''_3\Phi_4\Phi_5\Phi_6^3\Phi'_9\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$Z_3\otimes Z_3:1..1...$	$\frac{\zeta_3}{6}x^6\Phi_1^2\Phi_2^3\Phi''_3\Phi_4^2\Phi_5\Phi'_6\Phi_8\Phi'_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3
$Z_3\otimes Z_3:1.....1$	$\frac{-\zeta_3}{6}x^6\Phi_1^2\Phi_2^3\Phi''_3\Phi_4^2\Phi_5\Phi_6^2\Phi'_6\Phi_8\Phi'_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{18}\Phi'_{24}\Phi_{30}$	ζ_3
$Z_3:\phi''_{3,13}$	$\frac{-\zeta_3}{6}x^6\Phi_1\Phi_2{\Phi'_3}^2\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_8\Phi'_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$Z_3:\phi''_{6,4}$	$\frac{-\zeta_3}{6}x^6\Phi_1\Phi_2^3{\Phi'_3}^2\Phi_4\Phi_5\Phi_6^2\Phi''_6\Phi_8\Phi'_9\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi_{15}\Phi_{18}\Phi''_{24}\Phi_{30}$	ζ_3^2
$\phi_{45,10}$	$\frac{-\zeta_3}{6}x^6\Phi_3{\Phi''_3}^2\Phi_5\Phi_6\Phi'_6\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	1
$G_4:\phi_{1,16}$	$\frac{\zeta_3}{6}x^6\Phi_1^2\Phi_3^3\Phi_4\Phi_5\Phi_6\Phi'_6\Phi_9\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	-1
$G_{25}[\zeta_3]:-\zeta_3^2$	$\frac{-\zeta_3}{6}x^6\Phi_1^3\Phi_2^2\Phi''_3\Phi_4^2\Phi_5\Phi'_6\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	ζ_3
$G_{25}[-\zeta_3]:\zeta_3^2$	$\frac{\zeta_3}{6}x^6\Phi_1^3\Phi_2^2\Phi_3^2\Phi''_3\Phi_4^2\Phi_5\Phi'_6\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi''_{18}\Phi'_{24}\Phi_{30}$	$-\zeta_3$
$Z_3:\phi_{9,7}$	$\frac{-\zeta_3}{6}x^6\Phi_1\Phi_2{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^2\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$G_4\otimes Z_3:-\zeta_3^2$	$\frac{-\zeta_3}{6}x^6\Phi_1^3\Phi_2\Phi_3^2\Phi'_3\Phi_4\Phi_5{\Phi''_6}^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi_{15}\Phi''_{18}\Phi''_{24}\Phi_{30}$	$-\zeta_3^2$
* $\phi_{64,8}$	$\frac{3-\sqrt{-3}}{12}x^8\Phi_2^4\Phi'_3\Phi_4^2\Phi_6^4\Phi_8\Phi_{10}\Phi_{12}^2\Phi''''_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{64,11}$	$\frac{3-\sqrt{-3}}{12}x^8\Phi_2^4\Phi'_3\Phi_4^2\Phi_6^4\Phi_8\Phi_{10}\Phi_{12}^2\Phi''''_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$G_{32}^2[i]$	$\frac{3-\sqrt{-3}}{12}x^8\Phi_1^4\Phi_3^4\Phi_4^2\Phi_5\Phi''_6\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{24}\Phi'''_{30}$	$ix^{1/2}$
$G_{32}^2[-i]$	$\frac{3-\sqrt{-3}}{12}x^8\Phi_1^4\Phi_3^4\Phi_4^2\Phi_5\Phi''_6\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{24}\Phi'''_{30}$	$-ix^{1/2}$

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
	$\phi_{64,16}$	$\frac{3+\sqrt{-3}}{12}x^8\Phi_2^4\Phi_3''\Phi_4^2\Phi_6^4\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{15}'''\Phi_{18}\Phi_{24}\Phi_{30}$	1
#	$\phi_{64,13}$	$\frac{3+\sqrt{-3}}{12}x^8\Phi_2^4\Phi_3''\Phi_4^2\Phi_6^4\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{15}'''\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$G_{32}^3[i]$	$\frac{3+\sqrt{-3}}{12}x^8\Phi_1^4\Phi_3^4\Phi_4^2\Phi_5\Phi_6'\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{24}\Phi_{30}''''$	$ix^{1/2}$
	$G_{32}^3[-i]$	$\frac{3+\sqrt{-3}}{12}x^8\Phi_1^4\Phi_3^4\Phi_4^2\Phi_5\Phi_6'\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{24}\Phi_{30}''''$	$-ix^{1/2}$
	$Z_3:\phi_{8,3}$	$\frac{-\sqrt{-3}}{6}x^8\Phi_1\Phi_2^4\Phi_4^2\Phi_5\Phi_6^4\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
	$Z_3:\phi_{8,6}'$	$\frac{-\sqrt{-3}}{6}x^8\Phi_1\Phi_2^4\Phi_4^2\Phi_5\Phi_6^4\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
	$G_{32}[\zeta_{12}^{11}]$	$\frac{\sqrt{-3}}{6}x^8\Phi_1^4\Phi_2\Phi_3^4\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{24}$	$\zeta_{12}^{11}x^{1/2}$
	$G_{32}[\zeta_{12}^5]$	$\frac{\sqrt{-3}}{6}x^8\Phi_1^4\Phi_2\Phi_3^4\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{24}$	$\zeta_{12}^5x^{1/2}$
*	$\phi_{60,9}$	$\frac{-\zeta_3}{3}x^9\Phi_3\Phi_4\Phi_5\Phi_6\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{12}''''\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$\phi_{60,13}$	$\frac{1}{3}x^9\Phi_3'^2\Phi_4\Phi_5\Phi_6''^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{12}'''\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$Z_3:\phi_{6,8}'$	$\frac{\zeta_3^2}{3}x^9\Phi_1\Phi_2\Phi_3'\Phi_4^2\Phi_5\Phi_6''\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
	$\phi_{60,11}''$	$\frac{1}{3}x^9\Phi_3''^2\Phi_4\Phi_5\Phi_6'^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{12}''''\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
#	$\phi_{60,15}'$	$\frac{-\zeta_3^2}{3}x^9\Phi_3\Phi_4\Phi_5\Phi_6\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{12}'''\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$Z_3:\phi_{6,10}$	$\frac{-\zeta_3}{3}x^9\Phi_1\Phi_2\Phi_3''\Phi_4^2\Phi_5\Phi_6'\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
	$Z_3:\phi_{3,4}$	$\frac{\zeta_3^2}{3}x^9\Phi_1\Phi_2\Phi_3''\Phi_4\Phi_5\Phi_6'\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{12}''''\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
	$Z_3:\phi_{3,8}'$	$\frac{-\zeta_3}{3}x^9\Phi_1\Phi_2\Phi_3'\Phi_4\Phi_5\Phi_6''\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{12}'''\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
	$Z_3\otimes Z_3:..1...1$	$\frac{-1}{3}x^9\Phi_1^2\Phi_2^2\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3
	$\phi_{81,12}$	$\frac{1}{3}x^{10}\Phi_3^4\Phi_6^4\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$G_{32}[\zeta_9^5]$	$\frac{\zeta_3}{3}x^{10}\Phi_1^4\Phi_2^4\Phi_3'^4\Phi_4^2\Phi_5\Phi_6''^4\Phi_8\Phi_9\Phi_{10}\Phi_{12}'''^2\Phi_{15}''''\Phi_{18}\Phi_{24}''\Phi_{30}'''$	$\zeta_9^5x^{2/3}$
	$G_{32}[\zeta_9^2]$	$\frac{\zeta_3^2}{3}x^{10}\Phi_1^4\Phi_2^4\Phi_3''^4\Phi_4^2\Phi_5\Phi_6'^4\Phi_8\Phi_9\Phi_{10}\Phi_{12}''''^2\Phi_{15}'''\Phi_{18}\Phi_{24}'\Phi_{30}''''$	$\zeta_9^2x^{1/3}$
	$\phi_{81,14}$	$\frac{1}{3}x^{10}\Phi_3^4\Phi_6^4\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$G_{32}[\zeta_9^8]$	$\frac{\zeta_3}{3}x^{10}\Phi_1^4\Phi_2^4\Phi_3'^4\Phi_4^2\Phi_5\Phi_6''^4\Phi_8\Phi_9\Phi_{10}\Phi_{12}'''^2\Phi_{15}''''\Phi_{18}\Phi_{24}''\Phi_{30}'''$	$\zeta_9^8x^{2/3}$
	$G_{32}^2[\zeta_9^8]$	$\frac{\zeta_3^2}{3}x^{10}\Phi_1^4\Phi_2^4\Phi_3''^4\Phi_4^2\Phi_5\Phi_6'^4\Phi_8\Phi_9\Phi_{10}\Phi_{12}''''^2\Phi_{15}'''\Phi_{18}\Phi_{24}'\Phi_{30}''''$	$\zeta_9^8x^{1/3}$
*	$\phi_{81,10}$	$\frac{1}{3}x^{10}\Phi_3^4\Phi_6^4\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$G_{32}^2[\zeta_9^2]$	$\frac{\zeta_3}{3}x^{10}\Phi_1^4\Phi_2^4\Phi_3'^4\Phi_4^2\Phi_5\Phi_6''^4\Phi_8\Phi_9\Phi_{10}\Phi_{12}'''^2\Phi_{15}''''\Phi_{18}\Phi_{24}''\Phi_{30}'''$	$\zeta_9^2x^{2/3}$
	$G_{32}^2[\zeta_9^5]$	$\frac{\zeta_3^2}{3}x^{10}\Phi_1^4\Phi_2^4\Phi_3''^4\Phi_4^2\Phi_5\Phi_6'^4\Phi_8\Phi_9\Phi_{10}\Phi_{12}''''^2\Phi_{15}'''\Phi_{18}\Phi_{24}'\Phi_{30}''''$	$\zeta_9^5x^{1/3}$
*	$\phi_{30,12}'$	$\frac{-\sqrt{-3}}{18}x^{12}\Phi_3^2\Phi_4\Phi_5\Phi_6^2\Phi_8\Phi_9''\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}''\Phi_{24}\Phi_{30}$	1
	$\phi_{10,34}$	$\frac{-\sqrt{-3}}{18}x^{12}\Phi_3''\Phi_4\Phi_5\Phi_6'^4\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}'\Phi_{24}\Phi_{30}$	1
	$\phi_{30,20}'$	$\frac{\sqrt{-3}}{18}x^{12}\Phi_3'^4\Phi_4\Phi_5\Phi_6''\Phi_8\Phi_9'\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$\phi_{60,16}$	$\frac{-\sqrt{-3}}{18}x^{12}\Phi_3''^4\Phi_4^2\Phi_5\Phi_6'\Phi_8\Phi_9'\Phi_{10}\Phi_{12}\Phi_{12}''''\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$Z_3\otimes Z_3:...2..$	$\frac{\sqrt{-3}}{18}x^{12}\Phi_1^2\Phi_2^2\Phi_3''^2\Phi_4^2\Phi_5\Phi_6'^2\Phi_8\Phi_9''\Phi_{10}\Phi_{12}\Phi_{12}''''\Phi_{15}\Phi_{18}''\Phi_{24}\Phi_{30}$	ζ_3
	$Z_3:\phi_{8,9}''$	$\frac{-\sqrt{-3}}{18}x^{12}\Phi_1\Phi_2^4\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{12}''''\Phi_{15}\Phi_{18}'\Phi_{24}\Phi_{30}$	ζ_3^2
	$\phi_{20,20}$	$\frac{-\sqrt{-3}}{18}x^{12}\Phi_3'\Phi_4^2\Phi_5\Phi_6''^4\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{12}'''\Phi_{15}\Phi_{18}'\Phi_{24}\Phi_{30}$	1
	$G_{32}[\zeta_3^2]$	$\frac{-\sqrt{-3}}{18}x^{12}\Phi_1^4\Phi_2\Phi_4^2\Phi_5\Phi_8\Phi_9'\Phi_{10}\Phi_{12}\Phi_{12}'''\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
	$Z_3\otimes Z_3:.11....$	$\frac{\sqrt{-3}}{18}x^{12}\Phi_1^2\Phi_2^2\Phi_3'^2\Phi_4^2\Phi_5\Phi_6''^2\Phi_8\Phi_9''\Phi_{10}\Phi_{12}\Phi_{12}'''\Phi_{15}\Phi_{18}''\Phi_{24}\Phi_{30}$	ζ_3
	$\phi_{60,15}''$	$\frac{1}{6}x^{12}\Phi_2^2\Phi_3^2\Phi_5\Phi_6^4\Phi_8\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$\phi_{20,13}$	$\frac{-1}{6}x^{12}\Phi_2^2\Phi_3''\Phi_5\Phi_6^4\Phi_8\Phi_9''\Phi_{10}\Phi_{12}''''^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$\phi_{20,29}'$	$\frac{-1}{6}x^{12}\Phi_2^2\Phi_3'\Phi_5\Phi_6^4\Phi_8\Phi_9'\Phi_{10}\Phi_{12}'''^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
	$\phi_{80,13}$	$\frac{1}{6}x^{12}\Phi_2^4\Phi_3''\Phi_5\Phi_6^2\Phi_6'^2\Phi_8\Phi_9'\Phi_{10}\Phi_{12}''''^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$Z_3 \otimes Z_3 : 1...1..$	$\frac{-1}{6}x^{12}\Phi_1^2\Phi_2^4{\Phi''_3}^2\Phi_4\Phi_5\Phi_6^2{\Phi'_6}^2\Phi_8\Phi_{10}\Phi''''_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3
$Z_3 : \phi_{8,12}$	$\frac{1}{6}x^{12}\Phi_1\Phi_2^4\Phi_4^2\Phi_5\Phi_6^2{\Phi'_6}^2\Phi_8\Phi''_9\Phi_{10}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$\phi_{80,17}$	$\frac{1}{6}x^{12}\Phi_2^4\Phi'_3\Phi_5\Phi_6^2{\Phi''_6}^2\Phi_8\Phi''_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$Z_3 : \phi''_{8,6}$	$\frac{-1}{6}x^{12}\Phi_1\Phi_2^4\Phi_4^2\Phi_5\Phi_6^2{\Phi''_6}^2\Phi_8\Phi'_9\Phi_{10}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$Z_3 \otimes Z_3 : .1...1.$	$\frac{-1}{6}x^{12}\Phi_1^2\Phi_2^4{\Phi'_3}^2\Phi_4\Phi_5\Phi_6^2{\Phi''_6}^2\Phi_8\Phi_{10}\Phi'''_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3
$G_4 : \phi_{3,6}$	$\frac{-1}{6}x^{12}\Phi_1^2\Phi_3^4\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{24}\Phi_{30}$	-1
$G_4 : \phi_{3,4}$	$\frac{-1}{6}x^{12}\Phi_1^2\Phi_3^4\Phi_5\Phi'_6\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	-1
$G_4 : \phi_{3,2}$	$\frac{-1}{6}x^{12}\Phi_1^2\Phi_3^4\Phi_5\Phi''_6\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	-1
$G_{32}^3[-1]$	$\frac{1}{6}x^{12}\Phi_1^4\Phi_3^2{\Phi''_3}^2\Phi_5\Phi'_6\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	-1
$G_{32}^2[-\zeta_3]$	$\frac{-1}{6}x^{12}\Phi_1^4\Phi_2^2\Phi_3^2{\Phi''_3}^2\Phi_4\Phi_5{\Phi'_6}^2\Phi_8\Phi_9\Phi_{10}\Phi''''_{12}\Phi_{15}\Phi_{24}\Phi_{30}$	$-\zeta_3$
$G_{32}^2[-\zeta_3^2]$	$\frac{1}{6}x^{12}\Phi_1^4\Phi_2\Phi_3^2{\Phi''_3}^2\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	$-\zeta_3^2$
$G_{32}^2[-1]$	$\frac{1}{6}x^{12}\Phi_1^4\Phi_3^2{\Phi'_3}^2\Phi_5\Phi''_6\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	-1
$G_{32}[-\zeta_3^2]$	$\frac{1}{6}x^{12}\Phi_1^4\Phi_2\Phi_3^2{\Phi'_3}^2\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	$-\zeta_3^2$
$G_{32}[-\zeta_3]$	$\frac{-1}{6}x^{12}\Phi_1^4\Phi_2^2\Phi_3^2{\Phi'_3}^2\Phi_4\Phi_5{\Phi''_6}^2\Phi_8\Phi_9\Phi_{10}\Phi'''_{12}\Phi_{15}\Phi_{24}\Phi_{30}$	$-\zeta_3$
$Z_3 : \phi_{1,9}$	$\frac{-\sqrt{-3}}{9}x^{12}\Phi_1\Phi_2\Phi_4\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$Z_3 : \phi'_{3,13}$	$\frac{-\sqrt{-3}}{9}x^{12}\Phi_1\Phi_2\Phi_3\Phi''_3\Phi_4\Phi_5\Phi_6\Phi'_6\Phi_8\Phi'_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$Z_3 : \phi_{3,17}$	$\frac{-\sqrt{-3}}{9}x^{12}\Phi_1\Phi_2\Phi_3\Phi'_3\Phi_4\Phi_5\Phi_6\Phi''_6\Phi_8\Phi''_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$Z_3 : \phi'_{6,7}$	$\frac{-\sqrt{-3}}{9}x^{12}\Phi_1\Phi_2\Phi_3\Phi''_3\Phi_4^2\Phi_5\Phi_6\Phi'_6\Phi_8\Phi''_9\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$\phi_{40,14}$	$\frac{-\sqrt{-3}}{9}x^{12}\Phi'_3\Phi_4^2\Phi_5\Phi''_6\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$Z_3 \otimes Z_3 : .1..1..$	$\frac{\sqrt{-3}}{9}x^{12}\Phi_1^2\Phi_2^2\Phi_3\Phi_4^2\Phi_5\Phi_6\Phi_8\Phi'_9\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	ζ_3
$Z_3 : \phi_{6,5}$	$\frac{-\sqrt{-3}}{9}x^{12}\Phi_1\Phi_2\Phi_3\Phi'_3\Phi_4^2\Phi_5\Phi_6\Phi''_6\Phi_8\Phi'_9\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$Z_3 \otimes Z_3 : 1....1.$	$\frac{-\sqrt{-3}}{9}x^{12}\Phi_1^2\Phi_2^2\Phi_3\Phi_4^2\Phi_5\Phi_6\Phi_8\Phi''_9\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	ζ_3
$\phi_{40,22}$	$\frac{\sqrt{-3}}{9}x^{12}\Phi''_3\Phi_4^2\Phi_5\Phi'_6\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
# $\phi_{30,24}$	$\frac{\sqrt{-3}}{18}x^{12}\Phi_3^2\Phi_4\Phi_5\Phi_6^2\Phi_8\Phi'_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{30,16}$	$\frac{-\sqrt{-3}}{18}x^{12}{\Phi''_3}^4\Phi_4\Phi_5\Phi'_6\Phi_8\Phi''_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{10,14}$	$\frac{\sqrt{-3}}{18}x^{12}\Phi'_3\Phi_4\Phi_5{\Phi''_6}^4\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{20,16}$	$\frac{\sqrt{-3}}{18}x^{12}\Phi''_3\Phi_4^2\Phi_5{\Phi'_6}^4\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	1
$Z_3 \otimes Z_3 : 11.....$	$\frac{-\sqrt{-3}}{18}x^{12}\Phi_1^2\Phi_2^2{\Phi''_3}^2\Phi_4^2\Phi_5{\Phi'_6}^2\Phi_8\Phi'_9\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	ζ_3
$G_{32}^2[\zeta_3^2]$	$\frac{-\sqrt{-3}}{18}x^{12}\Phi_1^4\Phi_2\Phi_4^2\Phi_5\Phi_8\Phi''_9\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$\phi_{60,20}$	$\frac{\sqrt{-3}}{18}x^{12}{\Phi'_3}^4\Phi_4^2\Phi_5\Phi''_6\Phi_8\Phi''_9\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$Z_3 : \phi'_{8,9}$	$\frac{-\sqrt{-3}}{18}x^{12}\Phi_1\Phi_2^4\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$Z_3 \otimes Z_3 : ....2.$	$\frac{-\sqrt{-3}}{18}x^{12}\Phi_1^2\Phi_2^2{\Phi'_3}^2\Phi_4^2\Phi_5{\Phi''_6}^2\Phi_8\Phi'_9\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	ζ_3
$\phi_{4,51}$	$\frac{1}{30}x^{15}\Phi_4^2\Phi_5{\Phi'_6}^4\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi'''_{30}$	1
$\phi_{4,21}$	$\frac{1}{30}x^{15}\Phi_4^2\Phi_5{\Phi''_6}^4\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi''''_{30}$	1
* $\phi_{36,15}$	$\frac{-1}{30}x^{15}{\Phi''_3}^4\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi''''_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
# $\phi_{36,27}$	$\frac{-1}{30}x^{15}{\Phi'_3}^4\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi'''_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{60,19}$	$\frac{-\zeta_3^2}{6}x^{15}\Phi_3\Phi''_3\Phi_4^2\Phi_5\Phi_6{\Phi'_6}^2\Phi_8\Phi''_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{60,17}$	$\frac{-\zeta_3}{6}x^{15}\Phi_3\Phi'_3\Phi_4^2\Phi_5\Phi_6{\Phi''_6}^2\Phi_8\Phi'_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{36,25}$	$\frac{-\zeta_3^2}{6}x^{15}\Phi_3{\Phi''_3}^2\Phi_4^2\Phi_6\Phi'_6\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	1

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$\phi_{36,17}$	$\frac{-\zeta_3}{6}x^{15}\Phi_3\Phi_3'^2\Phi_4^2\Phi_6\Phi_6''\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}''\Phi_{24}\Phi_{30}$	1
$\phi_{24,16}$	$\frac{\zeta_3^2}{6}x^{15}\Phi_2^2\Phi_3\Phi_3''\Phi_4\Phi_6^3\Phi_8\Phi_9''\Phi_{10}\Phi_{12}\Phi_{12}''''\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{24,26}$	$\frac{\zeta_3}{6}x^{15}\Phi_2^2\Phi_3\Phi_3'\Phi_4\Phi_6^3\Phi_8\Phi_9'\Phi_{10}\Phi_{12}\Phi_{12}'''\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{40,24}$	$\frac{1}{6}x^{15}\Phi_2^2\Phi_4\Phi_5\Phi_6^2\Phi_6''^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{12}'''\Phi_{15}'''\Phi_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{40,18}$	$\frac{1}{6}x^{15}\Phi_2^2\Phi_4\Phi_5\Phi_6^2\Phi_6'^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{12}''''\Phi_{15}''''\Phi_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{64,18}$	$\frac{1}{6}x^{15}\Phi_2^4\Phi_4^2\Phi_6^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{64,21}$	$\frac{1}{30}x^{15}\Phi_2^4\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$Z_3:\phi_{3,8}''$	$\frac{-\zeta_3}{6}x^{15}\Phi_1\Phi_2\Phi_3''^2\Phi_4^2\Phi_5\Phi_6'^3\Phi_8\Phi_9'\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}''''$	ζ_3^2
$Z_3:\phi_{3,16}''$	$\frac{\zeta_3^2}{6}x^{15}\Phi_1\Phi_2\Phi_3'^2\Phi_4^2\Phi_5\Phi_6''^3\Phi_8\Phi_9''\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}'''$	ζ_3^2
$Z_3:\phi_{9,8}$	$\frac{-\zeta_3}{6}x^{15}\Phi_1\Phi_2\Phi_3''^3\Phi_4^2\Phi_5\Phi_6'^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}'''\Phi_{18}''\Phi_{24}\Phi_{30}$	ζ_3^2
$Z_3:\phi_{9,10}$	$\frac{\zeta_3^2}{6}x^{15}\Phi_1\Phi_2\Phi_3'^3\Phi_4^2\Phi_5\Phi_6''^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}''''\Phi_{18}'\Phi_{24}\Phi_{30}$	ζ_3^2
$Z_3:\phi_{6,11}''$	$\frac{-\zeta_3}{6}x^{15}\Phi_1\Phi_2^3\Phi_3''^2\Phi_4\Phi_5\Phi_6^2\Phi_6'\Phi_8\Phi_9'\Phi_{10}\Phi_{12}\Phi_{12}''''\Phi_{15}'''\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$Z_3:\phi_{6,7}''$	$\frac{\zeta_3^2}{6}x^{15}\Phi_1\Phi_2^3\Phi_3'^2\Phi_4\Phi_5\Phi_6^2\Phi_6''\Phi_8\Phi_9''\Phi_{10}\Phi_{12}\Phi_{12}'''\Phi_{15}''''\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$G_4:\phi_{2,3}''$	$\frac{-1}{6}x^{15}\Phi_1^2\Phi_3^2\Phi_3'^2\Phi_4\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{12}'''\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}''''$	-1
$G_4:\phi_{2,3}'$	$\frac{-1}{6}x^{15}\Phi_1^2\Phi_3^2\Phi_3''^2\Phi_4\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{12}''''\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}'''$	-1
$G_4:\phi_{2,1}$	$\frac{\zeta_3^2}{6}x^{15}\Phi_1^2\Phi_3^3\Phi_4\Phi_5\Phi_6\Phi_6'\Phi_8\Phi_9\Phi_{12}\Phi_{12}''''\Phi_{15}\Phi_{18}'\Phi_{24}\Phi_{30}$	-1
$G_4:\phi_{2,5}''$	$\frac{\zeta_3}{6}x^{15}\Phi_1^2\Phi_3^3\Phi_4\Phi_5\Phi_6\Phi_6''\Phi_8\Phi_9\Phi_{12}\Phi_{12}'''\Phi_{15}\Phi_{18}''\Phi_{24}\Phi_{30}$	-1
$Z_3\otimes Z_3:..1..1.$	$\frac{\zeta_3}{6}x^{15}\Phi_1^2\Phi_2^3\Phi_3'\Phi_4^2\Phi_5\Phi_6^2\Phi_6''\Phi_8\Phi_9'\Phi_{10}\Phi_{12}\Phi_{15}''''\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3
$Z_3\otimes Z_3:...1..1$	$\frac{\zeta_3^2}{6}x^{15}\Phi_1^2\Phi_2^3\Phi_3''\Phi_4^2\Phi_5\Phi_6^2\Phi_6'\Phi_8\Phi_9''\Phi_{10}\Phi_{12}\Phi_{15}'''\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3
$Z_3\otimes Z_3:....1.1$	$\frac{-\zeta_3}{6}x^{15}\Phi_1^2\Phi_2^3\Phi_3'\Phi_4^2\Phi_5\Phi_6''\Phi_8\Phi_9'\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}'''$	ζ_3
$Z_3\otimes Z_3:..1.1..$	$\frac{-\zeta_3^2}{6}x^{15}\Phi_1^2\Phi_2^3\Phi_3''\Phi_4^2\Phi_5\Phi_6'\Phi_8\Phi_9''\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}''''$	ζ_3
$G_4\otimes Z_3:\zeta_3$	$\frac{-\zeta_3}{6}x^{15}\Phi_1^3\Phi_2\Phi_3^2\Phi_3''\Phi_4\Phi_5\Phi_6'^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{12}''''\Phi_{15}\Phi_{18}''\Phi_{24}\Phi_{30}''''$	$-\zeta_3^2$
$G_4\otimes Z_3:\zeta_3^2$	$\frac{\zeta_3^2}{6}x^{15}\Phi_1^3\Phi_2\Phi_3^2\Phi_3'\Phi_4\Phi_5\Phi_6''^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{12}'''\Phi_{15}\Phi_{18}'\Phi_{24}\Phi_{30}'''$	$-\zeta_3^2$
$G_{25}[-\zeta_3]:-\zeta_3$	$\frac{\zeta_3}{6}x^{15}\Phi_1^3\Phi_2^2\Phi_3^2\Phi_3'\Phi_4^2\Phi_5\Phi_6''\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{18}''\Phi_{24}\Phi_{30}'''$	$-\zeta_3$
$G_{25}[-\zeta_3]:-\zeta_3^2$	$\frac{-\zeta_3^2}{6}x^{15}\Phi_1^3\Phi_2^2\Phi_3^2\Phi_3''\Phi_4^2\Phi_5\Phi_6'\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{18}'\Phi_{24}\Phi_{30}''''$	$-\zeta_3$
$G_{25}[\zeta_3]:\zeta_3$	$\frac{-\zeta_3}{6}x^{15}\Phi_1^3\Phi_2^2\Phi_3'\Phi_4^2\Phi_5\Phi_6''\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}''''\Phi_{18}''\Phi_{24}\Phi_{30}$	ζ_3
$G_{25}[\zeta_3]:\zeta_3^2$	$\frac{\zeta_3^2}{6}x^{15}\Phi_1^3\Phi_2^2\Phi_3''\Phi_4^2\Phi_5\Phi_6'\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}'''\Phi_{18}'\Phi_{24}\Phi_{30}$	ζ_3
$G_{32}^3[1]$	$\frac{-1}{30}x^{15}\Phi_1^4\Phi_4^2\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$G_{32}^4[-1]$	$\frac{1}{6}x^{15}\Phi_1^4\Phi_3^2\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{12}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	-1
$G_{32}[\zeta_5]$	$\frac{1}{5}x^{15}\Phi_1^4\Phi_2^4\Phi_3^4\Phi_4^2\Phi_6^4\Phi_8\Phi_9\Phi_{12}^2\Phi_{18}\Phi_{24}$	ζ_5
$G_{32}[\zeta_5^2]$	$\frac{1}{5}x^{15}\Phi_1^4\Phi_2^4\Phi_3^4\Phi_4^2\Phi_6^4\Phi_8\Phi_9\Phi_{12}^2\Phi_{18}\Phi_{24}$	ζ_5^2
$G_{32}[\zeta_5^3]$	$\frac{1}{5}x^{15}\Phi_1^4\Phi_2^4\Phi_3^4\Phi_4^2\Phi_6^4\Phi_8\Phi_9\Phi_{12}^2\Phi_{18}\Phi_{24}$	ζ_5^3
$G_{32}[\zeta_5^4]$	$\frac{1}{5}x^{15}\Phi_1^4\Phi_2^4\Phi_3^4\Phi_4^2\Phi_6^4\Phi_8\Phi_9\Phi_{12}^2\Phi_{18}\Phi_{24}$	ζ_5^4
* $\phi_{30,20}''$	$\frac{-\zeta_3}{6}x^{20}\Phi_3''^2\Phi_4^2\Phi_5\Phi_6'^2\Phi_9\Phi_{10}\Phi_{12}''''^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{45,22}$	$\frac{3-\sqrt{-3}}{12}x^{20}\Phi_3'^3\Phi_5\Phi_6''^3\Phi_8\Phi_9\Phi_{10}\Phi_{12}'''\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{15,24}$	$\frac{1}{3}x^{20}\Phi_3\Phi_5\Phi_6\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{45,26}$	$\frac{3+\sqrt{-3}}{12}x^{20}\Phi_3''^3\Phi_5\Phi_6'^3\Phi_8\Phi_9\Phi_{10}\Phi_{12}''''\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
# $\phi_{30,28}$	$\frac{-\zeta_3^2}{6}x^{20}\Phi_3'^2\Phi_4^2\Phi_5\Phi_6''^2\Phi_9\Phi_{10}\Phi_{12}'''^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$Z_3\otimes Z_3:..11...$	$\frac{1}{6}x^{20}\Phi_1^2\Phi_2^2\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}''''^2\Phi_{15}\Phi_{18}\Phi_{24}''\Phi_{30}$	ζ_3

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$G_4 : \phi'_{2,5}$	$\frac{-3-\sqrt{-3}}{12}x^{20}\Phi_1^2\Phi_2^2\Phi_3^2\Phi''_3\Phi_5\Phi_6^2\Phi'_6\Phi_9\Phi_{10}\Phi''''_{12}\Phi_{15}\Phi_{18}\Phi'_{24}\Phi_{30}$	-1
$Z_3 : \phi_{6,13}$	$\frac{\zeta_3^2}{3}x^{20}\Phi_1\Phi_2\Phi''_3\Phi_4^2\Phi_5\Phi'_6\Phi_8\Phi_9\Phi_{10}\Phi''''^2_{12}\Phi_{15}\Phi_{18}\Phi'_{24}\Phi_{30}$	ζ_3^2
$G_{25}[-\zeta_3] : -1$	$\frac{\sqrt{-3}}{6}x^{20}\Phi_1^3\Phi_2^3\Phi_3^2\Phi_4\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{10}\Phi_{15}\Phi_{18}\Phi_{30}$	$-\zeta_3$
$\phi_{15,22}$	$\frac{\zeta_3^2}{6}x^{20}\Phi'^2_3\Phi_5\Phi''^2_6\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi''_{24}\Phi_{30}$	1
$G_{25}[\zeta_3] : -1$	$\frac{\sqrt{-3}}{6}x^{20}\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_3
$Z_3 : \phi'_{6,11}$	$\frac{-\zeta_3}{3}x^{20}\Phi_1\Phi_2\Phi'_3\Phi_4^2\Phi_5\Phi''_6\Phi_8\Phi_9\Phi_{10}\Phi'''^2_{12}\Phi_{15}\Phi_{18}\Phi''_{24}\Phi_{30}$	ζ_3^2
$\phi_{15,38}$	$\frac{\zeta_3}{6}x^{20}\Phi''^2_3\Phi_5\Phi'^2_6\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi'_{24}\Phi_{30}$	1
$G_4 : \phi'_{2,7}$	$\frac{-3+\sqrt{-3}}{12}x^{20}\Phi_1^2\Phi_2^2\Phi_3^2\Phi'_3\Phi_5\Phi_6^2\Phi''_6\Phi_9\Phi_{10}\Phi'''_{12}\Phi_{15}\Phi_{18}\Phi''_{24}\Phi_{30}$	-1
$Z_3 \otimes Z_3 : .....11$	$\frac{1}{6}x^{20}\Phi_1^2\Phi_2^2\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi'''^2_{12}\Phi_{15}\Phi_{18}\Phi'_{24}\Phi_{30}$	ζ_3
$*$ $\phi_{20,25}$	$\frac{-\zeta_3}{3}x^{25}\Phi''_3\Phi_4\Phi_5\Phi'_6\Phi_8\Phi'_9\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	1
$\phi''_{20,29}$	$\frac{1}{3}x^{25}\Phi'_3\Phi_4\Phi_5\Phi''_6\Phi_8\Phi'_9\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	1
$Z_3 : \phi_{2,12}$	$\frac{\zeta_3^2}{3}x^{25}\Phi_1\Phi_2\Phi_4^2\Phi_5\Phi_8\Phi'_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$\phi_{20,31}$	$\frac{1}{3}x^{25}\Phi''_3\Phi_4\Phi_5\Phi'_6\Phi_8\Phi''_9\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	1
# $\phi_{20,35}$	$\frac{-\zeta_3^2}{3}x^{25}\Phi'_3\Phi_4\Phi_5\Phi''_6\Phi_8\Phi''_9\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	1
$Z_3 : \phi_{2,18}$	$\frac{-\zeta_3}{3}x^{25}\Phi_1\Phi_2\Phi_4^2\Phi_5\Phi_8\Phi''_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	ζ_3^2
$Z_3 : \phi'_{3,16}$	$\frac{\zeta_3^2}{3}x^{25}\Phi_1\Phi_2\Phi_3\Phi''_3\Phi_4\Phi_5\Phi_6\Phi'_6\Phi_8\Phi_{10}\Phi_{12}\Phi''''_{12}\Phi_{15}\Phi_{24}\Phi_{30}$	ζ_3^2
$Z_3 : \phi_{3,20}$	$\frac{-\zeta_3}{3}x^{25}\Phi_1\Phi_2\Phi_3\Phi'_3\Phi_4\Phi_5\Phi_6\Phi''_6\Phi_8\Phi_{10}\Phi_{12}\Phi'''_{12}\Phi_{15}\Phi_{24}\Phi_{30}$	ζ_3^2
$Z_3 \otimes Z_3 : ...1.1.$	$\frac{-1}{3}x^{25}\Phi_1^2\Phi_2^2\Phi_3\Phi_4^2\Phi_5\Phi_6\Phi_8\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{24}\Phi_{30}$	ζ_3
$*$ $\phi_{10,30}$	$\frac{-\sqrt{-3}}{6}x^{30}\Phi_4\Phi_5\Phi_8\Phi''_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{20,33}$	$\frac{1}{2}x^{30}\Phi_2^2\Phi_5\Phi_6^2\Phi_8\Phi_{10}\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	1
$G_4 : \phi_{1,0}$	$\frac{-1}{2}x^{30}\Phi_1^2\Phi_3^2\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{15}\Phi_{24}\Phi_{30}$	-1
$Z_3 : \phi_{3,15}$	$\frac{-\sqrt{-3}}{3}x^{30}\Phi_1\Phi_2\Phi_3\Phi_4\Phi_5\Phi_6\Phi_8\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{24}\Phi_{30}$	ζ_3^2
# $\phi_{10,42}$	$\frac{\sqrt{-3}}{6}x^{30}\Phi_4\Phi_5\Phi_8\Phi'_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{5,52}$	$\frac{-\zeta_3^2}{6}x^{40}\Phi''_3\Phi_5\Phi'_6\Phi_8\Phi'_9\Phi''''^2_{12}\Phi'''_{15}\Phi'_{18}\Phi_{24}\Phi_{30}$	1
$\phi_{4,41}$	$\frac{-3-\sqrt{-3}}{12}x^{40}\Phi_2^2\Phi'_3\Phi_4\Phi_6^2\Phi''_6\Phi_{10}\Phi_{12}\Phi''''_{15}\Phi_{18}\Phi''_{24}\Phi_{30}$	1
$\phi_{6,48}$	$\frac{1}{3}x^{40}\Phi_3^2\Phi_6^2\Phi_8\Phi_{12}\Phi_{15}\Phi_{24}\Phi_{30}$	1
$\phi_{4,61}$	$\frac{-3+\sqrt{-3}}{12}x^{40}\Phi_2^2\Phi''_3\Phi_4\Phi_6^2\Phi'_6\Phi_{10}\Phi_{12}\Phi'''_{15}\Phi_{18}\Phi'_{24}\Phi_{30}$	1
$\phi_{5,44}$	$\frac{-\zeta_3}{6}x^{40}\Phi'_3\Phi_5\Phi''_6\Phi_8\Phi''_9\Phi'''^2_{12}\Phi''''_{15}\Phi''_{18}\Phi_{24}\Phi_{30}$	1
$Z_3 : \phi_{1,33}$	$\frac{1}{6}x^{40}\Phi_1\Phi_2\Phi_4^2\Phi_5\Phi_8\Phi''_9\Phi_{10}\Phi'''_{15}\Phi''_{18}\Phi_{24}\Phi'''_{30}$	ζ_3^2
$G_4 : \phi''_{1,4}$	$\frac{-3+\sqrt{-3}}{12}x^{40}\Phi_1^2\Phi_3^2\Phi''_3\Phi_4\Phi_5\Phi'_6\Phi_9\Phi_{12}\Phi_{15}\Phi'_{24}\Phi''''_{30}$	-1
$Z_3 \otimes Z_3 : ...11..$	$\frac{\zeta_3}{3}x^{40}\Phi_1^2\Phi_2^2\Phi''^2_3\Phi_4\Phi_5\Phi'^2_6\Phi_8\Phi_{10}\Phi''''_{12}\Phi'''_{15}\Phi_{24}\Phi''''_{30}$	ζ_3
$G_4 \otimes Z_3 : -1$	$\frac{\sqrt{-3}}{6}x^{40}\Phi_1^3\Phi_2\Phi_3^2\Phi_4\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}$	$-\zeta_3^2$
# $\phi_{1,80}$	$\frac{-\zeta_3}{6}x^{40}\Phi'_3\Phi''_6\Phi_8\Phi'_9\Phi_{10}\Phi'''^2_{12}\Phi_{15}\Phi'_{18}\Phi_{24}\Phi'''_{30}$	1
$Z_3 : \phi_{2,24}$	$\frac{-\sqrt{-3}}{6}x^{40}\Phi_1\Phi_2^3\Phi_4\Phi_5\Phi_6^2\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}\Phi_{30}$	ζ_3^2
$Z_3 \otimes Z_3 : ....11.$	$\frac{\zeta_3^2}{3}x^{40}\Phi_1^2\Phi_2^2\Phi'^2_3\Phi_4\Phi_5\Phi''^2_6\Phi_8\Phi_{10}\Phi'''_{12}\Phi''''_{15}\Phi_{24}\Phi'''_{30}$	ζ_3
$*$ $\phi_{1,40}$	$\frac{-\zeta_3^2}{6}x^{40}\Phi''_3\Phi'_6\Phi_8\Phi''_9\Phi_{10}\Phi''''^2_{12}\Phi_{15}\Phi''_{18}\Phi_{24}\Phi''''_{30}$	1
$G_4 : \phi'''_{1,8}$	$\frac{-3-\sqrt{-3}}{12}x^{40}\Phi_1^2\Phi_3^2\Phi'_3\Phi_4\Phi_5\Phi''_6\Phi_9\Phi_{12}\Phi_{15}\Phi''_{24}\Phi'''_{30}$	-1
$Z_3 : \phi_{1,21}$	$\frac{-1}{6}x^{40}\Phi_1\Phi_2\Phi_4^2\Phi_5\Phi_8\Phi'_9\Phi_{10}\Phi''''_{15}\Phi'_{18}\Phi_{24}\Phi''''_{30}$	ζ_3^2

A.17. Unipotent characters for G_{33}

Some principal ζ-series

$\zeta_5 : \mathcal{H}_{Z_{10}}(\zeta_5 x^9, -\zeta_5 x^{9/2}, \zeta_5 x^5, -\zeta_5 x^3, \zeta_5 x^6, -\zeta_5 x^4, \zeta_5 x^{9/2}, -\zeta_5, \zeta_5 x^3, -\zeta_5 x^6)$
$\zeta_5^3 : \mathcal{H}_{Z_{10}}(\zeta_5^3 x^9, -\zeta_5^3, \zeta_5^3 x^6, -\zeta_5^3 x^{9/2}, \zeta_5^3 x^3, -\zeta_5^3 x^4, \zeta_5^3 x^5, -\zeta_5^3 x^6, \zeta_5^3 x^{9/2}, -\zeta_5^3 x^3)$
$\zeta_9^5 : \mathcal{H}_{Z_{18}}(\zeta_9^2 x^5, -\zeta_9^8 x^3, \zeta_9^2 x^{5/2}, -\zeta_9^5 x^2, \zeta_9^5 x^3, -\zeta_9^5 x^4, \zeta_9^2 x^2, -\zeta_9^2 x^3, \zeta_9^8 x, -\zeta_9^8 x^2, \zeta_9^8 x^3, -\zeta_9^2 x^{5/2}, \zeta_9^5 x^2, -\zeta_9^2, \zeta_9^2 x, -\zeta_9^2 x^2, \zeta_9^2 x^3, -\zeta_9^2 x^4)$
$\zeta_9 : \mathcal{H}_{Z_{18}}(\zeta_9^4 x^5, -\zeta_9 x^4, \zeta_9^7 x^3, -\zeta_9^4 x^2, \zeta_9^4 x^{5/2}, -\zeta_9^4 x^3, \zeta_9 x^2, -\zeta_9^4 x^4, \zeta_9 x^3, -\zeta_9^7 x^2, \zeta_9^4 x, -\zeta_9^7 x^3, \zeta_9^4 x^2, -\zeta_9^4 x^{5/2}, \zeta_9^4 x^3, -\zeta_9 x^2, \zeta_9^7 x, -\zeta_9^4)$
$\zeta_6 : \mathcal{H}_{G_{26}}(-\zeta_3 x, -1; \zeta_3^2 x^2, \zeta_3, -x; \zeta_3^2 x^2, \zeta_3, -x)$
$\zeta_3 : \mathcal{H}_{G_{26}}(\zeta_3^2 x, -1; \zeta_3 x^2, x, \zeta_3^2; \zeta_3 x^2, x, \zeta_3^2)$

Non-principal 1-Harish-Chandra series

$\mathcal{H}_{G_{33}}(G_{3,3,3}[\zeta_3]) = \mathcal{H}_{G_4}(x^3, \zeta_3, \zeta_3^2; x^3, \zeta_3, \zeta_3^2)$
$\mathcal{H}_{G_{33}}(G_{3,3,3}[\zeta_3^2]) = \mathcal{H}_{G_4}(x^3, \zeta_3 x^3, \zeta_3^2; x^3, \zeta_3 x^3, \zeta_3^2)$
$\mathcal{H}_{G_{33}}(D_4) = \mathcal{H}_{Z_6}(x^5, -\zeta_3^2 x^4, \zeta_3 x, -1, \zeta_3^2 x, -\zeta_3 x^4)$

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
*	$\phi_{1,0}$	1	1
*	$\phi_{5,1}$	$\frac{3+\sqrt{-3}}{6} x\Phi_5\Phi_9'\Phi_{10}\Phi_{12}'''\Phi_{18}''$	1
#	$\phi_{5,3}$	$\frac{3-\sqrt{-3}}{6} x\Phi_5\Phi_9''\Phi_{10}\Phi_{12}''''\Phi_{18}'$	1
	$G_{3,3,3}[\zeta_3] : \phi_{1,0}$	$\frac{\sqrt{-3}}{3} x\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_{10}$	ζ_3
*	$\phi_{15,2}$	$x^2\Phi_5\Phi_9\Phi_{10}\Phi_{18}$	1
*	$\phi_{30,3}$	$\frac{1}{2} x^3\Phi_4^2\Phi_5\Phi_9\Phi_{12}\Phi_{18}$	1
	$\phi_{6,5}$	$\frac{1}{2} x^3\Phi_4^2\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$	1
	$\phi_{24,4}$	$\frac{1}{2} x^3\Phi_2^4\Phi_6^2\Phi_9\Phi_{10}\Phi_{18}$	1
	$D_4 : 1$	$\frac{1}{2} x^3\Phi_1^4\Phi_3^2\Phi_5\Phi_9\Phi_{18}$	-1
*	$\phi_{30,4}$	$\frac{-\zeta_3^2}{6} x^4{\Phi_3'}^3\Phi_4^2\Phi_5\Phi_9''\Phi_{10}\Phi_{12}\Phi_{18}$	1
	$\phi_{40,5}''$	$\frac{3+\sqrt{-3}}{12} x^4\Phi_2^4\Phi_5\Phi_6^2\Phi_6'\Phi_9'\Phi_{10}\Phi_{12}''''\Phi_{18}$	1
	$\phi_{20,6}$	$\frac{1}{3} x^4\Phi_4^2\Phi_5\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$	1
	$\phi_{40,5}'$	$\frac{3-\sqrt{-3}}{12} x^4\Phi_2^4\Phi_5\Phi_6^2\Phi_6''\Phi_9''\Phi_{10}\Phi_{12}'''\Phi_{18}$	1
#	$\phi_{30,6}$	$\frac{-\zeta_3}{6} x^4{\Phi_3''}^3\Phi_4^2\Phi_5\Phi_9'\Phi_{10}\Phi_{12}\Phi_{18}$	1
	$G_{3,3,3}[\zeta_3^2] : \phi_{1,4}$	$\frac{-1}{6} x^4\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_9''\Phi_{10}\Phi_{12}\Phi_{18}''$	ζ_3^2
	$D_4 : -\zeta_3$	$\frac{-3+\sqrt{-3}}{12} x^4\Phi_1^4\Phi_3^2\Phi_3'\Phi_5\Phi_9\Phi_{10}\Phi_{12}'''\Phi_{18}'$	-1
	$G_{3,3,3}[\zeta_3] : \phi_{2,3}$	$\frac{\zeta_3}{3} x^4\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_9''\Phi_{10}\Phi_{12}\Phi_{18}'$	ζ_3
	$G_{33}[-\zeta_3^2]$	$\frac{\sqrt{-3}}{6} x^4\Phi_1^5\Phi_2^3\Phi_3^2\Phi_4\Phi_5\Phi_9\Phi_{10}$	$-\zeta_3^2$
	$\phi_{10,8}''$	$\frac{\zeta_3}{6} x^4\Phi_4^2\Phi_5{\Phi_6'}^3\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}''$	1

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$G_{3,3,3}[\zeta_3^2]:\phi_{2,1}$	$\frac{-\sqrt{-3}}{6}x^4\Phi_1^3\Phi_2^5\Phi_4\Phi_5\Phi_6^2\Phi_{10}\Phi_{18}$	ζ_3^2
$G_{3,3,3}[\zeta_3]:\phi_{2,1}$	$\frac{-\zeta_3^2}{3}x^4\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_9'\Phi_{10}\Phi_{12}\Phi_{18}''$	ζ_3
$\phi_{10,8}'$	$\frac{\zeta_3^2}{6}x^4\Phi_4^2\Phi_5\Phi_6''^3\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}'$	1
$D_4:-\zeta_3^2$	$\frac{-3-\sqrt{-3}}{12}x^4\Phi_1^4\Phi_3^2\Phi_3''\Phi_5\Phi_9\Phi_{10}\Phi_{12}''''\Phi_{18}''$	-1
$G_{3,3,3}[\zeta_3^2]:\phi_{1,0}$	$\frac{1}{6}x^4\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_9'\Phi_{10}\Phi_{12}\Phi_{18}'$	ζ_3^2
* $\phi_{81,6}$	$x^6\Phi_3^3\Phi_6^3\Phi_9\Phi_{12}\Phi_{18}$	1
* $\phi_{60,7}$	$x^7\Phi_4^2\Phi_5\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$	1
* $\phi_{45,7}$	$\frac{3+\sqrt{-3}}{6}x^7\Phi_3''^3\Phi_5\Phi_6'^3\Phi_9\Phi_{10}\Phi_{12}''''\Phi_{18}$	1
# $\phi_{45,9}$	$\frac{3-\sqrt{-3}}{6}x^7\Phi_3'^3\Phi_5\Phi_6''^3\Phi_9\Phi_{10}\Phi_{12}'''\Phi_{18}$	1
$G_{3,3,3}[\zeta_3]:\phi_{3,2}$	$\frac{\sqrt{-3}}{3}x^7\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_9\Phi_{10}\Phi_{18}$	ζ_3
* $\phi_{64,8}$	$\frac{1}{2}x^8\Phi_2^5\Phi_4^2\Phi_6^3\Phi_{10}\Phi_{12}\Phi_{18}$	1
# $\phi_{64,9}$	$\frac{1}{2}x^8\Phi_2^5\Phi_4^2\Phi_6^3\Phi_{10}\Phi_{12}\Phi_{18}$	1
$G_{33}[i]$	$\frac{1}{2}x^8\Phi_1^5\Phi_3^3\Phi_4^2\Phi_5\Phi_9\Phi_{12}$	$ix^{1/2}$
$G_{33}[-i]$	$\frac{1}{2}x^8\Phi_1^5\Phi_3^3\Phi_4^2\Phi_5\Phi_9\Phi_{12}$	$-ix^{1/2}$
* $\phi_{15,9}$	$x^9\Phi_5\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$	1
* $\phi_{45,10}$	$\frac{3-\sqrt{-3}}{6}x^{10}\Phi_3'^3\Phi_5\Phi_6''^3\Phi_9\Phi_{10}\Phi_{12}'''\Phi_{18}$	1
# $\phi_{45,12}$	$\frac{3+\sqrt{-3}}{6}x^{10}\Phi_3''^3\Phi_5\Phi_6'^3\Phi_9\Phi_{10}\Phi_{12}''''\Phi_{18}$	1
$G_{3,3,3}[\zeta_3^2]:\phi_{3,2}$	$\frac{-\sqrt{-3}}{3}x^{10}\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_9\Phi_{10}\Phi_{18}$	ζ_3^2
* $\phi_{81,11}$	$x^{11}\Phi_3^3\Phi_6^3\Phi_9\Phi_{12}\Phi_{18}$	1
* $\phi_{60,10}$	$x^{10}\Phi_4^2\Phi_5\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$	1
* $\phi_{15,12}$	$x^{12}\Phi_5\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$	1
* $\phi_{30,13}$	$\frac{-\zeta_3}{6}x^{13}\Phi_3''^3\Phi_4^2\Phi_5\Phi_9'\Phi_{10}\Phi_{12}\Phi_{18}$	1
$\phi_{40,14}''$	$\frac{3-\sqrt{-3}}{12}x^{13}\Phi_2^4\Phi_5\Phi_6^2\Phi_6''\Phi_9''\Phi_{10}\Phi_{12}'''\Phi_{18}$	1
$\phi_{20,15}$	$\frac{1}{3}x^{13}\Phi_4^2\Phi_5\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$	1
$\phi_{40,14}'$	$\frac{3+\sqrt{-3}}{12}x^{13}\Phi_2^4\Phi_5\Phi_6^2\Phi_6'\Phi_9'\Phi_{10}\Phi_{12}''''\Phi_{18}$	1
# $\phi_{30,15}$	$\frac{-\zeta_3^2}{6}x^{13}\Phi_3'^3\Phi_4^2\Phi_5\Phi_9''\Phi_{10}\Phi_{12}\Phi_{18}$	1
$G_{3,3,3}[\zeta_3]:\phi_{1,8}$	$\frac{-1}{6}x^{13}\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_9'\Phi_{10}\Phi_{12}\Phi_{18}'$	ζ_3
$D_4:\zeta_3$	$\frac{-3-\sqrt{-3}}{12}x^{13}\Phi_1^4\Phi_3^2\Phi_3''\Phi_5\Phi_9\Phi_{10}\Phi_{12}''''\Phi_{18}''$	-1
$G_{3,3,3}[\zeta_3^2]:\phi_{2,5}$	$\frac{\zeta_3^2}{3}x^{13}\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_9'\Phi_{10}\Phi_{12}\Phi_{18}''$	ζ_3^2
$G_{33}[-\zeta_3]$	$\frac{-\sqrt{-3}}{6}x^{13}\Phi_1^5\Phi_2^3\Phi_3^2\Phi_4\Phi_5\Phi_9\Phi_{10}$	$-\zeta_3$
$\phi_{10,17}''$	$\frac{\zeta_3^2}{6}x^{13}\Phi_4^2\Phi_5\Phi_6''^3\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}'$	1
$G_{3,3,3}[\zeta_3]:\phi_{2,5}$	$\frac{\sqrt{-3}}{6}x^{13}\Phi_1^3\Phi_2^5\Phi_4\Phi_5\Phi_6^2\Phi_{10}\Phi_{18}$	ζ_3
$G_{3,3,3}[\zeta_3^2]:\phi_{2,3}$	$\frac{-\zeta_3}{3}x^{13}\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_9''\Phi_{10}\Phi_{12}\Phi_{18}'$	ζ_3^2
$\phi_{10,17}'$	$\frac{\zeta_3}{6}x^{13}\Phi_4^2\Phi_5\Phi_6'^3\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}''$	1
$D_4:\zeta_3^2$	$\frac{-3+\sqrt{-3}}{12}x^{13}\Phi_1^4\Phi_3^2\Phi_3'\Phi_5\Phi_9\Phi_{10}\Phi_{12}'''\Phi_{18}'$	-1
$G_{3,3,3}[\zeta_3]:\phi_{1,4}$	$\frac{1}{6}x^{13}\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_9''\Phi_{10}\Phi_{12}\Phi_{18}''$	ζ_3
* $\phi_{30,18}$	$\frac{1}{2}x^{18}\Phi_4^2\Phi_5\Phi_9\Phi_{12}\Phi_{18}$	1
$\phi_{6,20}$	$\frac{1}{2}x^{18}\Phi_4^2\Phi_9\Phi_{10}\Phi_{12}\Phi_{18}$	1
$\phi_{24,19}$	$\frac{1}{2}x^{18}\Phi_2^4\Phi_6^2\Phi_9\Phi_{10}\Phi_{18}$	1

γ		$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
	$D_4 : -1$	$\frac{1}{2}x^{18}\Phi_1^4\Phi_3^2\Phi_5\Phi_9\Phi_{18}$	-1
*	$\phi_{15,23}$	$x^{23}\Phi_5\Phi_9\Phi_{10}\Phi_{18}$	1
*	$\phi_{5,28}$	$\frac{3-\sqrt{-3}}{6}x^{28}\Phi_5\Phi_9''\Phi_{10}\Phi_{12}''''\Phi_{18}'$	1
#	$\phi_{5,30}$	$\frac{3+\sqrt{-3}}{6}x^{28}\Phi_5\Phi_9'\Phi_{10}\Phi_{12}'''\Phi_{18}''$	1
	$G_{3,3,3}[\zeta_3^2] : \phi_{1,8}$	$\frac{-\sqrt{-3}}{3}x^{28}\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_{10}$	ζ_3^2
*	$\phi_{1,45}$	x^{45}	1

A.18. Unipotent characters for G_{34}

Some principal ζ-series

ζ_7^6 : $\mathcal{H}_{Z_{42}}(\zeta_7^6x^6, -\zeta_{21}^{11}x^{7/2}, \zeta_{21}^4x^{10/3}, -x^3, \zeta_{21}^{11}x^3, -\zeta_{21}^4x^4, \zeta_7^6x^{8/3}, -\zeta_{21}^{11}x^{5/2}, \zeta_{21}^4x^{7/2},$
$-\zeta_7x^3, \zeta_{21}^{11}x^2, -\zeta_{21}^4x^3, \zeta_7^6x^4, -\zeta_{21}^{11}x^5, \zeta_{21}^4x^{5/2}, -\zeta_7^2x^3, \zeta_{21}^{11}x^{10/3}, -\zeta_{21}^4x^2, \zeta_7^6x^3, -\zeta_{21}^{11}x^4,$
$\zeta_{21}^4x^{8/3}, -\zeta_7^3x^3, \zeta_{21}^{11}x^{7/2}, -\zeta_{21}^4x, \zeta_7^6x^2, -\zeta_{21}^{11}x^3, \zeta_{21}^4x^4, -\zeta_7^4x^3, \zeta_{21}^{11}x^{5/2}, -\zeta_{21}^4x^{7/2}, \zeta_7^6x^{10/3},$
$-\zeta_{21}^{11}x^2, \zeta_{21}^4x^3, -\zeta_7^5x^3, \zeta_{21}^{11}x^{8/3}, -\zeta_{21}^4x^{5/2}, \zeta_7^6, -\zeta_{21}^{11}x, \zeta_{21}^4x^2, -\zeta_7^6x^3, \zeta_{21}^{11}x^4, -\zeta_{21}^4x^5)$

ζ_7^5 : $\mathcal{H}_{Z_{42}}(\zeta_7^5x^6, -\zeta_{21}^8x^3, \zeta_{21}x^{7/2}, -\zeta_7^3x^3, \zeta_{21}^8x^{10/3}, -\zeta_{21}x^5, \zeta_7^5x^2, -\zeta_{21}^8x^{5/2}, \zeta_{21}x^3,$
$-\zeta_7^4x^3, \zeta_{21}^8x^4, -\zeta_{21}x, \zeta_7^5x^{8/3}, -\zeta_{21}^8x^2, \zeta_{21}x^{5/2}, -\zeta_7^5x^3, \zeta_{21}^8x^{7/2}, -\zeta_{21}x^4, \zeta_7^5x^{10/3}, -\zeta_{21}^8x^5,$
$\zeta_{21}x^2, -\zeta_7^6x^3, \zeta_{21}^8x^3, -\zeta_{21}x^{7/2}, \zeta_7^5x^4, -\zeta_{21}^8x, \zeta_{21}x^{8/3}, -x^3, \zeta_{21}^8x^{5/2}, -\zeta_{21}x^3, \zeta_7^5, -\zeta_{21}^8x^4,$
$\zeta_{21}x^{10/3}, -\zeta_7x^3, \zeta_{21}^8x^2, -\zeta_{21}x^{5/2}, \zeta_7^5x^3, -\zeta_{21}^8x^{7/2}, \zeta_{21}x^4, -\zeta_7^2x^3, \zeta_{21}^8x^{8/3}, -\zeta_{21}x^2)$

ζ_7^4 : $\mathcal{H}_{Z_{42}}(\zeta_7^4x^6, -\zeta_{21}^5x^4, \zeta_{21}^{19}x^2, -\zeta_7^6x^3, \zeta_{21}^5x^{8/3}, -\zeta_{21}^{19}x^3, \zeta_7^4x^{10/3}, -\zeta_{21}^5x^{5/2}, \zeta_{21}^{19}x^4,$
$-x^3, \zeta_{21}^5x^{7/2}, -\zeta_{21}^{19}x^5, \zeta_7^4x^3, -\zeta_{21}^5x, \zeta_{21}^{19}x^{5/2}, -\zeta_7x^3, \zeta_{21}^5x^2, -\zeta_{21}^{19}x^{7/2}, \zeta_7^4x^{8/3}, -\zeta_{21}^5x^3,$
$\zeta_{21}^{19}x^{10/3}, -\zeta_7^2x^3, \zeta_{21}^5x^4, -\zeta_{21}^{19}x^2, \zeta_7^4, -\zeta_{21}^5x^5, \zeta_{21}^{19}x^3, -\zeta_7^3x^3, \zeta_{21}^5x^{5/2}, -\zeta_{21}^{19}x^4, \zeta_7^4x^2, -\zeta_{21}^5x^{7/2},$
$\zeta_{21}^{19}x^{8/3}, -\zeta_7^4x^3, \zeta_{21}^5x^{10/3}, -\zeta_{21}^{19}x^{5/2}, \zeta_7^4x^4, -\zeta_{21}^5x^2, \zeta_{21}^{19}x^{7/2}, -\zeta_7^5x^3, \zeta_{21}^5x^3, -\zeta_{21}^{19}x)$

ζ_7 : $\mathcal{H}_{Z_{42}}(\zeta_7x^6, -\zeta_{21}^{17}x^5, \zeta_{21}^{10}x^4, -\zeta_7x^3, \zeta_{21}^{17}x^2, -\zeta_{21}^{10}x, \zeta_7, -\zeta_{21}^{17}x^{5/2}, \zeta_{21}^{10}x^{8/3}, -\zeta_7^2x^3,$
$\zeta_{21}^{17}x^3, -\zeta_{21}^{10}x^2, \zeta_7x^{10/3}, -\zeta_{21}^{17}x^{7/2}, \zeta_{21}^{10}x^{5/2}, -\zeta_7^3x^3, \zeta_{21}^{17}x^4, -\zeta_{21}^{10}x^3, \zeta_7x^2, -\zeta_{21}^{17}x, \zeta_{21}^{10}x^{7/2},$
$-\zeta_7^4x^3, \zeta_{21}^{17}x^{8/3}, -\zeta_{21}^{10}x^4, \zeta_7x^3, -\zeta_{21}^{17}x^2, \zeta_{21}^{10}x^{10/3}, -\zeta_7^5x^3, \zeta_{21}^{17}x^{5/2}, -\zeta_{21}^{10}x^5, \zeta_7x^4, -\zeta_{21}^{17}x^3,$
$\zeta_{21}^{10}x^2, -\zeta_7^6x^3, \zeta_{21}^{17}x^{7/2}, -\zeta_{21}^{10}x^{5/2}, \zeta_7x^{8/3}, -\zeta_{21}^{17}x^4, \zeta_{21}^{10}x^3, -x^3, \zeta_{21}^{17}x^{10/3}, -\zeta_{21}^{10}x^{7/2})$

Non-principal 1-Harish-Chandra series

$\mathcal{H}_{G_{34}}(G_{3,3,3}[\zeta_3]) = \mathcal{H}_{G_{26}}(x, -1; x^3, \zeta_3, \zeta_3^2; x^3, \zeta_3, \zeta_3^2)$
$\mathcal{H}_{G_{34}}(G_{3,3,3}[\zeta_3^2]) = \mathcal{H}_{G_{26}}(x, -1; x^3, \zeta_3x^3, \zeta_3^2; x^3, \zeta_3x^3, \zeta_3^2)$
$\mathcal{H}_{G_{34}}(D_4) = \mathcal{H}_{G_{6,1,2}}(x^5, -\zeta_3^2x^4, \zeta_3x, -1, \zeta_3^2x, -\zeta_3x^4; x^4, -1)$
$\mathcal{H}_{G_{34}}(G_{33}[i]) = \mathcal{H}_{Z_6}(x^5, -\zeta_3^2, \zeta_3x^7, -x^2, \zeta_3^2x^7, -\zeta_3)$
$\mathcal{H}_{G_{34}}(G_{33}[-i]) = \mathcal{H}_{Z_6}(x^5, -\zeta_3^2, \zeta_3x^7, -x^2, \zeta_3^2x^7, -\zeta_3)$
$\mathcal{H}_{G_{34}}(G_{33}[-\zeta_3]) = \mathcal{H}_{Z_6}(x^3, -\zeta_3^2x^8, \zeta_3x^7, -1, \zeta_3^2x^7, -\zeta_3x^8)$
$\mathcal{H}_{G_{34}}(G_{33}[-\zeta_3^2]) = \mathcal{H}_{Z_6}(x^8, -\zeta_3^2x, \zeta_3, -x^5, \zeta_3^2, -\zeta_3x)$

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
*	$\phi_{1,0}$	1	1
*	$\phi_{6,1}$	$\frac{3+\sqrt{-3}}{6}x\Phi_3'^3\Phi_6''^3\Phi_8\Phi_{12}'''\Phi_{15}''''\Phi_{21}'\Phi_{24}\Phi_{30}'''\Phi_{42}''$	1
#	$\phi_{6,5}$	$\frac{3-\sqrt{-3}}{6}x\Phi_3''^3\Phi_6'^3\Phi_8\Phi_{12}''''\Phi_{15}'''\Phi_{21}''\Phi_{24}\Phi_{30}''''\Phi_{42}'$	1
$G_{3,3,3}[\zeta_3]:$	$\phi_{1,0}$	$\frac{\sqrt{-3}}{3}x\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{14}\Phi_{24}$	ζ_3
*	$\phi_{21,2}$	$\frac{3+\sqrt{-3}}{6}x^2\Phi_3''^3\Phi_6'^3\Phi_7\Phi_{12}''''\Phi_{14}\Phi_{15}'''\Phi_{21}\Phi_{24}'\Phi_{30}''''\Phi_{42}$	1
#	$\phi_{21,4}$	$\frac{3-\sqrt{-3}}{6}x^2\Phi_3'^3\Phi_6''^3\Phi_7\Phi_{12}'''\Phi_{14}\Phi_{15}''''\Phi_{21}\Phi_{24}''\Phi_{30}'''\Phi_{42}$	1
$G_{3,3,3}[\zeta_3]:$	$\phi_{1,9}$	$\frac{\sqrt{-3}}{3}x^2\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{14}\Phi_{21}\Phi_{42}$	ζ_3
*	$\phi_{56,3}$	$\frac{1}{2}x^3\Phi_2^4\Phi_6^4\Phi_7\Phi_{10}\Phi_{14}\Phi_{18}\Phi_{21}\Phi_{30}\Phi_{42}$	1
	$\phi_{21,6}$	$\frac{1}{2}x^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$\phi_{35,6}$	$\frac{1}{2}x^3\Phi_5\Phi_7\Phi_8\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{42}$	1

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$D_4 : 2.....$	$\frac{1}{2}x^3\Phi_1^4\Phi_3^4\Phi_5\Phi_7\Phi_9\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{42}$	-1
* $\phi_{105,4}$	$\frac{-\zeta_3^2}{6}x^4{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi'_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{15}\Phi'_{18}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{120,7}$	$\frac{3-\sqrt{-3}}{12}x^4\Phi_2^4{\Phi''_3}^3\Phi_5\Phi_6^4\Phi'_6\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi''_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	1
$D_4 : .....2$	$\frac{-3+\sqrt{-3}}{12}x^4\Phi_1^4\Phi_3^4\Phi''_3\Phi_5{\Phi'_6}^3\Phi_7\Phi_9\Phi_{10}\Phi''''_{12}\Phi_{15}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi'_{42}$	-1
$G_{3,3,3}[\zeta_3] : \phi_{3,1}$	$\frac{-\zeta_3^2}{3}x^4\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi''_{42}$	ζ_3
$\phi_{15,16}$	$\frac{\zeta_3^2}{6}x^4{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_8\Phi''_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi'_{42}$	1
$\phi_{15,14}$	$\frac{\zeta_3}{6}x^4{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_8\Phi'_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi''_{42}$	1
$\phi_{120,5}$	$\frac{3+\sqrt{-3}}{12}x^4\Phi_2^4{\Phi'_3}^3\Phi_5\Phi_6^4\Phi''_6\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi'_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	1
$D_4 : .2....$	$\frac{-3-\sqrt{-3}}{12}x^4\Phi_1^4\Phi_3^4\Phi'_3\Phi_5{\Phi''_6}^3\Phi_7\Phi_9\Phi_{10}\Phi'''_{12}\Phi_{15}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi''_{42}$	-1
$G_{3,3,3}[\zeta_3] : \phi'_{3,5}$	$\frac{\zeta_3}{3}x^4\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi'_{42}$	ζ_3
# $\phi'_{105,8}$	$\frac{-\zeta_3}{6}x^4{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi''_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{15}\Phi''_{18}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3^2] : \phi_{1,0}$	$\frac{1}{6}x^4\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi'_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi'_{18}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi'_{42}$	ζ_3^2
$G_{3,3,3}[\zeta_3^2] : \phi_{2,3}$	$\frac{-\sqrt{-3}}{6}x^4\Phi_1^3\Phi_2^5\Phi_4\Phi_5\Phi_6^4\Phi_7\Phi_8\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{30}\Phi_{42}$	ζ_3^2
$G_{33}[-\zeta_3^2] : 1$	$\frac{\sqrt{-3}}{6}x^4\Phi_1^5\Phi_2^3\Phi_3^4\Phi_4\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{30}$	$-\zeta_3^2$
$\phi_{90,6}$	$\frac{1}{3}x^4\Phi_3^3\Phi_5\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3^2] : \phi_{1,12}$	$\frac{-1}{6}x^4\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi''_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi''_{18}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi''_{42}$	ζ_3^2
$\phi_{56,9}$	$\frac{1}{3}x^5\Phi_4^2\Phi_7\Phi_8\Phi_9\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
# $\phi_{126,7}$	$\frac{1}{3}x^5\Phi_3^3\Phi_6^3\Phi_7\Phi_8\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
* $\phi_{126,5}$	$\frac{1}{3}x^5\Phi_3^3\Phi_6^3\Phi_7\Phi_8\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi''_{70,9}$	$\frac{-\zeta_3^2}{3}x^5\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3] : \phi_{3,4}$	$\frac{-\zeta_3^2}{3}x^5\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi'''_{15}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	ζ_3
$G_{3,3,3}[\zeta_3^2] : \phi''_{3,5}$	$\frac{\zeta_3^2}{3}x^5\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi'''_{15}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	ζ_3^2
$\phi'''_{70,9}$	$\frac{-\zeta_3}{3}x^5\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3^2] : \phi_{3,1}$	$\frac{-\zeta_3}{3}x^5\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi''''_{15}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	ζ_3^2
$G_{3,3,3}[\zeta_3] : \phi'_{3,8}$	$\frac{\zeta_3}{3}x^5\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi''''_{15}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	ζ_3
* $\phi_{315,6}$	$\frac{1}{3}x^6\Phi_3^3\Phi_5\Phi_6^3\Phi_7\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{210,10}$	$\frac{1}{3}x^6{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi''_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{210,8}$	$\frac{1}{3}x^6{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi'_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	1
$\phi''_{105,8}$	$\frac{1}{3}x^6{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_7\Phi''_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3] : \phi_{2,3}$	$\frac{1}{3}x^6\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi'_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	ζ_3
$G_{3,3,3}[\zeta_3^2] : \phi_{3,4}$	$\frac{1}{3}x^6\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
$\phi_{105,10}$	$\frac{1}{3}x^6{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_7\Phi'_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3^2] : \phi''_{3,8}$	$\frac{-1}{3}x^6\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
$G_{3,3,3}[\zeta_3] : \phi_{2,9}$	$\frac{-1}{3}x^6\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi''_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	ζ_3
* $\phi_{420,7}$	$\frac{-\zeta_3^2}{6}x^7{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi''_9\Phi_{12}^2\Phi_{14}\Phi'''_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{336,10}$	$\frac{3-\sqrt{-3}}{12}x^7\Phi_2^4{\Phi''_3}^3\Phi_6^4\Phi'_6\Phi_7\Phi_8\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$D_4 : 1.1....$	$\frac{3-\sqrt{-3}}{12}x^7\Phi_1^4\Phi_3^4\Phi''_3\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	-1
$G_{3,3,3}[\zeta_3] : \phi'_{6,4}$	$\frac{-\zeta_3^2}{3}x^7\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi''''_{15}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	ζ_3
$\phi_{84,13}$	$\frac{-\zeta_3^2}{6}x^7{\Phi''_3}^3\Phi_4^2{\Phi'_6}^3\Phi_7\Phi_8\Phi'_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	1
$\phi_{84,17}$	$\frac{-\zeta_3}{6}x^7{\Phi'_3}^3\Phi_4^2{\Phi''_6}^3\Phi_7\Phi_8\Phi''_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	1
$\phi_{336,8}$	$\frac{3+\sqrt{-3}}{12}x^7\Phi_2^4{\Phi'_3}^3\Phi_6^4\Phi''_6\Phi_7\Phi_8\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$D_4 : 1.....1$	$\frac{3+\sqrt{-3}}{12}x^7\Phi_1^4\Phi_3^4\Phi'_3\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	-1
$G_{3,3,3}[\zeta_3] : \phi_{6,2}$	$\frac{\zeta_3}{3}x^7\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi'''_{15}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	ζ_3

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
#	$\phi_{420,11}$	$\frac{-\zeta_3}{6}x^7{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi'_9\Phi_{12}^2\Phi_{14}\Phi''''_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{3,3,3}[\zeta_3^2]:\phi_{1,21}$	$\frac{1}{6}x^7\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi''_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi''''_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	ζ_3^2
	$G_{3,3,3}[\zeta_3^2]:\phi_{2,12}$	$\frac{-\sqrt{-3}}{6}x^7\Phi_1^3\Phi_2^5\Phi_4\Phi_5\Phi_6^4\Phi_7\Phi_8\Phi_{10}\Phi_{14}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
	$G_{33}[-\zeta_3^2]:-1$	$\frac{\sqrt{-3}}{6}x^7\Phi_1^5\Phi_2^3\Phi_3^4\Phi_4\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{42}$	$-\zeta_3^2$
	$\phi_{504,9}$	$\frac{1}{3}x^7\Phi_3^3\Phi_4^2\Phi_6^3\Phi_7\Phi_8\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{3,3,3}[\zeta_3^2]:\phi_{1,9}$	$\frac{-1}{6}x^7\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi'_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi'''_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	ζ_3^2
*	$\phi_{384,8}$	$\frac{3+\sqrt{-3}}{12}x^8\Phi_2^5{\Phi''_3}^3\Phi_4^2\Phi_6^5\Phi'_6\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$\phi_{384,11}$	$\frac{3+\sqrt{-3}}{12}x^8\Phi_2^5{\Phi''_3}^3\Phi_4^2\Phi_6^5\Phi'_6\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{33}[i]:\zeta_3$	$\frac{3+\sqrt{-3}}{12}x^8\Phi_1^5\Phi_3^5\Phi''_3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi'_{42}$	$ix^{1/2}$
	$G_{33}[-i]:\zeta_3$	$\frac{3+\sqrt{-3}}{12}x^8\Phi_1^5\Phi_3^5\Phi''_3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi'_{42}$	$-ix^{1/2}$
	$\phi_{384,10}$	$\frac{3-\sqrt{-3}}{12}x^8\Phi_2^5{\Phi'_3}^3\Phi_4^2\Phi_6^5\Phi''_6\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
#	$\phi_{384,13}$	$\frac{3-\sqrt{-3}}{12}x^8\Phi_2^5{\Phi'_3}^3\Phi_4^2\Phi_6^5\Phi''_6\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{33}[i]:\zeta_3^2$	$\frac{3-\sqrt{-3}}{12}x^8\Phi_1^5\Phi_3^5\Phi'_3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi''_{42}$	$ix^{1/2}$
	$G_{33}[-i]:\zeta_3^2$	$\frac{3-\sqrt{-3}}{12}x^8\Phi_1^5\Phi_3^5\Phi'_3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi''_{42}$	$-ix^{1/2}$
	$G_{3,3,3}[\zeta_3]:\phi_{8,3}$	$\frac{\sqrt{-3}}{6}x^8\Phi_1^3\Phi_2^6\Phi_4^2\Phi_5\Phi_6^5\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
	$G_{3,3,3}[\zeta_3]:\phi'_{8,6}$	$\frac{\sqrt{-3}}{6}x^8\Phi_1^3\Phi_2^6\Phi_4^2\Phi_5\Phi_6^5\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
	$G_{34}[\zeta_{12}^7]$	$\frac{\sqrt{-3}}{6}x^8\Phi_1^6\Phi_2^3\Phi_3^5\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}$	$\zeta_{12}^7x^{1/2}$
	$G_{34}[\zeta_{12}]$	$\frac{\sqrt{-3}}{6}x^8\Phi_1^6\Phi_2^3\Phi_3^5\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}$	$\zeta_{12}x^{1/2}$
*	$\phi'_{70,9}$	$x^9\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
*	$\phi_{560,9}$	$\frac{1}{2}x^9\Phi_2^4\Phi_5\Phi_6^4\Phi_7\Phi_8\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$\phi_{140,12}$	$\frac{1}{2}x^9\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$\phi_{420,12}$	$\frac{1}{2}x^9\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$D_4:.1....1$	$\frac{1}{2}x^9\Phi_1^4\Phi_3^4\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	-1
	$\phi_{35,18}$	$\frac{1}{3}x^{10}\Phi_5\Phi_7\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
#	$\phi_{315,14}$	$\frac{1}{3}x^{10}\Phi_3^3\Phi_5\Phi_6^3\Phi_7\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
*	$\phi_{315,10}$	$\frac{1}{3}x^{10}\Phi_3^3\Phi_5\Phi_6^3\Phi_7\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$\phi'_{280,12}$	$\frac{-\zeta_3^2}{3}x^{10}\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	1
	$G_{3,3,3}[\zeta_3]:\phi_{6,5}$	$\frac{-\zeta_3^2}{3}x^{10}\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	ζ_3
	$G_{3,3,3}[\zeta_3^2]:\phi''_{6,4}$	$\frac{\zeta_3^2}{3}x^{10}\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
	$\phi''_{280,12}$	$\frac{-\zeta_3}{3}x^{10}\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	1
	$G_{3,3,3}[\zeta_3^2]:\phi_{6,2}$	$\frac{-\zeta_3}{3}x^{10}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
	$G_{3,3,3}[\zeta_3]:\phi'_{6,7}$	$\frac{\zeta_3}{3}x^{10}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	ζ_3
#	$\phi_{729,12}$	$\frac{1}{3}x^{10}\Phi_3^6\Phi_6^6\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{34}^2[\zeta_9^4]$	$\frac{1}{3}x^{10}\Phi_1^6\Phi_2^6{\Phi''_3}^6\Phi_4^2\Phi_5{\Phi'_6}^6\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi''_{21}\Phi'_{24}\Phi''''_{30}\Phi'_{42}$	$\zeta_9^4x^{2/3}$
	$G_{34}^2[\zeta_9^7]$	$\frac{1}{3}x^{10}\Phi_1^6\Phi_2^6{\Phi'_3}^6\Phi_4^2\Phi_5{\Phi''_6}^6\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi'_{21}\Phi''_{24}\Phi'''_{30}\Phi''_{42}$	$\zeta_9^7x^{1/3}$
	$\phi_{729,14}$	$\frac{1}{3}x^{10}\Phi_3^6\Phi_6^6\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{34}^2[\zeta_9]$	$\frac{1}{3}x^{10}\Phi_1^6\Phi_2^6{\Phi''_3}^6\Phi_4^2\Phi_5{\Phi'_6}^6\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi''_{21}\Phi'_{24}\Phi''''_{30}\Phi'_{42}$	$\zeta_9x^{2/3}$
	$G_{34}[\zeta_9]$	$\frac{1}{3}x^{10}\Phi_1^6\Phi_2^6{\Phi'_3}^6\Phi_4^2\Phi_5{\Phi''_6}^6\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi'_{21}\Phi''_{24}\Phi'''_{30}\Phi''_{42}$	$\zeta_9x^{1/3}$
*	$\phi_{729,10}$	$\frac{1}{3}x^{10}\Phi_3^6\Phi_6^6\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{34}[\zeta_9^7]$	$\frac{1}{3}x^{10}\Phi_1^6\Phi_2^6{\Phi''_3}^6\Phi_4^2\Phi_5{\Phi'_6}^6\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi''_{21}\Phi'_{24}\Phi''''_{30}\Phi'_{42}$	$\zeta_9^7x^{2/3}$
	$G_{34}[\zeta_9^4]$	$\frac{1}{3}x^{10}\Phi_1^6\Phi_2^6{\Phi'_3}^6\Phi_4^2\Phi_5{\Phi''_6}^6\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi'_{21}\Phi''_{24}\Phi'''_{30}\Phi''_{42}$	$\zeta_9^4x^{1/3}$
*	$\phi_{630,11}$	$\frac{3+\sqrt{-3}}{6}x^{11}{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
#	$\phi_{630,13}$	$\frac{3-\sqrt{-3}}{6}x^{11}{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
	$G_{3,3,3}[\zeta_3] : \phi_{3,6}$	$\frac{\sqrt{-3}}{3}x^{11}\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
	$\phi_{630,15}$	$\frac{1}{3}x^{11}\Phi_3^3\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
#	$\phi'_{840,13}$	$\frac{1}{3}x^{11}\Phi_3'^3\Phi_4^2\Phi_5\Phi_6''^3\Phi_7\Phi_8\Phi_9'\Phi_{10}\Phi_{12}'''^2\Phi_{14}\Phi_{15}\Phi_{18}''\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
*	$\phi_{840,11}$	$\frac{1}{3}x^{11}\Phi_3''^3\Phi_4^2\Phi_5\Phi_6'^3\Phi_7\Phi_8\Phi_9''\Phi_{10}\Phi_{12}''''^2\Phi_{14}\Phi_{15}\Phi_{18}'\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$\phi_{210,17}$	$\frac{-\zeta_3^2}{3}x^{11}\Phi_3''^3\Phi_5\Phi_6'^3\Phi_7\Phi_8\Phi_9'\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}''\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{3,3,3}[\zeta_3] : \phi_{2,18}$	$\frac{\zeta_3^2}{3}x^{11}\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9''\Phi_{10}\Phi_{12}'''^2\Phi_{14}\Phi_{15}\Phi_{18}'\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
	$G_{3,3,3}[\zeta_3^2] : \phi''_{6,7}$	$\frac{\zeta_3^2}{3}x^{11}\Phi_1^3\Phi_2^3\Phi_3''^3\Phi_4^2\Phi_5\Phi_6'^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}''''^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
	$\phi_{210,13}$	$\frac{-\zeta_3}{3}x^{11}\Phi_3'^3\Phi_5\Phi_6''^3\Phi_7\Phi_8\Phi_9''\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}'\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{3,3,3}[\zeta_3^2] : \phi_{6,5}$	$\frac{-\zeta_3}{3}x^{11}\Phi_1^3\Phi_2^3\Phi_3'^3\Phi_4^2\Phi_5\Phi_6''^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}'''^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
	$G_{3,3,3}[\zeta_3] : \phi_{2,12}$	$\frac{-\zeta_3}{3}x^{11}\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9'\Phi_{10}\Phi_{12}''''^2\Phi_{14}\Phi_{15}\Phi_{18}''\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
*	$\phi_{210,12}$	$x^{12}\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
*	$\phi_{896,12}$	$\frac{1}{2}x^{12}\Phi_2^5\Phi_4^2\Phi_6^5\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
#	$\phi_{896,15}$	$\frac{1}{2}x^{12}\Phi_2^5\Phi_4^2\Phi_6^5\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{33}[i] : 1$	$\frac{1}{2}x^{12}\Phi_1^5\Phi_3^5\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{42}$	$ix^{1/2}$
	$G_{33}[-i] : 1$	$\frac{1}{2}x^{12}\Phi_1^5\Phi_3^5\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{42}$	$-ix^{1/2}$
	$\phi_{189,18}$	$\frac{1}{6}x^{13}\Phi_3^3\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$\phi_{336,19}$	$\frac{-\zeta_3}{6}x^{13}\Phi_2^4\Phi_3''^3\Phi_6^4\Phi_6'^2\Phi_7\Phi_8\Phi_9''\Phi_{10}\Phi_{12}''''^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}'\Phi_{30}\Phi_{42}$	1
	$\phi_{756,14}$	$\frac{-\zeta_3^2}{6}x^{13}\Phi_3'^6\Phi_4^2\Phi_6''^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}'''^2\Phi_{14}\Phi_{15}\Phi_{18}''\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$\phi_{504,15}$	$\frac{1}{6}x^{13}\Phi_2^4\Phi_3^3\Phi_6^5\Phi_7\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$\phi_{756,16}$	$\frac{-\zeta_3}{6}x^{13}\Phi_3''^6\Phi_4^2\Phi_6'^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}''''^2\Phi_{14}\Phi_{15}\Phi_{18}'\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$\phi_{336,17}$	$\frac{-\zeta_3^2}{6}x^{13}\Phi_2^4\Phi_3'^3\Phi_6^4\Phi_6''^2\Phi_7\Phi_8\Phi_9'\Phi_{10}\Phi_{12}'''^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}''\Phi_{30}\Phi_{42}$	1
	$\phi_{105,20}$	$\frac{-1}{6}x^{13}\Phi_3'^3\Phi_5\Phi_6''^6\Phi_7\Phi_8\Phi_9''\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}'''\Phi_{42}$	1
	$G_{34}^2[-\zeta_3^2]$	$\frac{-\zeta_3}{6}x^{13}\Phi_1^6\Phi_2^3\Phi_3^4\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}''\Phi_{21}\Phi_{24}'\Phi_{30}'''\Phi_{42}$	$-\zeta_3^2$
	$G_{3,3,3}[\zeta_3] : \phi''_{3,13}$	$\frac{\zeta_3^2}{6}x^{13}\Phi_1^3\Phi_2^3\Phi_3'^3\Phi_4^2\Phi_5\Phi_6''^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}'''^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}'''\Phi_{42}$	ζ_3
	$D_4 : ....2.$	$\frac{-1}{6}x^{13}\Phi_1^4\Phi_3^4\Phi_3'^2\Phi_5\Phi_6''^3\Phi_7\Phi_9\Phi_{10}\Phi_{12}'''^2\Phi_{14}\Phi_{15}\Phi_{18}'\Phi_{21}\Phi_{24}\Phi_{30}'''\Phi_{42}$	-1
	$G_{3,3,3}[\zeta_3^2] : \phi'_{8,6}$	$\frac{-\zeta_3}{6}x^{13}\Phi_1^3\Phi_2^6\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9'\Phi_{10}\Phi_{12}''''^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}'''\Phi_{42}$	ζ_3^2
	$G_{33}[-\zeta_3] : -\zeta_3^2$	$\frac{\zeta_3^2}{6}x^{13}\Phi_1^5\Phi_2^3\Phi_3^4\Phi_3'\Phi_4\Phi_5\Phi_6''^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}'''\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}''\Phi_{30}'''\Phi_{42}$	$-\zeta_3$
	$\phi_{945,16}$	$\frac{1}{6}x^{13}\Phi_3''^6\Phi_5\Phi_6'^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}'''\Phi_{18}''\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{3,3,3}[\zeta_3] : \phi''_{6,8}$	$\frac{\zeta_3}{6}x^{13}\Phi_1^3\Phi_2^5\Phi_3''^3\Phi_4\Phi_5\Phi_6^4\Phi_6'\Phi_7\Phi_8\Phi_{10}\Phi_{12}''''\Phi_{14}\Phi_{15}'''\Phi_{18}\Phi_{21}\Phi_{24}'\Phi_{30}\Phi_{42}$	ζ_3
	$G_{34}^2[\zeta_3^2]$	$\frac{\zeta_3^2}{6}x^{13}\Phi_1^6\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}'''^2\Phi_{14}\Phi_{15}'''\Phi_{18}'\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
*	$\phi''_{840,13}$	$\frac{1}{6}x^{13}\Phi_2^4\Phi_3''^3\Phi_5\Phi_6^4\Phi_6'^2\Phi_7\Phi_9'\Phi_{10}\Phi_{12}''''^2\Phi_{14}\Phi_{15}'''\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{3,3,3}[\zeta_3] : \phi_{9,5}$	$\frac{\zeta_3}{6}x^{13}\Phi_1^3\Phi_2^3\Phi_3''^3\Phi_4^2\Phi_5\Phi_6'^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}''''^2\Phi_{14}\Phi_{15}'''\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
	$G_{3,3,3}[\zeta_3^2] : \phi'_{8,9}$	$\frac{\zeta_3^2}{6}x^{13}\Phi_1^3\Phi_2^6\Phi_4^2\Phi_5\Phi_6^4\Phi_7\Phi_8\Phi_9''\Phi_{10}\Phi_{14}\Phi_{15}'''\Phi_{18}\Phi_{21}\Phi_{24}''\Phi_{30}\Phi_{42}$	ζ_3^2
	$\phi_{315,18}$	$\frac{1}{6}x^{13}\Phi_3^3\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$D_4 : 1....1.$	$\frac{\zeta_3}{6}x^{13}\Phi_1^4\Phi_3^4\Phi_3''^2\Phi_5\Phi_6'^3\Phi_7\Phi_8\Phi_9\Phi_{12}''''^2\Phi_{14}\Phi_{15}\Phi_{18}'\Phi_{21}\Phi_{24}'\Phi_{30}\Phi_{42}$	-1
	$\phi_{420,14}$	$\frac{\zeta_3^2}{6}x^{13}\Phi_3'^3\Phi_4^2\Phi_5\Phi_6''^6\Phi_7\Phi_8\Phi_9'\Phi_{12}'''^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$D_4 : 11.....$	$\frac{1}{6}x^{13}\Phi_1^4\Phi_3^5\Phi_5\Phi_6^3\Phi_7\Phi_9\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	-1
	$\phi_{420,16}$	$\frac{\zeta_3}{6}x^{13}\Phi_3''^3\Phi_4^2\Phi_5\Phi_6'^6\Phi_7\Phi_8\Phi_9''\Phi_{12}''''^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$D_4 : 1..1...$	$\frac{\zeta_3^2}{6}x^{13}\Phi_1^4\Phi_3^4\Phi_3'^2\Phi_5\Phi_6''^3\Phi_7\Phi_8\Phi_9\Phi_{12}'''^2\Phi_{14}\Phi_{15}\Phi_{18}''\Phi_{21}\Phi_{24}''\Phi_{30}\Phi_{42}$	-1
	$\phi_{945,14}$	$\frac{1}{6}x^{13}\Phi_3'^6\Phi_5\Phi_6''^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}''''\Phi_{18}'\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{3,3,3}[\zeta_3^2] : \phi_{8,3}$	$\frac{-\zeta_3}{6}x^{13}\Phi_1^3\Phi_2^6\Phi_4^2\Phi_5\Phi_6^4\Phi_7\Phi_8\Phi_9'\Phi_{10}\Phi_{14}\Phi_{15}''''\Phi_{18}\Phi_{21}\Phi_{24}'\Phi_{30}\Phi_{42}$	ζ_3^2
	$G_{3,3,3}[\zeta_3] : \phi_{9,7}$	$\frac{-\zeta_3^2}{6}x^{13}\Phi_1^3\Phi_2^3\Phi_3'^3\Phi_4^2\Phi_5\Phi_6''^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}'''^2\Phi_{14}\Phi_{15}''''\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
#	$\phi_{840,17}$	$\frac{1}{6}x^{13}\Phi_2^4\Phi_3'^3\Phi_5\Phi_6^4\Phi_6''^2\Phi_7\Phi_9''\Phi_{10}\Phi_{12}'''^2\Phi_{14}\Phi_{15}''''\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$G_{34}[\zeta_3^2]$	$\frac{-\zeta_3}{6}x^{13}\Phi_1^6\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi''''_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
$G_{3,3,3}[\zeta_3]:\phi''_{6,4}$	$\frac{-\zeta_3^2}{6}x^{13}\Phi_1^3\Phi_2^5{\Phi'_3}^3\Phi_4\Phi_5\Phi_6^4\Phi''_6\Phi_7\Phi_8\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	ζ_3
$\phi_{105,22}$	$\frac{-1}{6}x^{13}{\Phi''_3}^3\Phi_5{\Phi'_6}^6\Phi_7\Phi_8\Phi'_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	1
$G_{33}[-\zeta_3]:-\zeta_3$	$\frac{-\zeta_3}{6}x^{13}\Phi_1^5\Phi_2^3\Phi_3^4\Phi''_3\Phi_4\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi'_{24}\Phi''''_{30}\Phi_{42}$	$-\zeta_3$
$G_{3,3,3}[\zeta_3^2]:\phi''_{8,6}$	$\frac{\zeta_3^2}{6}x^{13}\Phi_1^3\Phi_2^6\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi''_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	ζ_3^2
$D_4:..2...$	$\frac{-1}{6}x^{13}\Phi_1^4\Phi_3^4{\Phi''_3}^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	-1
$G_{3,3,3}[\zeta_3]:\phi''_{3,5}$	$\frac{-\zeta_3}{6}x^{13}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	ζ_3
$G_{34}[-\zeta_3^2]$	$\frac{\zeta_3^2}{6}x^{13}\Phi_1^6\Phi_2^3\Phi_3^4\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi''_{24}\Phi''''_{30}\Phi_{42}$	$-\zeta_3^2$
* $\phi_{630,14}$	$\frac{3-\sqrt{-3}}{6}x^{14}{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
# $\phi_{630,16}$	$\frac{3+\sqrt{-3}}{6}x^{14}{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3^2]:\phi_{3,6}$	$\frac{-\sqrt{-3}}{3}x^{14}\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
* $\phi_{1280,15}$	$\frac{1}{42}x^{15}\Phi_2^6\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{34}[1]$	$\frac{-1}{42}x^{15}\Phi_1^6\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi''_{540,21}$	$\frac{-1}{42}x^{15}{\Phi'_3}^6\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi'_{540,21}$	$\frac{-1}{42}x^{15}{\Phi''_3}^6\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi''_{20,33}$	$\frac{1}{42}x^{15}\Phi_4^2\Phi_5{\Phi''_6}^6\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi'_{42}$	1
$\phi'_{20,33}$	$\frac{1}{42}x^{15}\Phi_4^2\Phi_5{\Phi'_6}^6\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi''_{42}$	1
$\phi''_{120,21}$	$\frac{1}{7}x^{15}\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{1260,17}$	$\frac{1}{6}x^{15}\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{540,17}$	$\frac{1}{6}x^{15}\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3]:\phi''_{3,8}$	$\frac{1}{6}x^{15}\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi''_{42}$	ζ_3
$G_{3,3,3}[\zeta_3]:\phi_{9,8}$	$\frac{1}{6}x^{15}\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
$G_{3,3,3}[\zeta_3^2]:\phi'_{3,5}$	$\frac{1}{6}x^{15}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi'_{42}$	ζ_3^2
$G_{3,3,3}[\zeta_3^2]:\phi_{9,5}$	$\frac{1}{6}x^{15}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
$\phi_{720,20}$	$\frac{1}{6}x^{15}\Phi_2^4\Phi_3^3\Phi_5\Phi_6^5\Phi_8\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$D_4:.1...1.$	$\frac{1}{6}x^{15}\Phi_1^4\Phi_3^5\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	-1
$G_{3,3,3}[\zeta_3]:\phi''_{6,11}$	$\frac{-1}{6}x^{15}\Phi_1^3\Phi_2^5{\Phi'_3}^3\Phi_4\Phi_5\Phi_6^4\Phi''_6\Phi_7\Phi_8\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
$G_{33}[-\zeta_3]:\zeta_3$	$\frac{-1}{6}x^{15}\Phi_1^5\Phi_2^3\Phi_3^4\Phi'_3\Phi_4\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi''_{42}$	$-\zeta_3$
$G_{3,3,3}[\zeta_3^2]:\phi'_{6,8}$	$\frac{-1}{6}x^{15}\Phi_1^3\Phi_2^5{\Phi''_3}^3\Phi_4\Phi_5\Phi_6^4\Phi'_6\Phi_7\Phi_8\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
$G_{33}[-\zeta_3^2]:-\zeta_3$	$\frac{-1}{6}x^{15}\Phi_1^5\Phi_2^3\Phi_3^4\Phi''_3\Phi_4\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi'_{42}$	$-\zeta_3^2$
$\phi_{720,16}$	$\frac{1}{6}x^{15}\Phi_2^4\Phi_3^3\Phi_5\Phi_6^5\Phi_8\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$D_4:..1...1$	$\frac{1}{6}x^{15}\Phi_1^4\Phi_3^5\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	-1
$G_{3,3,3}[\zeta_3^2]:\phi'_{6,4}$	$\frac{1}{6}x^{15}\Phi_1^3\Phi_2^5{\Phi'_3}^3\Phi_4\Phi_5\Phi_6^4\Phi''_6\Phi_7\Phi_8\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
$G_{33}[-\zeta_3^2]:-\zeta_3^2$	$\frac{1}{6}x^{15}\Phi_1^5\Phi_2^3\Phi_3^4\Phi'_3\Phi_4\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi''_{42}$	$-\zeta_3^2$
$G_{3,3,3}[\zeta_3]:\phi''_{6,7}$	$\frac{1}{6}x^{15}\Phi_1^3\Phi_2^5{\Phi''_3}^3\Phi_4\Phi_5\Phi_6^4\Phi'_6\Phi_7\Phi_8\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
$G_{33}[-\zeta_3]:\zeta_3^2$	$\frac{1}{6}x^{15}\Phi_1^5\Phi_2^3\Phi_3^4\Phi''_3\Phi_4\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi'_{42}$	$-\zeta_3$
$\phi_{1280,18}$	$\frac{1}{6}x^{15}\Phi_2^6\Phi_4^2\Phi_5\Phi_6^4\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{34}[-1]$	$\frac{-1}{6}x^{15}\Phi_1^6\Phi_3^4\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	-1
$\phi''_{560,18}$	$\frac{1}{6}x^{15}\Phi_2^4\Phi_5\Phi_6^4{\Phi'_6}^2\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$D_4:.1.1...$	$\frac{-1}{6}x^{15}\Phi_1^4\Phi_3^4{\Phi''_3}^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi''_{42}$	-1
$\phi'''_{560,18}$	$\frac{1}{6}x^{15}\Phi_2^4\Phi_5\Phi_6^4{\Phi''_6}^2\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$D_4:....1.1$	$\frac{-1}{6}x^{15}\Phi_1^4\Phi_3^4{\Phi'_3}^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi'_{42}$	-1
$\phi_{1260,19}$	$\frac{1}{6}x^{15}\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$\phi_{540,19}$	$\frac{1}{6}x^{15}\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3^2]:\phi'_{3,13}$	$\frac{-1}{6}x^{15}\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi''_{42}$	ζ_3^2
$G_{3,3,3}[\zeta_3^2]:\phi_{9,7}$	$\frac{-1}{6}x^{15}\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
$G_{3,3,3}[\zeta_3]:\phi''_{3,16}$	$\frac{-1}{6}x^{15}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi'_{42}$	ζ_3
$G_{3,3,3}[\zeta_3]:\phi_{9,10}$	$\frac{-1}{6}x^{15}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
$\phi'_{120,21}$	$\frac{1}{7}x^{15}\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{34}[\zeta_7]$	$\frac{1}{7}x^{15}\Phi_1^6\Phi_2^6\Phi_3^6\Phi_4^2\Phi_5\Phi_6^6\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_7
$G_{34}[\zeta_7^2]$	$\frac{1}{7}x^{15}\Phi_1^6\Phi_2^6\Phi_3^6\Phi_4^2\Phi_5\Phi_6^6\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_7^2
$G_{34}[\zeta_7^3]$	$\frac{1}{7}x^{15}\Phi_1^6\Phi_2^6\Phi_3^6\Phi_4^2\Phi_5\Phi_6^6\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_7^3
$G_{34}[\zeta_7^4]$	$\frac{1}{7}x^{15}\Phi_1^6\Phi_2^6\Phi_3^6\Phi_4^2\Phi_5\Phi_6^6\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_7^4
$G_{34}[\zeta_7^5]$	$\frac{1}{7}x^{15}\Phi_1^6\Phi_2^6\Phi_3^6\Phi_4^2\Phi_5\Phi_6^6\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_7^5
$G_{34}[\zeta_7^6]$	$\frac{1}{7}x^{15}\Phi_1^6\Phi_2^6\Phi_3^6\Phi_4^2\Phi_5\Phi_6^6\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{24}\Phi_{30}$	ζ_7^6
* $\phi'_{560,18}$	$\frac{1}{2}x^{18}\Phi_2^4\Phi_5\Phi_6^4\Phi_7\Phi_8\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{140,21}$	$\frac{1}{2}x^{18}\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{420,21}$	$\frac{1}{2}x^{18}\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$D_4:1...1..$	$\frac{1}{2}x^{18}\Phi_1^4\Phi_3^4\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	-1
* $\phi_{630,20}$	$\frac{3+\sqrt{-3}}{6}x^{20}{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
# $\phi_{630,22}$	$\frac{3-\sqrt{-3}}{6}x^{20}{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3]:\phi_{3,15}$	$\frac{\sqrt{-3}}{3}x^{20}\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
$\phi_{504,21}$	$\frac{1}{6}x^{19}\Phi_2^4\Phi_3^3\Phi_6^5\Phi_7\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{756,20}$	$\frac{-\zeta_3}{6}x^{19}{\Phi''_3}^6\Phi_4^2{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{336,25}$	$\frac{-\zeta_3^2}{6}x^{19}\Phi_2^4{\Phi'_3}^3\Phi_6^4{\Phi''_6}^2\Phi_7\Phi_8\Phi'_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{189,24}$	$\frac{1}{6}x^{19}\Phi_3^3\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{336,23}$	$\frac{-\zeta_3}{6}x^{19}\Phi_2^4{\Phi''_3}^3\Phi_6^4{\Phi'_6}^2\Phi_7\Phi_8\Phi''_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{756,22}$	$\frac{-\zeta_3^2}{6}x^{19}{\Phi'_3}^6\Phi_4^2{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{105,26}$	$\frac{-1}{6}x^{19}{\Phi''_3}^3\Phi_5{\Phi'_6}^6\Phi_7\Phi_8\Phi'_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	1
$G_{33}[-\zeta_3^2]:\zeta_3^2$	$\frac{\zeta_3}{6}x^{19}\Phi_1^5\Phi_2^3\Phi_3^4\Phi''_3\Phi_4\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi'_{24}\Phi''''_{30}\Phi_{42}$	$-\zeta_3^2$
$G_{3,3,3}[\zeta_3]:\phi'_{8,9}$	$\frac{-\zeta_3^2}{6}x^{19}\Phi_1^3\Phi_2^6\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi''_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	ζ_3
$D_4:.11....$	$\frac{-1}{6}x^{19}\Phi_1^4\Phi_3^4{\Phi''_3}^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	-1
$G_{3,3,3}[\zeta_3^2]:\phi'_{3,16}$	$\frac{\zeta_3}{6}x^{19}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	ζ_3^2
$G_{34}^2[-\zeta_3]$	$\frac{-\zeta_3^2}{6}x^{19}\Phi_1^6\Phi_2^3\Phi_3^4\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi''_{24}\Phi''''_{30}\Phi_{42}$	$-\zeta_3$
* $\phi_{840,19}$	$\frac{1}{6}x^{19}\Phi_2^4{\Phi'_3}^3\Phi_5\Phi_6^4{\Phi''_6}^2\Phi_7\Phi''_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{34}^2[\zeta_3]$	$\frac{\zeta_3}{6}x^{19}\Phi_1^6\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi''''_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
$G_{3,3,3}[\zeta_3^2]:\phi'_{6,11}$	$\frac{\zeta_3^2}{6}x^{19}\Phi_1^3\Phi_2^5{\Phi'_3}^3\Phi_4\Phi_5\Phi_6^4\Phi''_6\Phi_7\Phi_8\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
$\phi_{945,22}$	$\frac{1}{6}x^{19}{\Phi'_3}^6\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi''''_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3]:\phi_{8,12}$	$\frac{\zeta_3}{6}x^{19}\Phi_1^3\Phi_2^6\Phi_4^2\Phi_5\Phi_6^4\Phi_7\Phi_8\Phi'_9\Phi_{10}\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	ζ_3
$G_{3,3,3}[\zeta_3^2]:\phi_{9,8}$	$\frac{\zeta_3^2}{6}x^{19}\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi''''_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
$\phi_{315,24}$	$\frac{1}{6}x^{19}\Phi_3^3\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$D_4:...1..1$	$\frac{\zeta_3}{6}x^{19}\Phi_1^4\Phi_3^4{\Phi''_3}^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	-1
$\phi_{420,22}$	$\frac{\zeta_3^2}{6}x^{19}{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^6\Phi_7\Phi_8\Phi'_9{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$D_4:...2..$	$\frac{1}{6}x^{19}\Phi_1^4\Phi_3^5\Phi_5\Phi_6^3\Phi_7\Phi_9\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	-1
$\phi_{420,20}$	$\frac{\zeta_3}{6}x^{19}{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^6\Phi_7\Phi_8\Phi''_9{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$D_4:.1..1..$	$\frac{\zeta_3^2}{6}x^{19}\Phi_1^4\Phi_3^4{\Phi'_3}^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	-1

	γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
#	$\phi'_{840,23}$	$\frac{1}{6}x^{19}\Phi_2^4{\Phi''_3}^3\Phi_5\Phi_6^4{\Phi'_6}^2\Phi_7\Phi'_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{3,3,3}[\zeta_3^2]:\phi_{9,10}$	$\frac{-\zeta_3}{6}x^{19}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi'''_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
	$G_{3,3,3}[\zeta_3]:\phi''_{8,6}$	$\frac{-\zeta_3^2}{6}x^{19}\Phi_1^3\Phi_2^6\Phi_4^2\Phi_5\Phi_6^4\Phi_7\Phi_8\Phi''_9\Phi_{10}\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	ζ_3
	$\phi_{945,20}$	$\frac{1}{6}x^{19}{\Phi''_3}^6\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi'''_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{3,3,3}[\zeta_3^2]:\phi'_{6,7}$	$\frac{-\zeta_3}{6}x^{19}\Phi_1^3\Phi_2^5{\Phi''_3}^3\Phi_4\Phi_5\Phi_6^4\Phi'_6\Phi_7\Phi_8\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
	$G_{34}[\zeta_3]$	$\frac{-\zeta_3^2}{6}x^{19}\Phi_1^6\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi'''_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
	$\phi_{105,28}$	$\frac{-1}{6}x^{19}{\Phi'_3}^3\Phi_5{\Phi''_6}^6\Phi_7\Phi_8\Phi''_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	1
	$G_{34}[-\zeta_3]$	$\frac{\zeta_3}{6}x^{19}\Phi_1^6\Phi_2^3\Phi_3^4\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi'_{24}\Phi'''_{30}\Phi_{42}$	$-\zeta_3$
	$G_{3,3,3}[\zeta_3^2]:\phi'_{3,8}$	$\frac{-\zeta_3^2}{6}x^{19}\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	ζ_3^2
	$D_4:.....11$	$\frac{-1}{6}x^{19}\Phi_1^4\Phi_3^4{\Phi'_3}^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	-1
	$G_{3,3,3}[\zeta_3]:\phi''_{8,9}$	$\frac{\zeta_3}{6}x^{19}\Phi_1^3\Phi_2^6\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi'_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	ζ_3
	$G_{33}[-\zeta_3^2]:\zeta_3$	$\frac{-\zeta_3^2}{6}x^{19}\Phi_1^5\Phi_2^3\Phi_3^4\Phi'_3\Phi_4\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi''_{24}\Phi'''_{30}\Phi_{42}$	$-\zeta_3^2$
*	$\phi_{896,21}$	$\frac{1}{2}x^{21}\Phi_2^5\Phi_4^2\Phi_6^5\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
#	$\phi_{896,24}$	$\frac{1}{2}x^{21}\Phi_2^5\Phi_4^2\Phi_6^5\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{33}[i]:-1$	$\frac{1}{2}x^{21}\Phi_1^5\Phi_3^5\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{42}$	$ix^{1/2}$
	$G_{33}[-i]:-1$	$\frac{1}{2}x^{21}\Phi_1^5\Phi_3^5\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{42}$	$-ix^{1/2}$
*	$\phi_{210,30}$	$x^{30}\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$\phi_{630,27}$	$\frac{1}{3}x^{23}\Phi_3^3\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$\phi_{210,29}$	$\frac{-\zeta_3}{3}x^{23}{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi''_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$\phi_{210,25}$	$\frac{-\zeta_3^2}{3}x^{23}{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi'_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
*	$\phi''_{840,23}$	$\frac{1}{3}x^{23}{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi'_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{3,3,3}[\zeta_3]:\phi_{6,10}$	$\frac{\zeta_3}{3}x^{23}\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
	$G_{3,3,3}[\zeta_3^2]:\phi_{2,9}$	$\frac{-\zeta_3^2}{3}x^{23}\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi''_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
#	$\phi_{840,25}$	$\frac{1}{3}x^{23}{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi''_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{3,3,3}[\zeta_3^2]:\phi_{2,15}$	$\frac{\zeta_3}{3}x^{23}\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi'_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
	$G_{3,3,3}[\zeta_3]:\phi'_{6,8}$	$\frac{-\zeta_3^2}{3}x^{23}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
*	$\phi_{630,23}$	$\frac{3-\sqrt{-3}}{6}x^{23}{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
#	$\phi_{630,25}$	$\frac{3+\sqrt{-3}}{6}x^{23}{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{3,3,3}[\zeta_3^2]:\phi_{3,15}$	$\frac{-\sqrt{-3}}{3}x^{23}\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
*	$\phi_{729,24}$	$\frac{1}{3}x^{24}\Phi_3^6\Phi_6^6\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{34}[\zeta_9^5]$	$\frac{\zeta_3^2}{3}x^{24}\Phi_1^6\Phi_2^6{\Phi'_3}^6\Phi_4^2\Phi_5{\Phi''_6}^6\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi'_{21}\Phi''_{24}\Phi'''_{30}\Phi''_{42}$	$\zeta_9^5x^{2/3}$
	$G_{34}^2[\zeta_9^2]$	$\frac{\zeta_3}{3}x^{24}\Phi_1^6\Phi_2^6{\Phi''_3}^6\Phi_4^2\Phi_5{\Phi'_6}^6\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi''_{21}\Phi'_{24}\Phi''''_{30}\Phi'_{42}$	$\zeta_9^2x^{1/3}$
#	$\phi_{729,26}$	$\frac{1}{3}x^{24}\Phi_3^6\Phi_6^6\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{34}^2[\zeta_9^8]$	$\frac{\zeta_3^2}{3}x^{24}\Phi_1^6\Phi_2^6{\Phi'_3}^6\Phi_4^2\Phi_5{\Phi''_6}^6\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi'_{21}\Phi''_{24}\Phi'''_{30}\Phi''_{42}$	$\zeta_9^8x^{2/3}$
	$G_{34}^6[\zeta_9^8]$	$\frac{\zeta_3}{3}x^{24}\Phi_1^6\Phi_2^6{\Phi''_3}^6\Phi_4^2\Phi_5{\Phi'_6}^6\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi''_{21}\Phi'_{24}\Phi''''_{30}\Phi'_{42}$	$\zeta_9^8x^{1/3}$
	$\phi_{729,28}$	$\frac{1}{3}x^{24}\Phi_3^6\Phi_6^6\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{34}[\zeta_9^2]$	$\frac{\zeta_3^2}{3}x^{24}\Phi_1^6\Phi_2^6{\Phi'_3}^6\Phi_4^2\Phi_5{\Phi''_6}^6\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi'_{21}\Phi''_{24}\Phi'''_{30}\Phi''_{42}$	$\zeta_9^2x^{2/3}$
	$G_{34}^2[\zeta_9^5]$	$\frac{\zeta_3}{3}x^{24}\Phi_1^6\Phi_2^6{\Phi''_3}^6\Phi_4^2\Phi_5{\Phi'_6}^6\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi''_{21}\Phi'_{24}\Phi''''_{30}\Phi'_{42}$	$\zeta_9^5x^{1/3}$
	$\phi_{35,36}$	$\frac{1}{3}x^{28}\Phi_5\Phi_7\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$\phi'_{280,30}$	$\frac{-\zeta_3}{3}x^{28}\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	1
	$\phi''_{280,30}$	$\frac{-\zeta_3^2}{3}x^{28}\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	1
*	$\phi_{315,28}$	$\frac{1}{3}x^{28}\Phi_3^3\Phi_5\Phi_6^3\Phi_7\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
	$G_{3,3,3}[\zeta_3]:\phi_{6,13}$	$\frac{\zeta_3}{3}x^{28}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	ζ_3

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$G_{3,3,3}[\zeta_3^2] : \phi_{6,10}$	$\frac{\zeta_3^2}{3}x^{28}\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
# $\phi_{315,32}$	$\frac{1}{3}x^{28}\Phi_3^3\Phi_5\Phi_6^3\Phi_7\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3^2] : \phi''_{6,8}$	$\frac{-\zeta_3}{3}x^{28}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
$G_{3,3,3}[\zeta_3] : \phi'_{6,11}$	$\frac{-\zeta_3^2}{3}x^{28}\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	ζ_3
* $\phi_{560,27}$	$\frac{1}{2}x^{27}\Phi_2^4\Phi_5\Phi_6^4\Phi_7\Phi_8\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{140,30}$	$\frac{1}{2}x^{27}\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{420,30}$	$\frac{1}{2}x^{27}\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$D_4 : ..1..1.$	$\frac{1}{2}x^{27}\Phi_1^4\Phi_3^4\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	-1
* $\phi'_{70,45}$	$x^{45}\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
* $\phi_{384,29}$	$\frac{3-\sqrt{-3}}{12}x^{29}\Phi_2^5{\Phi'_3}^3\Phi_4^2\Phi_6^5\Phi''_6\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{384,32}$	$\frac{3-\sqrt{-3}}{12}x^{29}\Phi_2^5{\Phi'_3}^3\Phi_4^2\Phi_6^5\Phi''_6\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{33}[i] : -\zeta_3$	$\frac{3-\sqrt{-3}}{12}x^{29}\Phi_1^5\Phi_3^5\Phi'_3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi''_{42}$	$ix^{1/2}$
$G_{33}[-i] : -\zeta_3$	$\frac{3-\sqrt{-3}}{12}x^{29}\Phi_1^5\Phi_3^5\Phi'_3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi''_{42}$	$-ix^{1/2}$
$\phi_{384,31}$	$\frac{3+\sqrt{-3}}{12}x^{29}\Phi_2^5{\Phi''_3}^3\Phi_4^2\Phi_6^5\Phi'_6\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
# $\phi_{384,34}$	$\frac{3+\sqrt{-3}}{12}x^{29}\Phi_2^5{\Phi''_3}^3\Phi_4^2\Phi_6^5\Phi'_6\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{33}[i] : -\zeta_3^2$	$\frac{3+\sqrt{-3}}{12}x^{29}\Phi_1^5\Phi_3^5\Phi''_3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi'_{42}$	$ix^{1/2}$
$G_{33}[-i] : -\zeta_3^2$	$\frac{3+\sqrt{-3}}{12}x^{29}\Phi_1^5\Phi_3^5\Phi''_3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi_{12}^2\Phi_{15}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi'_{42}$	$-ix^{1/2}$
$G_{3,3,3}[\zeta_3^2] : \phi_{8,12}$	$\frac{-\sqrt{-3}}{6}x^{29}\Phi_1^3\Phi_2^6\Phi_4^2\Phi_5\Phi_6^5\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
$G_{3,3,3}[\zeta_3^2] : \phi''_{8,9}$	$\frac{-\sqrt{-3}}{6}x^{29}\Phi_1^3\Phi_2^6\Phi_4^2\Phi_5\Phi_6^5\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{18}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
$G_{34}^7[\zeta_{12}^{11}]$	$\frac{-\sqrt{-3}}{6}x^{29}\Phi_1^6\Phi_2^3\Phi_3^5\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}$	$\zeta_{12}^{11}x^{1/2}$
$G_{34}[\zeta_{12}^5]$	$\frac{-\sqrt{-3}}{6}x^{29}\Phi_1^6\Phi_2^3\Phi_3^5\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}$	$\zeta_{12}^5x^{1/2}$
* $\phi_{420,31}$	$\frac{-\zeta_3}{6}x^{31}{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi'_9\Phi_{12}^2\Phi_{14}\Phi''''_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{336,34}$	$\frac{3+\sqrt{-3}}{12}x^{31}\Phi_2^4{\Phi'_3}^3\Phi_6^4\Phi''_6\Phi_7\Phi_8\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$D_4 : ...1.1.$	$\frac{3+\sqrt{-3}}{12}x^{31}\Phi_1^4\Phi_3^4\Phi'_3\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_9\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	-1
$G_{3,3,3}[\zeta_3^2] : \phi_{6,13}$	$\frac{-\zeta_3}{3}x^{31}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi'''_{15}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	ζ_3^2
$\phi_{84,37}$	$\frac{-\zeta_3}{6}x^{31}{\Phi'_3}^3\Phi_4^2{\Phi''_6}^3\Phi_7\Phi_8\Phi''_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	1
$\phi_{84,41}$	$\frac{-\zeta_3^2}{6}x^{31}{\Phi''_3}^3\Phi_4^2{\Phi'_6}^3\Phi_7\Phi_8\Phi'_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	1
$\phi_{336,32}$	$\frac{3-\sqrt{-3}}{12}x^{31}\Phi_2^4{\Phi''_3}^3\Phi_6^4\Phi'_6\Phi_7\Phi_8\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$D_4 : ..1.1..$	$\frac{3-\sqrt{-3}}{12}x^{31}\Phi_1^4\Phi_3^4\Phi''_3\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_9\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	-1
$G_{3,3,3}[\zeta_3^2] : \phi''_{6,11}$	$\frac{\zeta_3^2}{3}x^{31}\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4^2\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi''''_{15}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	ζ_3^2
# $\phi_{420,35}$	$\frac{-\zeta_3^2}{6}x^{31}{\Phi''_3}^3\Phi_4^2\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi''_9\Phi_{12}^2\Phi_{14}\Phi'''_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3] : \phi_{1,24}$	$\frac{1}{6}x^{31}\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi'_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi'''_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	ζ_3
$G_{3,3,3}[\zeta_3] : \phi_{2,15}$	$\frac{\sqrt{-3}}{6}x^{31}\Phi_1^3\Phi_2^5\Phi_4\Phi_5\Phi_6^4\Phi_7\Phi_8\Phi_{10}\Phi_{14}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	ζ_3
$G_{33}[-\zeta_3] : 1$	$\frac{-\sqrt{-3}}{6}x^{31}\Phi_1^5\Phi_2^3\Phi_3^4\Phi_4\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{42}$	$-\zeta_3$
$\phi_{504,33}$	$\frac{1}{3}x^{31}\Phi_3^3\Phi_4^2\Phi_6^3\Phi_7\Phi_8\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3] : \phi_{1,12}$	$\frac{-1}{6}x^{31}\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi''_9\Phi_{10}\Phi_{12}^2\Phi_{14}\Phi''''_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	ζ_3
* $\phi_{315,36}$	$\frac{1}{3}x^{36}\Phi_3^3\Phi_5\Phi_6^3\Phi_7\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{210,40}$	$\frac{1}{3}x^{36}{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi'_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{210,38}$	$\frac{1}{3}x^{36}{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi''_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{105,40}$	$\frac{1}{3}x^{36}{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_7\Phi''_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3] : \phi_{3,17}$	$\frac{-1}{3}x^{36}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	ζ_3
$G_{3,3,3}[\zeta_3^2] : \phi_{2,24}$	$\frac{-1}{3}x^{36}\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi'_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
$\phi_{105,38}$	$\frac{1}{3}x^{36}{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_7\Phi'_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1

γ	$\mathrm{Deg}(\gamma)$	$\mathrm{Fr}(\gamma)$
$G_{3,3,3}[\zeta_3^2] : \phi_{2,18}$	$\frac{1}{3}x^{36}\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi''_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	ζ_3^2
$G_{3,3,3}[\zeta_3] : \phi'_{3,13}$	$\frac{1}{3}x^{36}\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	ζ_3
$\phi_{56,45}$	$\frac{1}{3}x^{41}\Phi_4^2\Phi_7\Phi_8\Phi_9\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi''_{70,45}$	$\frac{-\zeta_3}{3}x^{41}\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi''''_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	1
$\phi'''_{70,45}$	$\frac{-\zeta_3^2}{3}x^{41}\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi'''_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	1
* $\phi_{126,41}$	$\frac{1}{3}x^{41}\Phi_3^3\Phi_6^3\Phi_7\Phi_8\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3] : \phi_{3,20}$	$\frac{\zeta_3}{3}x^{41}\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi''''_{15}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	ζ_3
$G_{3,3,3}[\zeta_3^2] : \phi_{3,17}$	$\frac{\zeta_3^2}{3}x^{41}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi'''_{15}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	ζ_3^2
# $\phi_{126,43}$	$\frac{1}{3}x^{41}\Phi_3^3\Phi_6^3\Phi_7\Phi_8\Phi_{12}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3^2] : \phi''_{3,13}$	$\frac{-\zeta_3}{3}x^{41}\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi''''_{15}\Phi_{21}\Phi_{24}\Phi'''_{30}\Phi_{42}$	ζ_3^2
$G_{3,3,3}[\zeta_3] : \phi'_{3,16}$	$\frac{-\zeta_3^2}{3}x^{41}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi'''_{15}\Phi_{21}\Phi_{24}\Phi''''_{30}\Phi_{42}$	ζ_3
* $\phi_{105,46}$	$\frac{-\zeta_3}{6}x^{46}{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi''_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{15}\Phi''_{18}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{120,49}$	$\frac{3+\sqrt{-3}}{12}x^{46}\Phi_2^4{\Phi'_3}^3\Phi_5\Phi_6^4\Phi''_6\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi'_{21}\Phi''_{24}\Phi_{30}\Phi_{42}$	1
$D_4 : ..11...$	$\frac{-3-\sqrt{-3}}{12}x^{46}\Phi_1^4\Phi_3^4\Phi'_3\Phi_5{\Phi''_6}^3\Phi_7\Phi_9\Phi_{10}\Phi'''_{12}\Phi_{15}\Phi_{21}\Phi''_{24}\Phi_{30}\Phi''_{42}$	-1
$G_{3,3,3}[\zeta_3^2] : \phi''_{3,16}$	$\frac{-\zeta_3}{3}x^{46}\Phi_1^3\Phi_2^3{\Phi''_3}^3\Phi_4\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi'_{42}$	ζ_3^2
$\phi_{15,58}$	$\frac{\zeta_3}{6}x^{46}{\Phi'_3}^3\Phi_5{\Phi''_6}^3\Phi_8\Phi'_9\Phi_{10}{\Phi'''_{12}}^2\Phi_{14}\Phi_{15}\Phi'_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi''_{42}$	1
$\phi_{15,56}$	$\frac{\zeta_3^2}{6}x^{46}{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_8\Phi''_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{14}\Phi_{15}\Phi''_{18}\Phi_{21}\Phi_{24}\Phi_{30}\Phi'_{42}$	1
$\phi_{120,47}$	$\frac{3-\sqrt{-3}}{12}x^{46}\Phi_2^4{\Phi''_3}^3\Phi_5\Phi_6^4\Phi'_6\Phi_{10}\Phi''''_{12}\Phi_{14}\Phi_{15}\Phi_{18}\Phi''_{21}\Phi'_{24}\Phi_{30}\Phi_{42}$	1
$D_4 : ....11.$	$\frac{-3+\sqrt{-3}}{12}x^{46}\Phi_1^4\Phi_3^4\Phi''_3\Phi_5{\Phi'_6}^3\Phi_7\Phi_9\Phi_{10}\Phi''''_{12}\Phi_{15}\Phi_{21}\Phi'_{24}\Phi_{30}\Phi'_{42}$	-1
$G_{3,3,3}[\zeta_3^2] : \phi_{3,20}$	$\frac{\zeta_3^2}{3}x^{46}\Phi_1^3\Phi_2^3{\Phi'_3}^3\Phi_4\Phi_5{\Phi''_6}^3\Phi_7\Phi_8\Phi_{10}\Phi'''_{12}\Phi_{14}\Phi_{15}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi''_{42}$	ζ_3^2
# $\phi_{105,50}$	$\frac{-\zeta_3^2}{6}x^{46}{\Phi''_3}^3\Phi_5{\Phi'_6}^3\Phi_7\Phi_8\Phi'_9\Phi_{10}{\Phi''''_{12}}^2\Phi_{15}\Phi'_{18}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3] : \phi_{1,21}$	$\frac{1}{6}x^{46}\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi''_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi''_{18}\Phi''_{21}\Phi_{24}\Phi_{30}\Phi''_{42}$	ζ_3
$G_{3,3,3}[\zeta_3] : \phi_{2,24}$	$\frac{\sqrt{-3}}{6}x^{46}\Phi_1^3\Phi_2^5\Phi_4\Phi_5\Phi_6^4\Phi_7\Phi_8\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{30}\Phi_{42}$	ζ_3
$G_{33}[-\zeta_3] : -1$	$\frac{-\sqrt{-3}}{6}x^{46}\Phi_1^5\Phi_2^3\Phi_3^4\Phi_4\Phi_5\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{30}$	$-\zeta_3$
$\phi_{90,48}$	$\frac{1}{3}x^{46}\Phi_3^3\Phi_5\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{15}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3] : \phi_{1,33}$	$\frac{-1}{6}x^{46}\Phi_1^3\Phi_2^3\Phi_4^2\Phi_5\Phi_7\Phi_8\Phi'_9\Phi_{10}\Phi_{14}\Phi_{15}\Phi'_{18}\Phi'_{21}\Phi_{24}\Phi_{30}\Phi'_{42}$	ζ_3
* $\phi_{56,57}$	$\frac{1}{2}x^{57}\Phi_2^4\Phi_6^4\Phi_7\Phi_{10}\Phi_{14}\Phi_{18}\Phi_{21}\Phi_{30}\Phi_{42}$	1
$\phi_{21,60}$	$\frac{1}{2}x^{57}\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{14}\Phi_{21}\Phi_{24}\Phi_{30}\Phi_{42}$	1
$\phi_{35,60}$	$\frac{1}{2}x^{57}\Phi_5\Phi_7\Phi_8\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{21}\Phi_{24}\Phi_{42}$	1
$D_4 : ...11..$	$\frac{1}{2}x^{57}\Phi_1^4\Phi_3^4\Phi_5\Phi_7\Phi_9\Phi_{14}\Phi_{15}\Phi_{21}\Phi_{42}$	-1
* $\phi_{21,68}$	$\frac{3-\sqrt{-3}}{6}x^{68}{\Phi'_3}^3{\Phi''_6}^3\Phi_7\Phi'''_{12}\Phi_{14}\Phi''''_{15}\Phi_{21}\Phi''_{24}\Phi'''_{30}\Phi_{42}$	1
# $\phi_{21,70}$	$\frac{3+\sqrt{-3}}{6}x^{68}{\Phi''_3}^3{\Phi'_6}^3\Phi_7\Phi''''_{12}\Phi_{14}\Phi'''_{15}\Phi_{21}\Phi'_{24}\Phi''''_{30}\Phi_{42}$	1
$G_{3,3,3}[\zeta_3^2] : \phi_{1,24}$	$\frac{-\sqrt{-3}}{3}x^{68}\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{14}\Phi_{21}\Phi_{42}$	ζ_3^2
* $\phi_{6,85}$	$\frac{3-\sqrt{-3}}{6}x^{85}{\Phi''_3}^3{\Phi'_6}^3\Phi_8\Phi''''_{12}\Phi'''_{15}\Phi''_{21}\Phi_{24}\Phi''''_{30}\Phi'_{42}$	1
# $\phi_{6,89}$	$\frac{3+\sqrt{-3}}{6}x^{85}{\Phi'_3}^3{\Phi''_6}^3\Phi_8\Phi'''_{12}\Phi''''_{15}\Phi'_{21}\Phi_{24}\Phi'''_{30}\Phi''_{42}$	1
$G_{3,3,3}[\zeta_3^2] : \phi_{1,33}$	$\frac{-\sqrt{-3}}{3}x^{85}\Phi_1^3\Phi_2^3\Phi_4\Phi_5\Phi_7\Phi_8\Phi_{10}\Phi_{14}\Phi_{24}$	ζ_3^2
* $\phi_{1,126}$	x^{126}	1

APPENDIX B

ERRATA FOR [BMM99].

- Proof of 1.17: this forgets the case of $G(2e,e,2)$ with 3 classes of hyperplanes. This case is still open.
- Page 184, generalized sign: let $\Delta_{\mathbb{G}}$ be the eigenvalue of ϕ on the discriminant Δ. Then change the definition of $\varepsilon_{\mathbb{G}}$ to $\varepsilon_{\mathbb{G}} = (-1)^r \zeta_1 \ldots \zeta_r \Delta_{\mathbb{G}}^*$ where $r = \dim V$.

  Most of the subsequent errata come from this, and are superceded by results in the current paper, see in particular 1.3.1.
- 3 lines below: $\{\zeta_1, \ldots, \zeta_r\}$ is the spectrum of $w\phi$ (in its action on V'), and $\Delta_{\mathbb{G}} = 1$. In particular we have then $\varepsilon_{\mathbb{G}} = (-1)^r \det_V(w\phi)$. In general, if $w\phi$ is ζ-regular then the spectrum of $w\phi$ is $\{\zeta_i \zeta^{-d_i+1}\}$ so $\det_V(w\phi) = \zeta_1 \ldots \zeta_r \zeta^{-N^\vee}$; and $\varepsilon_{\mathbb{G}} = (-1)^r \zeta^{-N} \det_V(w\phi)$.
- Second line of 3.3: "Moreover, if there exists $v\phi \in W\phi$ such that $v\phi$ admits a fixed point in $V - \bigcup_{H \in \mathcal{A}} H$, then $\varepsilon_{\mathbb{T}} = \varepsilon_{\mathbb{G}} \det_V(vw^{-1})^*$."
- 3.5: $|\mathbb{G}| = \varepsilon_{\mathbb{G}} x^N \prod_i (1 - \zeta_i^* x^{d_j}) = x^N \Delta_{\mathbb{G}}^* \prod_i (x^{d_i} - \zeta_i)$.
- 3.6 (i): $|\mathbb{G}| = x^N \Delta_{\mathbb{G}}^* \prod_\Phi \Phi^{a(\Phi)}$.
- 3.6 (ii): $|\mathbb{G}|(1/x) = \Delta_{\mathbb{G}}^* \varepsilon_{\mathbb{G}} x^{-(2N+N^\vee+r)} |\mathbb{G}|(x)^*$.
- 3.7 (ls.2): $\phi^{(a)}$ is the product of $(1, \ldots, 1, \phi)$ acting on $V \times \ldots \times V$ by the a-cycle which permutes cyclically the factors V of $V^{(a)}$. The ζ_i for $\mathbb{G}^{(a)}$ are $\{\zeta_a^j \zeta_i^{1/a}\}_{j=0..a-1, i=1..r}$ and $\Delta_{\mathbb{G}^{(a)}} = \Delta_{\mathbb{G}}$ so $|\mathbb{G}^{(a)}|(x) = |\mathbb{G}|(x^a)$.
- 3.8: The ζ_i for $\mathbb{G}^\zeta$ are $\zeta^{d_i} \zeta_i$ and $\Delta_{\mathbb{G}^\zeta} = \Delta_{\mathbb{G}} \zeta^{N+N^\vee}$ and thus we get $|\mathbb{G}^\zeta|(x) = \zeta^r |\mathbb{G}|(\zeta^{-1} x)$.
- 4.9: $\mathrm{Deg}(R_{w\phi}^{\mathbb{G}}) = \mathrm{tr}_{R\mathbb{G}}(w\phi)^*$.
- 4.25, second equality: $\mathrm{Deg}(\alpha \det_{V^*}) = \Delta_{\mathbb{G}} (-1)^r \varepsilon_{\mathbb{G}}^* x^{N^\vee} \mathrm{Deg}_{\mathbb{G}}(\alpha^*)(1/x)^*$.
- 4.26, first equality: $\mathrm{Deg}(\det_V^*) = (-1)^r \Delta_{\mathbb{G}} \varepsilon_{\mathbb{G}}^* x^{N^\vee}$.
- bottom of page 198: suppress the first $|\mathbb{G}| S_{\mathbb{G}}(\alpha)$.
- last equality in proof of 5.3: suppress TG.

– 5.4: In particular we have:

$$\varepsilon_{\mathbb{G}} x^{N(\mathbb{G})} \equiv \varepsilon_{\mathbb{L}} x^{N(\mathbb{L})} \pmod{\Phi},$$
$$\Delta_{\mathbb{G}} \varepsilon^*_{\mathbb{G}} x^{N^\vee(\mathbb{G})} \equiv \Delta_{\mathbb{L}} \varepsilon^*_{\mathbb{L}} x^{N^\vee(\mathbb{L})} \pmod{\Phi},$$
$$\Delta_{\mathbb{G}} x^{N(\mathbb{G})+N^\vee(\mathbb{G})} \equiv \Delta_{\mathbb{L}} x^{N(\mathbb{L})+N^\vee(\mathbb{L})} \pmod{\Phi}$$

– 6.1(b): the polynomial in t

$$\prod_{j=0}^{j=e_{\mathcal{C}}-1} (t - \zeta_{e_{\mathcal{C}}}^j y^{n_{\mathcal{C},j}|\mu(K)|})$$

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