CHARACTERIZING NORMAL SYLOW p-SUBGROUPS BY CHARACTER DEGREES

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Abstract. Suppose that G is a finite group, let p be a prime and let $P \in \mathrm{Syl}_p(G)$. We prove that P is normal in G if and only if all the irreducible constituents of the permutation character $(1_P)^G$ have degree not divisible by p.

1. Introduction

Let G be a finite group, and let $\mathrm{Irr}(G)$ be the set of the irreducible complex characters of G. The Itô-Michler theorem, a fundamental theorem on Character Degrees, asserts that if p is a prime and $P \in \mathrm{Syl}_p(G)$, then p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ if and only if P is normal in G and abelian. A consequence of the (yet unproven) Brauer Height Zero Conjecture is that we can characterize if P is abelian if instead of $\mathrm{Irr}(G)$ we only consider $\mathrm{Irr}(B_0)$, the characters in the principal p-block of G. That is, P is abelian if and only if p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(B_0)$. For some time we have been wondering if there is a canonical subset of $\mathrm{Irr}(G)$ that exactly captures the normality of P. Our aim in this paper is to give a solution to this problem: we only need to consider the irreducible constituents of the induced character $(1_P)^G$.

Theorem A. *Let G be a finite group, let p be a prime, and let $P \in \mathrm{Syl}_p(G)$. Then $P \lhd G$ if and only if p does not divide $\chi(1)$ for all irreducible constituents χ of $(1_P)^G$.*

But in fact, we prove more. It is well-known that a character of a group G of degree not divisible by p cannot take the value zero on any p-element of G (although the converse is not true). This non-vanishing hypothesis was used by Dolfi, Pacifici, Sanus and Spiga [4] to give a sufficient condition for a finite group to have a normal Sylow p-subgroup. Here, using again the irreducible constituents of the permutation character $(1_P)^G$, we can give the following characterization which both extends Theorem A and the main result in [4].

Theorem B. *Let G be a finite group, let p be a prime, and let $P \in \mathrm{Syl}_p(G)$. Then the following conditions are equivalent:*

(i) *p does not divide $\chi(1)$ for all irreducible constituents χ of $(1_P)^G$.*
(ii) *$\chi(x) \neq 0$ for all irreducible constituents χ of $(1_P)^G$ and all $x \in P$.*
(iii) *$P \lhd G$.*

Our proof uses the Classification of Finite Simple Groups.

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2. The Simple Group Case

We start by considering non-abelian simple groups.

Theorem 2.1. *Let S be a finite non-abelian simple group, p a prime, and $P \in \mathrm{Syl}_p(S)$. Then either S has a p-defect zero character, or there exists a constituent $\theta \in \mathrm{Irr}(S)$ of the permutation character $(1_P)^S$ such that θ extends to $\mathrm{Aut}(S)$ and $\theta(x) = 0$ on some p-element x of S.*

In order to prove the theorem, we use the following observation.

Lemma 2.2. *Let G be a finite group, p a prime, and $P \in \mathrm{Syl}_p(G)$. If $\chi \in \mathrm{Irr}(G)$ has p-defect zero, then χ is a constituent of $(1_P)^G$ and vanishes on the non-trivial p-elements of G.*

Proof. By Brauer's theorem (Theorem (8.17) of [6]), we have that $\chi(z) = 0$ for all $1 \neq z \in P$. Hence χ_P is a multiple of the regular character ρ_P of P. Thus

$$\chi_P = a\,\rho_P\,,$$

for some $a > 0$. Thus $[\chi_P, 1_P] \neq 0$ whence the lemma follows by Frobenius reciprocity. $\square$

Lemma 2.3. *Let $p = 3$. Then Theorem 2.1 holds for $S = \mathfrak{A}_n$, $n \geq 7$.*

Proof. For a partition $\lambda \vdash n$ we denote by χ_λ the corresponding irreducible character of $\mathfrak{S}_n$. First assume that $n = 3^k$ for some k. We claim that there exists some partition $\lambda \vdash n$ with the following properties: χ_λ occurs in $1_P^{\mathfrak{S}_n}$, λ is not a hook, and λ is not self-dual. When $k = 2$, so $n = 9$, the partition $\lambda = (7, 2)$ does the job. Now assume that the claim has been proven for $m = 3^{k-1}$, with corresponding partition $\mu \vdash m$. The imprimitive subgroup $H_1 = \mathfrak{S}_m \wr 3$ of $\mathfrak{S}_n$ contains a Sylow 3-subgroup P of $\mathfrak{S}_n$. By our inductive assumption, the character $\psi := \chi_\mu \odot \chi_{(m)} \odot \chi_{(m)}$ of the Young subgroup $H = \mathfrak{S}_m \times \mathfrak{S}_m \times \mathfrak{S}_m$ is a constituent of the permutation character on its Sylow 3-subgroup. It induces irreducibly to H_1, so all constituents of $\psi^{\mathfrak{S}_n}$ are constituents of $1_P^{\mathfrak{S}_n}$. But by Young's rule (see [7, 2.8.2]), any constituent of $\psi^{\mathfrak{S}_n}$ has the required properties if χ_μ has. This proves the claim.

Now $\chi = \chi_\lambda$ as in the claim vanishes on n-cycles by the Murnaghan-Nakayama-rule (as λ is not a hook; see [7, 2.4.7]), and restricts irreducibly to $\mathfrak{A}_n$ (as λ is not self-dual). Thus the assertion holds when $n = 3^k$.

Else, the 3-adic decomposition of n has at least two summands. Let m denote one of them. Then the Young subgroup $H = \mathfrak{S}_m \times \mathfrak{S}_{n-m}$ contains a Sylow 3-subgroup of $\mathfrak{S}_n$. Again by Young's rule, when $n \equiv 1 \pmod 3$, the permutation character $1_H^{\mathfrak{S}_n}$ contains a constituent parametrized by $(n-1, 1)$ of degree $n-1$, which by the Murnaghan-Nakayama formula vanishes on elements of cycle shape $(3^{(n-1)/3}, 1)$. When $n \equiv 2 \pmod 3$, it contains a constituent parametrized by $(n-2, 1^2)$ of degree $(n-1)(n-2)/2$, which vanishes on elements of cycle shape $(3^{(n-2)/3}, 1^2)$, and when $n \equiv 0 \pmod 3$, it contains a constituent parametrized by $(n-2, 2)$ of degree $n(n-3)/2$ which vanishes on elements of cycle shape $(3^{n/3})$. Since all of these restrict irreducibly to $\mathfrak{A}_n$, the assertion follows. $\square$

Lemma 2.4. *Let $p = 2$. Then Theorem 2.1 holds for $S = \mathfrak{A}_n$, $n \geq 7$.*

Proof. The proof is rather similar to the previous one. Again first assume that $n = 2^k$ for some k. We claim that χ_λ, with $\lambda = (n-3, 2, 1)$ occurs in $1_P^{\mathfrak{S}_n}$. When $k = 3$, so $n = 8$, this is checked by direct computation. Now let $m := 2^{k-1}$. By induction $\psi := \chi_{(m-3,2,1)} \odot \chi_{(m)}$ is a constituent of the permutation character of the Young subgroup $H = \mathfrak{S}_m \times \mathfrak{S}_m$ on its Sylow 2-subgroup. The imprimitive subgroup $H_1 = \mathfrak{S}_m \wr 2 > H$ of $\mathfrak{S}_n$ contains a Sylow 2-subgroup P of $\mathfrak{S}_n$. Now ψ^{H_1} is irreducible, so all constituents of $\psi^{\mathfrak{S}_n}$ are constituents of $1_P^{\mathfrak{S}_n}$. By Young's rule, χ_λ is a constituent.

Now χ_λ as in the claim vanishes on partitions with cycle shape (m^2) and restricts irreducibly to $\mathfrak{A}_n$. Thus the assertion holds when $n = 2^k$.

Else, the 2-adic decomposition of n has at least two summands. Let m denote one of them. Then the Young subgroup $H = \mathfrak{S}_m \times \mathfrak{S}_{n-m}$ contains a Sylow 2-subgroup P of $\mathfrak{S}_n$. When $n \equiv 1 \pmod 4$ the permutation character $1_H^{\mathfrak{S}_n}$ contains a constituent parametrized by $(n-1, 1)$, which vanishes on elements of cycle shape $(2^{(n-1)/2}, 1)$. When $n \equiv 3 \pmod 4$ it contains a constituent parametrized by $(n-2, 2)$ which vanishes on elements of cycle shapes $(4^{(n-3)/4}, 1^3)$ and $(4^{(n-3)/4}, 2, 1)$. When n is even, then by the claim above the permutation character of H on P contains $\chi_{(m-3,2,1)} \odot \chi_{(n-m)}$, so $1_P^{\mathfrak{S}_n}$ contains $\chi_{(n-3,2,1)}$ which vanishes on elements of cycle shapes $(4^{(n-2)/4}, 1^2)$ and $(8, 4^{(n-10)/4}, 1^2)$, respectively $(4^{n/4})$ and $(8, 4^{(n-8)/4})$. In all cases, one of these classes lies in $\mathfrak{A}_n$, and since all of these characters restrict irreducibly to $\mathfrak{A}_n$, the assertion follows. □

Proof of Theorem 2.1. By [5, Cor. 2], finite non-abelian simple groups S have characters of p-defect zero unless we are in one of the following cases: $p = 2$ and either S is one of the sporadic groups

$$M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_3, Co_1, BM,$$

or $S = \mathfrak{A}_n$ with $n \geq 7$ not of the form $2m^2 + m$ or $2m^2 + m + 2$ for any integer m; or $p = 3$ and either S is Suz or Co_3, or $S = \mathfrak{A}_n$ with $3n + 1$ divisible by some prime q congruent $2 \pmod 3$ to an odd power.

When $S = \mathfrak{A}_n$, $n \geq 7$, and $p \leq 3$, the claim follows by Lemmas 2.3 and 2.4. For the open cases in the sporadic groups S, it is easily seen from the Atlas [2] that there exists an $\mathrm{Aut}(S)$-invariant irreducible character θ of S above 1_P vanishing on some p-element of S. As $|\mathrm{Out}(S)| \leq 2$, any such θ extends to $\mathrm{Aut}(S)$. □

3. The General Case

We are ready now to prove Theorem B of the introduction.

Proof of Theorem B. If $P \triangleleft G$, then the irreducible constituents of $(1_P)^G$ are the irreducible characters of the p'-group G/P, and therefore (iii) implies (i).

In the ring of algebraic integers R, any root of unity of p-power degree is congruent to 1 mod pR. Hence, if $x \in P$ and χ is a character of G, then $\chi(x) \equiv \chi(1) \bmod pR$. In particular, $\chi(x) \neq 0$ if χ has p'-degree. This proves that (i) implies (ii).

So we assume that $\chi(x) \neq 0$ for all irreducible constituents χ of $(1_P)^G$ and all $x \in P$ and prove by induction on $|G|$ that $P \triangleleft G$.

Suppose that $1 < N \triangleleft G$. If $\chi \in \mathrm{Irr}(G)$ contains N in its kernel and lies over 1_{PN}, then it lies over 1_P and therefore $\chi(nx) = \chi(x) \neq 0$ for all $x \in P$ and all $x \in N$. Thus

$PN \triangleleft G$ by applying the inductive hypothesis in G/N. In particular, we may assume that $\mathbf{O}_p(G) = 1$.

Let N be a minimal normal subgroup of G. We have that $PN \triangleleft G$ by the previous paragraph. Suppose first that $|N|$ is divisible by p. Then we have that $N \cong S \times \cdots \times S$ for a non-abelian simple group S of order divisible by p. Assume that $\theta \in \mathrm{Irr}(S)$ has p-defect zero, and let $\eta = \theta \times \cdots \times \theta \in \mathrm{Irr}(N)$, which has p-defect zero. Let $Q = P \cap N > 1$. Then by Lemma 2.2 we know that $[\eta_Q, 1_Q] \neq 0$. Now $((1_P)^{NP})_N = (1_Q)^N$ contains η, so we can find $\tau \in \mathrm{Irr}(NP)$ over η and over 1_P. Let $\chi \in \mathrm{Irr}(G)$ be over τ. Then χ_N is a sum of G-conjugates of η, all of them vanishing on the non-trivial p-elements of N. So $\chi(x) = 0$ for every non-trivial p-element $x \in N$, and this is a contradiction.

So by Theorem 2.1 we may assume that S has an irreducible character θ that extends to $\mathrm{Aut}(S)$, that lies over $1_{S\cap Q}$ and such that $\theta(x) = 0$ for some non-trivial p-element $x \in S$. By Lemma 5 of [1], we have that $\eta = \theta \times \cdots \times \theta \in \mathrm{Irr}(N)$ extends to G. Also, it is straightforward to check that $[\eta_Q, 1_Q] \neq 0$, and that η vanishes on the p-element $y = (x, \ldots, x) \in N$. As $((1_P)^{NP})_N = (1_Q)^N$, we deduce that there is some $\tau \in \mathrm{Irr}(NP)$ lying over η and over 1_P. Let $\chi \in \mathrm{Irr}(G)$ be over τ. Then χ lies over η and by Gallagher, we have that $\chi = \hat{\eta}\rho$, where $\hat{\eta} \in \mathrm{Irr}(G)$ is an extension of η and $\rho \in \mathrm{Irr}(G/N)$. Then χ lies over 1_P and $\chi(y) = \rho(1)\eta(y) = 0$, which is a contradiction.

Hence we may assume that $|N|$ is not divisible by p. Let K/N be a chief factor of G contained in PN, and let $H = K \cap P > 1$. Now, $G = N\mathbf{N}_G(P)$ by the Frattini argument, and $\mathbf{C}_H(N) \triangleleft G$. Hence $\mathbf{C}_H(N) = 1$ and H acts faithfully on N, because $\mathbf{O}_p(G) = 1$. By Lemma 2.8 of [4], there exists some $\theta \in \mathrm{Irr}(N)$ such that $\eta = \theta^K \in \mathrm{Irr}(K)$. Notice that for every $g \in G$, the conjugate $\eta^g = (\theta^g)^K \in \mathrm{Irr}(K)$ vanishes on every non-trivial element of H. Now, let T be the stabilizer of θ in NP. Since $(|T/N|, |N|) = 1$, we know that θ has an extension $\hat{\theta} \in \mathrm{Irr}(T)$. Then $\hat{\theta}$ has p'-degree and therefore the character $\hat{\theta}_{T\cap P}$ contains a linear character $\lambda \in \mathrm{Irr}(T \cap P)$. Consider λ as a character of $T/N \cong T \cap P$ and let $\nu = \bar{\lambda}\hat{\theta} \in \mathrm{Irr}(T)$, where $\bar{\lambda}$ is the complex conjugate of λ. Then $\nu^{NP} \in \mathrm{Irr}(NP)$ by the Clifford correspondence, and $\nu_{T\cap P}$ contains $1_{T\cap P}$. Hence $(\nu^{NP})_P = (\nu_{T\cap P})^P$ contains 1_P. Let finally $\chi \in \mathrm{Irr}(G)$ be over ν^{NP}. Then χ lies over 1_P and over ν. In particular, it lies over θ, and therefore over $\eta = \theta^K$. Hence $\chi(x) = 0$ for every non-trivial element x of H. This is the final contradiction. $\square$

We conclude this note with an open question which might have some interest. We believe that Theorem B can even be refined to proving that the following assertions are equivalent:

(i) $P \triangleleft G$,

(ii) p does not divide the degrees of the irreducible constituents of $(1_P)^G$ *occurring with multiplicity not divisible by* p, and

(iii) $\chi(x) \neq 0$ for all $x \in P$ and all the irreducible constituents of $(1_P)^G$ *occurring with multiplicity not divisible by* p.

With some changes in the proof of Theorem B above, this question reduces to simple groups. Since p-defect zero characters clearly occur with p'-multiplicity in $(1_P)^G$, we are led to study alternating groups for the primes $p = 2$ and $p = 3$ and sporadic groups. While computer calculations take care of the sporadic groups, it seems non-trivial to solve this problem for the alternating groups. (We thank Thomas Breuer and Joern Olsson for conversations on this subject.) In particular, one sees that the statement mentioned above is true for all primes $p \geq 5$.

Finally we mention two recent related results ([3] and [8]) that guarantee that a group has a normal Sylow p-subgroup using fields of values of characters. However these two results do not provide full characterizations, as we do in this paper.

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