THE FINITE GROUPS WITH NO REAL p-ELEMENTS

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ABSTRACT. Given a prime p, we investigate the finite groups with no nontrivial real p-elements. Under certain natural hypotheses, we show that these groups have abelian Sylow p-subgroups.

1. Introduction

Reality questions are at the heart of finite group theory. In a finite group G, an element $x \in G$ is **real** if it is G-conjugate to its inverse x^{-1}. Our aim in this paper is to fix a prime p and investigate the finite groups having no nontrivial real p-elements.

Groups of odd order have no nontrivial real elements (in fact, the real elements of a finite group G are the real elements of $\mathbf{O}^{2'}(G)$), so some natural restrictions are necessary to study this problem. Since G has no real elements of order 2 if and only if G has odd order, we shall also assume that p is odd. Our main result is the following:

Theorem A. *Let G be a finite group with $\mathbf{O}^{2'}(G) = G$. Suppose that p is an odd prime dividing $|G|$ and such that G has no real elements of order p. Then $H = \mathbf{O}^{p'}(G/\mathbf{O}_{p'}(G))$ is a direct product of simple groups $H = S_1 \times S_2 \times \cdots \times S_m$, where S_i is normal in $G/\mathbf{O}_{p'}(G)$, and S_i is one of the groups listed in Theorem 2.1, for all $1 \le i \le m$.*

In particular, we get the following:

Corollary B. *Let G be a finite group with $\mathbf{O}^{2'}(G) = G$. Suppose that p is an odd prime dividing $|G|$. If G has no real elements of order p, then G is a non-p-solvable group with abelian Sylow p-subgroups.*

We have not attempted in this paper to give a complete classification of the finite groups with no nontrivial real p-elements. However, with no further work, we can write down now the cases for $p = 3$ and $p = 5$.

Theorem C. *Let G be a finite group of order divisible by 3, with $\mathbf{O}^{2'}(G) = G$. Then G has no real elements of order 3 if and only if*

$$G/\mathbf{O}_{3'}(G) \cong \mathrm{L}_2(3^{2f_1+1}) \times \cdots \times \mathrm{L}_2(3^{2f_k+1})$$

for some positive integers $f_1, \ldots, f_k$.

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Theorem D. *Let G be a finite group of order divisible by 5, with $\mathbf{O}^{2'}(G) = G$. Then G has real elements of order* 5.

Finally, we should mention that the conclusion of Corollary B is not true if we replace real elements by rational elements:

Example 1.1. Indeed, let S be any simple group of order divisible by $p \geq 5$, all of whose elements of order p have the property that their non-trivial powers fall into at least three distinct conjugacy classes. Set $H = S \wr D_p$, the wreath product of S with the dihedral group of order $2p$ in its permutation representation on p points. Let $G = \mathbf{O}^{2'}(H)$; note that $G = H$ when S is non-abelian and that G has index p in H when $S = C_p$. Then G has non-abelian Sylow p-subgroups, but no non-trivial rational p-elements.

Taking $S = C_5$ and $p = 5$, we get an example of order $2 \cdot 5^5$. Factoring out a normal subgroup of order 5^2 we get a smaller example of order $2 \cdot 5^3$; taking $S = \mathrm{L}_2(13)$ and $p = 7$ we get a non-solvable example.

2. Simple Groups

This section is devoted to the proof of a result on simple groups, which is a necessary ingredient for our main statement, using the classification. Here, for a prime p and a positive integer q prime to p, $e_p(q)$ denotes the order of q modulo p.

Theorem 2.1. *Let p be an odd prime, S a finite nonabelian simple group of order divisible by p but not containing any real element of order p. Then one of the following occurs:*

(1) $S = \mathfrak{A}_p$ *or* $\mathfrak{A}_{p+1}$ *with* $7 \leq p \equiv 3 \pmod 4$,
(2) $S = \mathrm{L}_2(q)$, $q = p^f \equiv 3 \pmod 4$,
(3) $S = \mathrm{L}_n(q)$, $e_p(q) = i$ *for some odd* $i > n/2$,
(4) $S = \mathrm{U}_n(q)$, $e_p(q) = 2i$ *for some odd* $i > n/2$,
(5) $S = \mathrm{O}_{2n}^+(q)$, $n \geq 5$ *odd*, $e_p(q) = n$,
(6) $S = \mathrm{O}_{2n}^-(q)$, $n \geq 5$ *odd*, $e_p(q) = 2n$,
(7) $S = E_6(q)$, $e_p(q) \in \{5, 9\}$,
(8) $S = {}^2E_6(q)$, $e_p(q) \in \{10, 18\}$, *or*
(9) S *is sporadic, with p as in Table 1.*

In particular, S has abelian Sylow p-subgroups, which are even cyclic in all cases but (2) above. Moreover, p neither divides the order of $\mathbf{O}^{2'}(\mathrm{Out}(S))$ nor of the Schur multiplier of S.

Proof. For the sporadic groups and ${}^2F_4(2)'$, the claim is easily checked from the Atlas [1]. Next assume that $S = \mathfrak{A}_n$ is an alternating group. Since $\mathfrak{S}_{n-2} \leq \mathfrak{A}_n$ and all elements of symmetric groups are rational, S contains real p-elements for all $p \leq n-2$. A p-cycle is inverted by a product of $(p-1)/2$ transpositions, which is even when $p \equiv 1 \pmod 4$. On the other hand, for $p = n-1$ or $p = n$ with $p \equiv 3 \pmod 4$ the normalizer of a p-cycle in $\mathfrak{A}_n$ has odd order, so p-elements are non-real.

If S is of Lie type, then there is a simple algebraic group $\mathbf{G}$ of simply connected type with a Steinberg endomorphism $F : \mathbf{G} \to \mathbf{G}$ such that $S \cong G/Z(G)$, where $G = \mathbf{G}^F$ denotes the group of fixed points. Clearly, if a group has a real element of order p, then

Table 1. Sporadic groups without real p-elements

S	p	S	p	S	p
M_{11}	11	McL	$7, 11$	Th	31
M_{12}	11	ON	31	Fi_{23}	23
M_{22}	$7, 11$	Co_3	$11, 23$	Co_1	23
M_{23}	$7, 11, 23$	Co_2	23	J_4	7
HS	11	Fi_{22}	11	Fi'_{24}	23
J_3	19	HN	19	B	$23, 31, 47$
M_{24}	$7, 23$	Ly	11	M	$23, 31, 47, 59, 71$

so has every factor group by a central subgroup. It hence suffices to consider our question for the perfect central extension G of S.

First assume that p is the defining prime for $\mathbf{G}$ and $q = p^f$. The normalizer of a Sylow p-subgroup P of $\mathrm{L}_2(q)$ is the extension of the additive group of $\mathbb{F}_q$ by the subgroup of $\mathbb{F}_q^\times$ of order $(q-1)/2$, so $\mathrm{L}_2(q)$ contains real elements of order p if and only if $q \equiv 1 \pmod 4$. (Note for this that $\mathbf{N}_G(P)$ controls fusion in the abelian Sylow subgroup P.) The groups $\mathrm{SL}_3(q)$, $\mathrm{SU}_3(q)$ and $\mathrm{Sp}_4(q)$ contain a Levi subgroup $\mathrm{GL}_2(q)$ or $\mathrm{GU}_2(q)$ and hence a real element of order p, since all non-trivial p-elements in the latter groups are conjugate. Now any group of simply connected type B_n or C_n $(n \geq 3)$, ${}^{(2)}A_n$ or ${}^{(2)}D_n$ $(n \geq 4)$, G_2, F_4, ${}^{(2)}E_6$, E_7 or E_8 over $\mathbb{F}_q$ contains at least one of $\mathrm{SL}_3(q)$, $\mathrm{SU}_3(q)$ or $\mathrm{Sp}_4(q)$ as a Levi subgroup of a suitable parabolic subgroup (use [4, Prop. 12.14]), hence a real p-element by what we just saw. Also, groups of type ${}^3D_4$ contain $G_2(q)$ as the centralizer of the graph-field automorphism of order 3. Since $\mathrm{L}_2(8) = {}^2G_2(3)'$ has a real 3-element, so does ${}^2G_2(3^{2f+1})$ for all $f \geq 1$. The Suzuki and Ree groups in characteristic 2 do not arise here as p was assumed to be odd.

It remains to consider non-defining primes p for simply connected groups G of Lie type. Thus, p-elements are semisimple. Any semisimple element of G lies in an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$, and $\mathbf{N}_{\mathbf{G}^F}(\mathbf{T})/\mathbf{T}^F$ is isomorphic to the centralizer of an element in the Weyl group W^F of G (see [4, Prop. 25.3(a)]). The irreducible Weyl groups not of type A_n $(n \geq 2)$, D_{2n+1} $(n \geq 2)$ and E_6 contain $-\mathrm{id}$ in their natural reflection representation ([4, Cor. B.23]), which acts by inversion on any torus $\mathbf{T}^F$ of $G = \mathbf{G}^F$, so every semisimple element in a twisted or untwisted group of that type is real.

For the remaining cases, first assume that $G = \mathrm{SL}_n(q)$ with $n \geq 3$, and let $s \in G$ be semisimple of order p. As $|G| = q^{n(n-1)/2} \prod_{i=2}^n (q^i - 1)$ we have that $p|(q^i-1)$ for some $i \leq n$. Let $i = e_p(q)$, the minimal i with this property. If i is even, resp. $i \leq n/2$, then the symplectic subgroup $\mathrm{Sp}_i(q) \leq \mathrm{SL}_n(q)$ resp. $\mathrm{Sp}_{2i}(q) \leq \mathrm{SL}_n(q)$ contains a real element of order p by our previous observation. On the other hand, if $i > n/2$ is odd then s lies in a subgroup $\mathrm{GL}_1(q^i) \cap \mathrm{SL}_n(q)$, and its eigenvalues under this embedding are of the form $\alpha, \alpha^q, \ldots, \alpha^{q^{i-1}}$ for some $\alpha \in \mathbb{F}_{q^i}$ of the same order as s. Now assume that $\alpha^{q^j} = \alpha^{-1}$ for some $j \leq i-1$. Then $\alpha^{q^j+1} = 1$, that is, the order of s divides $q^j + 1$, so $\gcd(q^{2j}-1, q^i-1)$, with $j < i$, in contradiction to the definition of i. Thus, the Jordan normal forms of s and s^{-1} are different, whence s cannot be real. For $G = \mathrm{SU}_n(q)$ we have $|G| = q^{n(n-1)/2}\prod_{i=2}^n(q^i-(-1)^i)$. Let i be minimal with $p|(q^i-(-1)^i)$. As before, if $i \le n/2$ or $i \le n$ is even, then a suitable symplectic subgroup of G contains a real p-element. This leaves the primes p dividing q^m+1 with $2m = e_p(q)$ for some odd $m > n/2$. Embedding $\mathrm{SU}_n(q)$ into $\mathrm{SL}_n(q^2)$ we see that they cannot be real by what we just showed for the latter group.

For $G = \mathrm{Spin}_{2n}^+(q)$, $n \ge 5$ odd, the order formulas (see e.g., [4, Table 24.1]) show that either p divides the order of the subgroup $\mathrm{Spin}_{2n-1}(q)$ of type B_{n-1}, or $e_p(q) = n$. Elements of the latter order lie in a unique, cyclic maximal torus, of order q^n-1, of index n in its normalizer, so they are non-real. The same argument applies to the twisted groups $\mathrm{Spin}_{2n}^-(q)$, now for primes p with with $e_p(q) = 2n$. Finally, for $G = E_6(q)$, the primes p not dividing the order of its subgroup $F_4(q)$ are those with $e_p(q) \in \{5, 9\}$ ([4, Table 24.1]), and in both cases, the automizer, i.e., the centralizer of a corresponding element of order 5 or 9 in the Weyl group $W(E_6)$, has odd order. Similarly, for ${}^2E_6(q)$ these are the primes p with $e_p(q) \in \{10, 18\}$. This completes the investigation of simple groups without real p-elements.

It is well known that the Sylow p-subgroup of $\mathfrak{A}_p$, $\mathfrak{A}_{p+1}$ and of $\mathrm{L}_2(p^f)$ are abelian. In all the other cases, the Sylow p-subgroups are abelian, and in fact cyclic, by [4, Thm. 25.14]. The Schur multipliers of sporadic groups and alternating groups are $\{2, 3\}$-groups, hence prime to p. The order of the Schur multiplier of $\mathrm{L}_n(q)$ is generically $\gcd(n, q-1)$, hence not divisible by primes p with $e_p(q) > 1$; the Schur multiplier of $\mathrm{U}_n(q)$ has generic order $\gcd(n, q+1)$, hence not divisible by primes p with $e_p(q) > 2$ (see e.g. [4, Table 24.2]). The Schur multipliers of orthogonal groups are 2-groups, and those of ${}^{(2)}E_6(q)$ have order dividing 3. The claim is easily seen to also hold for the finitely many exceptional multipliers (see [4, Table 24.3]).

To prove the claim on $\mathbf{O}^{2'}(\mathrm{Out}(S))$ note that for alternating and sporadic groups, $\mathrm{Out}(S)$ is a 2-group. Furthermore, the group of diagonal automorphisms in cases (2)–(8) has order divisible only by prime divisors of $2(q^2-1)$, hence in particular prime to p. But $\mathrm{Out}(S)$ modulo the group of inner-diagonal automorphisms is abelian (see [2, §2.5] for these statements). □

We now write $\mathcal{L}_p$ for the set of simple groups in Theorem 2.1(1)–(9) giving an exception for the prime p.

3. Proof of theorems A and B

We shall use several elementary properties of real elements. If r is a prime and $x \in G$, then x_r and $x_{r'}$ denote, respectively, the r-part and the r'-part of x.

Lemma 3.1. *Let G be a finite group.*

(a) *If $x \in G$ is real, then there is a 2-element $y \in G$ such that $x^y = x^{-1}$.*
(b) *If $x \in G$ is real, then x^m is real for every integer m.*
(c) *Suppose that $N \lhd G$ and that $Nx \in G/N$ is real. If $o(Nx)$ is odd in the group G/N, then there exists a real $y \in G$ such that $Nx = Ny$.*
(d) *If a 2-group Q acts nontrivially on G, then there is $1 \ne x \in G$ and $q \in Q$ such that $x^q = x^{-1}$.*

Proof. (a) If $x^y = x^{-1}$, then y^2 centralizes x. Since y normalizes x, it follows that $y_{2'}$ centralizes x, and therefore $x^{y_2} = x^{-1}$. To prove (b), notice that if $x^y = x^{-1}$, then $(x^m)^y = (x^m)^{-1}$.

Part (c) is Lemma (3.2) of [5].

In order to prove (d), now let $q\mathbf{C}_Q(G)$ be of order 2 in the group $G/\mathbf{C}_Q(G)$, and let $g \in G$ be with $g^q \neq g$. Then set $x = g^{-1}g^q$. □

Notice that by Lemma 3.1(b), a finite group G of order divisible by p has no real elements of order p if and only if it has no nontrivial real p-elements.

Remark 3.2. If the group G has no real elements of order p, p odd, then the same is true for every subgroup and every factor group of G (hence for every section of G). In fact, any real element of $H \leq G$ is clearly also a real element of G. If N is a normal subgroup of G and Nx is a real element of order p of G/N, then by Lemma 3.1(c) there exists a real element y of G such that $Ny = Nx$. Then p divides $o(y)$, and by Lemma 3.1(b) it follows that a suitable power of y is a real element of order p of G.

Now, we are going to prove Theorem A, which we state (in a slightly different form) as the following Theorem 3.3.

Theorem 3.3. *Suppose that p is an odd prime, and let G be a finite group with $\mathbf{O}^{2'}(G) = G$ and $\mathbf{O}_{p'}(G) = 1$. Suppose that p divides $|G|$ (or, equivalently, that $G \neq 1$). If G has no real elements of order p, then $\mathbf{O}^{p'}(G)$ is a direct product of simple groups $S \in \mathcal{L}_p$, S normal in G.*

Proof. We argue by induction on $|G|$. Observe that $|G|$ is even, as $\mathbf{O}^{2'}(G) = G$ and $G \neq 1$.

Let $M = \mathbf{O}_p(G)$, and let Q be a Sylow 2-subgroup of G. If Q does not centralize M, then G contains real elements of order p by Lemma 3.1(d). Thus $Q \subseteq \mathbf{C}_G(M)$, and then $G/\mathbf{C}_G(M)$ has odd order. Then $\mathbf{C}_G(M) = G$. We will show that $M = 1$.

Let N/M be a minimal normal subgroup of G/M. If N/M is a p'-group, then by Schur–Zassenhaus theorem we have that $N = M \times N_0$, for a p'-subgroup N_0 of N, and hence $\mathbf{O}_{p'}(G) > 1$, a contradiction. Therefore, $\mathbf{O}_{p'}(G/M) = 1$.

By Remark 3.2, G/M has no real elements of order p. Also, we have that p divides $|G/M|$ and that $\mathbf{O}^{2'}(G/M) = G/M$. Let $K = \mathbf{O}^{p'}(G)$; notice that $M \leq K$ and that $K/M = \mathbf{O}^{p'}(G/M)$. If $M > 1$, then induction yields that K/M is a direct product of simple groups $T/M \in \mathcal{L}_p$, $T/M \lhd G/M$. So, the Schur multiplier D of K/M is the direct product of the Schur multipliers of its direct factors T/M (see [3, Satz V.25.10]). As $T/M \in \mathcal{L}_p$, p does not divide the order of D by Theorem 2.1. Let $L = K'$ be the commutator subgroup of K. Since K/M is perfect, we see that $K = LM$. Write $Z = L \cap M$. We observe that L is a perfect group. In fact, since $K = LM$ and M is central in K, we have $L = [K, K] = [LM, LM] = [L, L] = L'$. Hence, $M \cap L \leq \mathbf{Z}(L) \cap L'$ and we conclude that $M \cap L$ is isomorphic to a quotient of the Schur multiplier D of $L/M \cap L \cong K/M$. This forces $M \cap L = 1$, as p does not divide $|D|$. Therefore, $K = L \times M$ and, in particular, $L \lhd G$. As ML/L is central in G/L, by Schur–Zassenhaus ML/L is a direct factor of G/L. Then, $\mathbf{O}^{2'}(G) = G$ implies $ML/L = 1$, so $M = 1$, a contradiction.

So, we have proved that $\mathbf{O}_p(G) = 1 = \mathbf{O}_{p'}(G)$. Hence, G has no nontrivial abelian normal subgroups and the generalized Fitting subgroup R of G is the direct product of all the (nonsolvable) minimal normal subgroups of G.

Let N be a minimal normal subgroup of G. In order to finish the proof, it is enough to show that N is a simple group, $N \in \mathcal{L}_p$ and that $G/N\mathbf{C}_G(N)$ is a p'-group. In fact, G embeds into the direct product

$$\hat{G} = \prod_{N \lhd_{\min} G} G/\mathbf{C}_G(N)$$

because $\bigcap_{N \lhd_{\min} G} \mathbf{C}_G(N) = \mathbf{C}_G(R) = \mathbf{Z}(R) = 1$ (recall that $\mathbf{C}_G(R) \leq R$). If $G/N\mathbf{C}_G(N)$ is a p'-group for every minimal normal subgroup N of G, then the image of R in the above mentioned embedding is a subgroup of p'-power index in $\hat{G}$. It follows that $R = \mathbf{O}^{p'}(G)$, and we are done.

Assume $\mathbf{C}_G(N) > 1$; write $\overline{G} = G/\mathbf{C}_G(N)$ and use the "bar convention". Note that $\overline{N}$ is the only minimal normal subgroup of $\overline{G}$. So, in particular, $\mathbf{O}_{p'}(\overline{G}) = 1$. Clearly, $\mathbf{O}^{2'}(\overline{G}) = \overline{G}$ and p divides the order of $\overline{G}$. By Remark 3.2, $\overline{G}$ has no real element of order p, so by induction we have that $N \cong \overline{N}$ is a simple group, $\overline{N} \in \mathcal{L}_p$ and that $\overline{N} = \mathbf{O}^{p'}(\overline{G})$.

Hence, we can assume that $\mathbf{C}_G(N) = 1$, so N is the only minimal normal subgroup of G. Write $N = S_1 \times S_2 \times \cdots \times S_m$, where the S_j's are isomorphic nonabelian simple groups transitively permuted by G. Let $B = \bigcap_j \mathbf{N}_G(S_j)$; so G/B is a permutation group on the set $\Omega = \{S_1, \ldots, S_m\}$.

If P denotes a Sylow p-subgroup of B then $\mathbf{N}_G(P)B = G$ by the Frattini argument. Let Q be a Sylow 2-subgroup of $N_G(P)$. By our assumption and Lemma 3.1(d), Q centralizes P, so in its action on $\{S_1, \ldots, S_m\}$ it fixes all factors. It follows that $Q \leq B$, so G/B is of odd order, whence $G = B$.

Let $S := S_1$. Then, by the above, S is normal in G and hence $S = N$. So G is an almost simple group with socle S. By Remark 3.2 and Theorem 2.1, $S \in \mathcal{L}_p$. Now, since G/S is isomorphic to a subgroup of $\mathrm{Out}(S)$ and $\mathbf{O}^{2'}(G/S) = G/S$, we deduce that G/S is isomorphic to a subgroup of $\mathbf{O}^{2'}(\mathrm{Out}(S))$. By Theorem 2.1, we conclude that G/S is a p'-group, and this finishes the proof of the theorem. □

We can give a characterization of the groups G with no real elements of order p, $p \neq 2$, in terms of the structure of the section $\mathbf{O}^{2'}(G/\mathbf{O}_{p'}(G))$. Recall that G has real elements of order p if and only if $\mathbf{O}^{2'}(G/\mathbf{O}_{p'}(G))$ does.

Corollary 3.4. *Let p be an odd prime and let G be a finite group such that $\mathbf{O}^{2'}(G) = G$ and p divides $|G|$. Then G has no real elements of order p if and only if $G/\mathbf{O}_{p'}(G)$ is isomorphic to a subgroup of a direct product of almost simple groups of order divisible by p and with no real elements of order p.*

Proof. Write $H = G/\mathbf{O}_{p'}(G)$. Since p divides $|G|$ and G has no real elements of order p, then the same is true for H. So by Theorem 3.3, $K = \mathbf{O}^{p'}(H) = S_1 \times S_2 \times \cdots \times S_n$ for suitable simple groups $S_i \in \mathcal{L}_p$, $S_i \lhd H$. Note that $\bigcap_i \mathbf{C}_H(S_i) = 1$, because it intersects trivially K and $\mathbf{O}_{p'}(H) = 1$. Hence, H is isomorphic to a subgroup of the direct product of the almost simple groups $T_i = H/\mathbf{C}_H(S_i)$. By Remark 3.2, T_i has no real element of order p and, clearly, p divides $|T_i|$.

Conversely, assume that H is a subgroup of a direct product L of groups having no real elements of order p. Note that L has no real elements of order p, so the same is true for H and hence for G. □

By Corollary 3.4, a complete classification of the finite groups with no real elements of order p is reduced to the classification of the almost simple groups with this property.

We are now ready to prove Corollary B and Theorems C and D.

Proof of Corollary B. Let G be a finite group and let p be an odd prime. Assume that p divides $|G|$ and that G has no real elements of order p and no nontrivial factor group of odd order. Then, the same is true for $H = G/\mathbf{O}_{p'}(G)$ (see Remark 3.2). By Theorem 3.3, $\mathbf{O}^{p'}(H)$ is a direct product of simple groups $S_i \in \mathcal{L}_p$. So, G is a non-p-solvable group. Finally, a Sylow p-subgroup P of G is isomorphic to a Sylow p-subgroup of $\mathbf{O}^{p'}(H)$. Hence P is abelian, because by Theorem 2.1 the groups S_i have abelian Sylow p-subgroups. □

Proof of Theorems C and D. By Theorem 2.1, it follows that S is a simple group of order divisible by 3 with no real elements of order 3 (i.e., $S \in \mathcal{L}_3$) if and only if $S = \mathrm{L}_2(3^f)$ with f odd, $f > 1$. So, one implication of Theorem C is clear.

Assume now that G is a finite group of order divisible by 3 such that $\mathbf{O}^{2'}(G) = G$ and G has no real elements of order 3. Write $H = G/\mathbf{O}_{p'}(G)$ and $K = \mathbf{O}^{p'}(H)$. By Theorem 3.3, $K = S_1 \times S_2 \times \cdots \times S_k$ with $S_i = \mathrm{L}_2(3^{f_i})$, f_i odd, $f_i > 1$ and $S_i \lhd H$. As in the proof of Corollary 3.4, we have that $H \leq T_1 \times T_2 \times \cdots \times T_k$, with $T_i \cong H/\mathbf{C}_H(S_i)$ almost simple groups, $S_i \leq T_i \leq \mathrm{Aut}(S_i)$. Since T_i is a section of G, then T_i has no real elements of order 3. If $|T_i/S_i|$ is even, then by the Frattini argument there is a 2-element $g \in T_i$, $g \notin S_i$, such that g normalizes a Sylow 3-subgroup P of S_i. Thus, Lemma 3.1(d) yields that g centralizes P; a contradiction, since it can easily be seen that no outer automorphism of $\mathrm{L}_2(3^{f_i})$ centralizes P. As $\mathbf{O}^{2'}(T_i) = T_i$, it follows that $T_i = S_i$ for all i, and hence we conclude that $H = K = S_1 \times S_2 \times \cdots \times S_k$.

For $p = 5$ no "exceptional" examples arise. In fact, Theorem 2.1 yields that $\mathcal{L}_5 = \emptyset$. Thus, Theorem D is implied by Theorem A. □

Finally, we remark that for $p = 7$ there are quite a few almost simple groups T with no real elements of order 7, such that 7 divides $|T|$ and $\mathbf{O}^{2'}(T) = T$. We mention, among others, $T = \mathrm{L}_3(4) : 2_2$ and $T = \mathrm{L}_3(4) : S_3$.

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