Projective modules in the intersection cohomology of Deligne–Lusztig varieties

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Abstract

We formulate a strong positivity conjecture on characters afforded by the Alvis-Curtis dual of the intersection cohomology of Deligne–Lusztig varieties. This conjecture provides a powerful tool to determine decomposition numbers of unipotent ℓ-blocks of finite reductive groups. *To cite this article: O. Dudas, G. Malle, C. R. Acad. Sci. Paris, Ser. I ??? (???).*

Résumé

Modules projectifs dans la cohomologie d'intersection des variétiés de Deligne–Lusztig. Nous formulons une conjecture de positivité sur le dual d'Alvis-Curtis du caractère obtenu à partir de la cohomologie d'intersection d'une variété de Deligne–Lusztig. Cette conjecture se révèle être un outil puissant pour déterminer les nombres de décompositions des ℓ-blocs unipotents des groupes réductifs finis. *Pour citer cet article : O. Dudas, G. Malle, C. R. Acad. Sci. Paris, Ser. I ??? (???).*

1. Deligne–Lusztig characters

Let $\mathbf{G}$ be a connected reductive linear algebraic group over an algebraically closed field of positive characteristic p, and $F : \mathbf{G} \to \mathbf{G}$ be a Steinberg endomorphism. There exists a positive integer δ such that F^δ defines a split $\mathbb{F}_{q^\delta}$-structure on $\mathbf{G}$ (with $q \in \mathbb{R}_+$), and we will choose δ minimal for this property. We set $G := \mathbf{G}^F$, the finite group of fixed points.

We fix a pair $(\mathbf{T}, \mathbf{B})$ consisting of a maximal torus contained in a Borel subgroup of $\mathbf{G}$, both of which are assumed to be F-stable. We denote by W the Weyl group of $\mathbf{G}$, and by S the set of simple reflections in W associated with $\mathbf{B}$. Then F^δ acts trivially on W.

Following Deligne–Lusztig, there exists a variety $\mathrm{X}(w)$, for any $w \in W$, endowed with an action of $G \times \langle F^\delta \rangle$, whose ℓ-adic cohomology gives rise to the so-called *unipotent* representations of G.

1.1. *Basic sets for finite reductive groups*

Let ℓ be a prime number and $(K, \mathcal{O}, k)$ be an ℓ-modular system. We assume that it is large enough for all the finite groups encountered. Furthermore, since we will be working with ℓ-adic cohomology we will assume throughout this note that K is a finite extension of $\mathbb{Q}_\ell$.

Let H be a finite group. Representations of H will always be assumed to be finite-dimensional. Recall that every projective kH-module lifts to a representation of H over K. The character afforded by such a representation will be referred to as a *projective character*. Integral linear combinations of projective characters will be called *virtual projective characters*.

Throughout this paper, we shall make the following assumptions on ℓ:
– $\ell \neq p$ (non-defining characteristic),
– ℓ is good for $\mathbf{G}$ and $\ell \nmid |(Z(\mathbf{G})/Z(\mathbf{G})^\circ)^F|$.

In this situation, the unipotent characters lying in a given unipotent ℓ-block of G form a *basic set* of this block [6,5]. Consequently, the restriction of the decomposition matrix of the block to the unipotent characters is invertible. In particular every (virtual) unipotent character is a virtual projective character, up to adding and removing some non-unipotent characters.

1.2. *A positivity conjecture*

Let D_G denote the Alvis–Curtis duality, with the convention that if ρ is a cuspidal unipotent character, then $D_G(\rho) = (-1)^{\mathrm{rk}_F(\mathbf{G})}\rho$ where $\mathrm{rk}_F(\mathbf{G}) = |S/F|$ is the F-semisimple rank of $\mathbf{G}$. For $w \in W$, we denote by $IH^i\big(\overline{\mathrm{X}}(w)\big)$ the ith intersection cohomology group of the Zariski closure $\overline{\mathrm{X}}(w)$ of $\mathrm{X}(w)$ in $\mathbf{G}/\mathbf{B}$, with coefficients in the trivial local system. We define Q_w to be the restriction to G of the virtual character

$$\sum_{i \in \mathbb{Z}} (-1)^i \big[D_G\big(IH^i\big(\overline{\mathrm{X}}(w)\big)\big)\big],$$

and by $Q_w[\lambda]$ the generalized λ-eigenspace of F^δ for $\lambda \in k^\times$. By [11, Thm. 3.8, 5.11.4 and 6.8.6] this character can be computed using almost characters together with values of characters of the Iwahori–Hecke algebra associated to W on the Kazhdan–Lusztig basis.

Lusztig proved [11, Prop. 6.9 and 6.10] that up to a global sign, Q_w (and therefore $Q_w[\lambda]$) is always a nonnegative linear combination of unipotent characters (note that the assumption on q can be removed by [3, Cor. 3.3.22]). The sign is given by the a-value $a(w)$ of the two-sided cell in which w lies.

Proposition 1.1 (Lusztig) *For all $w \in W$ and all $\lambda \in k^\times$, $(-1)^{a(w)}Q_w[\lambda]$ is a sum of unipotent characters.*

Unipotent characters are only the unipotent part of virtual projective characters in general. We conjecture that the modular analogue of Proposition 1.1 should hold in general, that is that Q_w is actually a proper projective character whenever ℓ is not too small.

Conjecture 1.2 *Under the assumption on ℓ in §1.1, for all $w \in W$ and all $\lambda \in k^\times$, $(-1)^{a(w)}Q_w[\lambda]$ is the unipotent part of a projective character.*

This conjecture has been checked on many decomposition matrices, including the unipotent blocks with cyclic defect groups for exceptional groups, and the matrices that are determined in [4,1]. We will give some examples in § 3.

Example 1 The closure of the Deligne–Lusztig variety $X(w_0)$ associated with the longest element w_0 of W is smooth and equal to $\mathbf{G}/\mathbf{B}$. Therefore its intersection cohomology $IH^\bullet\big(\overline{X}(w_0)\big)$ consists of copies of the trivial representation in degrees of a given parity. Since $a(w_0) = \ell(w_0)$, then $(-1)^{a(w_0)}Q_{w_0}$ is a nonnegative multiple of the Steinberg character. On the other hand, the Steinberg character is the unipotent part of a unique projective indecomposable module (given by a summand of a Gelfand–Graev module) and therefore the conjecture holds for w_0.

2. Applications

In this section we explain how to deduce properties of decomposition matrices using Conjecture 1.2.

2.1. *Families of simple unipotent modules*

Following [11, §5], we denote by $\leq_{\mathrm{LR}}$ the partial order on two-sided cells of W. Recall that to each cell Γ corresponds a two-sided ideal $I_{\leq\Gamma} := \mathrm{span}_{\mathbb{Q}}\{C_{w|v=1} \mid w \leq_{\mathrm{LR}} \Gamma\}$ of the group algebra $\mathbb{Q}W$, where $(C_w)_{w\in W}$ is the Kazhdan–Lusztig basis of the corresponding Hecke-algebra with parameter v. Moreover, given $\chi \in \mathrm{Irr}\, W$, there is a unique two-sided cell Γ_χ such that χ occurs in $I_{\leq\Gamma_\chi}/I_{<\Gamma_\chi}$. To each two-sided cell Γ one can attach a so-called family $\mathcal{F}_\Gamma$ of unipotent characters. They are defined as the constituents of R_χ for various χ such that $\Gamma_\chi = \Gamma$. By [11, Thm. 6.17], they form a partition of the set of unipotent characters of G.

We can use the characters Q_w to define families of simple unipotent kG-modules. For each Q_w we choose a virtual projective kG-module $\widetilde{Q}_w$ whose character coincides with Q_w up to adding/removing non-unipotent characters. We denote by $\mathrm{Irr}_{\leq\Gamma}kG$ (resp. $\mathrm{Irr}_{<\Gamma}kG$) the set of simple unipotent kG-modules N such that a projective cover P_N occurs in the virtual module $\widetilde{Q}_w$ for some $w \leq_{\mathrm{LR}} \Gamma$ (resp. $w <_{\mathrm{LR}} \Gamma$). We define the family of simple unipotent modules associated to Γ as $\mathrm{Irr}_\Gamma kG = \mathrm{Irr}_{\leq\Gamma}kG \setminus \mathrm{Irr}_{<\Gamma}kG$. Since the regular representation is uniform (see for example [2, Cor. 12.14]) then every indecomposable projective module lying in a unipotent block occurs in some $\widetilde{Q}_w$, therefore every simple unipotent module belongs to at least one family. However it is unclear whether this family is unique in general.

It is conjectured that, for ℓ not too small, the ℓ-modular decomposition matrix of G depends only on the order of q modulo ℓ. The following proposition gives some evidence toward this conjecture as well as to Geck's conjecture on the unitriangular shape of the decomposition matrix [7, Conj. 3.4].

Proposition 2.1 *Let b be a unipotent ℓ-block of G. Assume that*
  (i) *Conjecture 1.2 holds, and*
  (ii) *$|\mathrm{Irr}_\Gamma bkG| = |\mathcal{F}_\Gamma \cap \mathrm{Irr}\, b|$ for any two-sided cell Γ of W.*
Then the unipotent part of the decomposition matrix of b has the following shape:

$$D_{\mathrm{uni}} = \begin{pmatrix} D_{\mathcal{F}_1} & 0 & \cdots & 0 \\ * & D_{\mathcal{F}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & D_{\mathcal{F}_r} \end{pmatrix}$$

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where $\mathcal{F}_i$ runs over the families and where each $D_{\mathcal{F}_i}$ is a square matrix of size $|\mathcal{F}_i \cap \mathrm{Irr}\, b|$. Furthermore, the entries of D_{uni} are bounded above independently of q and ℓ.

Proof: Given any simple kG-module N in the block, there exists by (ii) a unique two-sided cell Γ such that $N \in \mathrm{Irr}_\Gamma bkG$ and P_N occurs in some Q_w for $w \in \Gamma$ (note that $w \not\preceq_{\mathrm{LR}} \Gamma$ otherwise N would belong to a smaller family). Since the matrix of the Q_w's is block triangular with respect to families, the proposition follows from (i). $\square$

Remark 1 Assumption (ii) assures that families define a partition of the set of simple unipotent kG-modules. This assumption will hold whenever each unipotent character in the block is a linear combination of the unipotent almost characters cut by the block. This condition can easily be checked on the Fourier matrices, for example, it holds whenever $\ell | (q \pm 1)$ and $\mathbf{G}$ is exceptional.

2.2. Determining decomposition numbers

The bounds on the entries of the decomposition matrix given by the Q_w's are often small enough to determine some of the decomposition numbers. We illustrate this on the principal Φ_2-block of the group $\mathrm{F}_4(q)$.

Proposition 2.2 *Assume that $(q,6) = 1$, $q \equiv -1 \pmod{\ell}$ and $\ell > 11$. If Conjecture 1.2 holds then*

$$
\begin{array}{c|ccccc}
\phi_{9,10} & 1 & & & & \\
\phi''_{2,16} & 1 & 1 & & & \\
\phi'_{2,16} & 1 & . & 1 & & \\
B_2, \varepsilon & 2 & . & . & 1 & \\
\phi_{1,24} & 3 & 2 & 2 & 4 & 1
\end{array}
$$

is a submatrix of the ℓ-modular decomposition matrix of $\mathrm{F}_4(q)$. (Here, "$\cdot$" denotes zero entries.)

Proof: By [10], the above matrix is a submatrix of the decomposition matrix of $\mathrm{F}_4(q)$, but with undetermined entries f, g, h, i, j in place of the entries larger than 1, which satisfy $f, h, i \geq 2$, $g \geq 4f - 5$ and $j \geq 4$. Let b be the block idempotent associated with the principal ℓ-block of $\mathrm{F}_4(q)$. With $w = s_2 s_3 s_2 s_3 s_4 s_3 s_2 s_1 s_3 s_2 s_4 s_3 s_2 s_1$ we have

$$bQ_w = 32\phi_{9,10} + 72\phi''_{2,16} + 72\phi'_{2,16} + 72B_2, \varepsilon + 288\phi_{1,24}.$$

If we denote by $P_1, \ldots, P_5$ the unipotent part of the characters of the PIMs corresponding to the last five columns of the decomposition matrix of $\mathrm{F}_4(q)$, we can decompose Q_w as

$$bQ_w = 32P_1 + 40P_2 + 40P_3 + (72 - 32f)P_4 + \big(288 - 32g - 40h - 40i - (70 - 32f)j\big)P_5.$$

By Conjecture 1.2 the multiplicity of each P_k should be nonnegative, that is $72 - 32f \geq 0$ and $288 - 32g - 40h - 40i - (70 - 32f)j \geq 0$. Since $f \geq 2$ the first relation forces $f = 2$ (and therefore $g \geq 3$). The second becomes $288 - 32g - 40h - 40i - 8j \geq 0$. Since $288 = 32 + 40 \times 2 + 40 \times 2 + 8 \times 4$ is the minimal value that the expression $32g + 40h + 40i + 8j$ can take, we deduce that $h = i = 2$, $g = 3$ and $j = 4$. $\square$

3. Further examples

3.1. A cuspidal module in the unitary group

We give here a non-trivial example for $\mathrm{SU}_n(q)$ where Conjecture 1.2 holds. The key point is to find a formula for the corresponding Kazhdan–Lusztig basis element in terms of well-identified elements of W.

This is done using the geometric description of Kazhdan–Lusztig polynomials. The approach given below could be adapted to other groups, even when the corresponding Schubert variety is no longer smooth, using Bott–Samelson varieties instead.

Recall that the set of unipotent characters of $\mathrm{SU}_n(q)$ is parametrized by partitions of n. Given such a partition λ, we denote by ρ_λ (resp. χ_λ) the corresponding unipotent character (resp. character of $\mathfrak{S}_n$), with the convention that ρ_{1^n} is the Steinberg character.

Proposition 3.1 *Let $\ell > n$ be a prime dividing $q + 1$. Then Conjecture 1.2 holds for $\mathrm{SU}_n(q)$ and $w = s_1 w_0 \in \mathfrak{S}_n$. Furthermore,*

$$\frac{(-1)^{a(w)}}{(n-1)!} Q_w = \rho_{21^{n-2}} + (n-1)\rho_{1^n}$$

is the unipotent part of a projective indecomposable $k\mathrm{SU}_n(q)$-module.

Here w_0 is the longest element of $W \cong \mathfrak{S}_n$ and s_i is the transposition $(i, i+1)$.

Proof: Let w_I denote the longest element of the parabolic subgroup of W generated by $s_2, \ldots, s_{n-2}$. Using the fact that the Schubert varieties asssociated with w_0, w_1 and $s_1 w_0$ are smooth, one finds the following relation between the corresponding Kazhdan-Lusztig basis elements:

$$C_{s_1 w_0 | v=1} = -C_{w_0 | v=1} + s_1 s_2 \cdots s_{n-1} C_{w_I | v=1}.$$

Given a partition λ, the multiplicity of ρ_λ in Q_w is $(-1)^{A(\lambda)}\chi_\lambda(C_{w|v=1}w_0)$, where $A(\lambda)$ denotes the degree in q of the degree polynomial of the unipotent character ρ_λ. For $w = s_1 w_0$ the value of χ_λ can be computed easily from the above formula. □

3.2. Groups of small rank

We finish by computing for several groups G of small rank the contribution to the principal ℓ-block b by various Q_w's. In the table below we give in the second column the minimal integer d such that $\ell \mid (q^d - 1)$, and in the last column the decomposition of Q_w in the basis of projective indecomposable modules which is obtained from the decomposition matrices in [9,4,1,8]. Note however that this might change when ℓ is small as the decomposition matrix will change (see for example $G_2(4)$ with $\ell = 5$). In any case, Q_w remains the unipotent part of a proper projective character. Our notation for the unipotent characters is as in the cited sources.

G	d	w	λ	$bQ_w[\lambda]$
$\mathrm{Sp}_6(q)$	2	$s_1 s_2 s_1 s_2 s_3$	1	$3(\rho_{1,1^2} + \mathrm{St}) + (B_2, 1^2 + 2\mathrm{St}) + (\rho_{1^3,\_} + 3\mathrm{St})$
$\mathrm{G}_2(q)$	2	$s_1 s_2$	-1	$G_2[-1] + 2\mathrm{St}$
${}^3\mathrm{D}_4(q)$	6	$s_1 s_2 s_3 s_2$	1	$2(\phi_{2,2} + \phi'_{1,3} + \mathrm{St}) + 2({}^3\mathrm{D}_4[-1] + 2\mathrm{St})$
$\mathrm{SU}_6(q)$	2	$s_1 s_3 s_4 s_3$	1	$2(\rho_{321} + 2\rho_{2^3} + 2\rho_{31^3} + 2\rho_{2^2 1^2} + 2\rho_{21^4} + 6\rho_{1^6})$
$\mathrm{F}_4(q)$	12	$s_1 s_2 s_1 s_3 s_2 s_1 s_3 s_3 s_2 s_4 s_3 s_2 s_3$	ζ_{12}^5	$2F_4[\theta] + 20B_2, \mathrm{St} + 4\mathrm{St}$
			ζ_{12}^4	$4F_4[\theta] + F_4[\mathrm{i}] + 11B_2, \mathrm{St} + 10\mathrm{St}$

One can even construct larger examples for which the conjecture holds. We give below five examples of computations of Q_w in $\mathrm{SU}_{10}(q)$ for $\ell \mid (q + 1)$. When $\ell > 17$, we can use [1, Thm. 6.2] to decompose them

on the basis of projective indecomposable modules. It turns out that at least four of them are (up to a scalar) the character of a projective indecomposable module.

w	Q_w
$s_1s_3s_4s_3s_5s_4s_3s_6s_5s_4s_3s_7s_6s_5s_4s_3s_9s_8s_7s_6s_5s_4s_3$	$1440\Psi_{321^5}$
$s_1s_2s_1s_4s_5s_4s_6s_5s_4s_7s_6s_5s_4s_8s_7s_6s_5s_4$	$96\Psi_{32^21^3}$
$s_5s_4s_3s_2s_1s_4s_3s_2s_4s_3s_4s_9s_8s_7s_9s_8s_9$	$576\Psi_{32^31}$
$s_1s_2s_1s_4s_5s_4s_6s_5s_4s_7s_6s_5s_4s_9$	$96\Psi_{32^221^2}$
$s_2s_1s_2s_3s_7s_6s_5s_7s_6s_7s_9$	$16\big(\Psi_{4321} + (1-\alpha)\Psi_{32^31} + \beta\Psi_{2^41^2}\big)$

Here $\alpha, \beta \in \{0, 1\}$ are as in [1, Thm. 6.2]

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