NEW DEVELOPMENTS IN THE REPRESENTATION THEORY OF FINITE GROUPS

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To I. M. Isaacs, in memoriam

Abstract. Two fundamental long-standing conjectures in the representation theory of finite groups have recently been proved: Brauer's Height Zero Conjecture from 1955, and the McKay Conjecture from 1971. We discuss how these conjectures have been solved and the stimulating implications of these proofs for the future of the subject.

Contents

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1. Introduction

These are exciting times for finite group representation theory. Several of the long-standing major conjectures in the field that for most of the past 50 or more years seemed inaccessible have recently been solved, and stunning progress is currently being made on others.

It is common knowledge that the global/local philosophy is ubiquitous in Mathematics. The investigation of number theoretic objects at a prime (or prime ideal) yields deep information on the subject; similarly, the local study of objects in algebraic geometry is an indispensable tool. The main conjectures and recent theorems show that this approach is also crucial and fruitful in finite group representation theory.

We can only imagine the reaction of Richard Brauer when in 1971, in a paper dedicated to him, J. McKay made a bold astonishing conjecture. In his address [13] at the 1950 International Congress of Mathematicians, Brauer had already asked to what extent the properties of the representations of a finite group G were determined by the representations of suitable proper subgroups. This was a main theme in Brauer's research. In his talk, he was identifying the characters of the finite groups (i.e., the traces of their representations over the complex numbers) by means of how they restricted to the so-called elementary subgroups. This had a fundamental application to Artin L-series. In general, Brauer was interested in how the properties of a finite group, from the point of view of a prime number p, were encoded in its so-called p-local subgroups. And the McKay conjecture was a startling example of this.

Let us begin by recalling that the first feature of a finite group G is its *order* $n = |G|$, which we readily decompose as the product of the distinct prime powers. If p is a prime, we use n_p to denote the largest power of p dividing n. If we are interested in subgroups of G related to p, the first place to look is in $\mathrm{Syl}_p(G)$, the set of Sylow p-subgroups of G. These are the subgroups P of G of order n_p; they always exist, they are conjugate in G, and they contain all the *p-subgroups of G* (the subgroups of order a power of p). But p-subgroups are not enough (they can't be enough): we want to see how these subgroups are embedded into the group. So given a p-subgroup $Q \neq 1$ of G, we consider its normaliser in G, which is

$$\mathbf{N}_G(Q) = \{g \in G \mid g^{-1}Qg = Q\},$$

the largest subgroup of G containing Q as a normal subgroup. These are the *local subgroups* of G. The interaction between a group and its local subgroups is a well-known and successfully exploited theme in finite group theory. For instance, Brauer and Fowler showed that there are only finitely many non-abelian simple groups with a fixed $\mathbf{N}_G(Q)$, where Q is a subgroup of order 2. (The existence of subgroups of order 2 is guaranteed by the Feit–Thompson theorem!) And this was one of the first ideas in the Classification of Finite Simple Groups. As a matter of fact, the main tool of the classification consisted in cleverly using and studying local subgroups. It is hence natural to ask:

Are there also global/local connections in the Representation Theory of Finite Groups?

The key notion when studying complex representations of groups is that of a character. Given a complex *representation* of a finite group G, that is, a multiplicative function

$$\mathcal{X} : G \to \mathrm{GL}_m(\mathbb{C}) \qquad \text{for some } m \geq 1,$$

the *character* χ *of* $\mathcal{X}$ is the function

$$G \to \mathbb{C}, \qquad g \mapsto \chi(g) := \mathrm{Trace}(\mathcal{X}(g))\,,$$

defined by taking the trace. Of course,

$$\chi(x^{-1}gx) = \chi(g) \quad \text{for } x, g \in G,$$

since similar matrices have identical traces. Therefore characters are class functions (constant on conjugacy classes). The number $\chi(1) = m$, the dimension of the underlying representation, is the *degree* of the character. This is the most important number associated to χ. By elementary linear algebra, if $\mathcal{X}$ is a representation and $g \in G$ has order k, then $\mathcal{X}(g)$ is similar to a diagonal matrix with k-th root of unity entries, and therefore $\chi(g)$ is a sum of roots of unity. It is a remarkable non-elementary fact that two representations $\mathcal{X}$ and $\mathcal{Y}$ have the same character if and only if there is $M \in \mathrm{GL}_m(\mathbb{C})$ such that $M^{-1}\mathcal{X}M = \mathcal{Y}$. (This fails, for example, for infinite groups.) Hence characters determine complex representations, up to similarity.

As can be seen by taking direct sums of representations, sums of characters are again characters. We arrive at a central definition: a complex character χ is *irreducible* if it is not the sum of two characters. The Fundamental Theorem of Character Theory establishes that the set $\mathrm{Irr}(G)$ of irreducible characters of G is an orthonormal basis of the space $\mathrm{cf}(G)$ of complex valued class functions with respect to the Hermitian inner product

$$[\alpha, \beta] := \frac{1}{|G|} \sum_{g \in G} \alpha(g)\overline{\beta(g)}\,.$$

In particular, $|\mathrm{Irr}(G)| = k(G)$ is the number of conjugacy classes of G. All this was developed by G. Frobenius around 1900 and it is usually proved using Wedderburn's theory on the complex group algebra $\mathbb{C}G$. For instance the equation

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$$

follows from the decomposition of the group algebra $\mathbb{C}G$ as a direct sum of matrix algebras. If we consider the alternating group $\mathfrak{A}_5$ on five letters, say, the smallest non-abelian simple group, of order 60, the degrees of its irreducible character are $1, 3, 3, 4$ and 5. And of course

$$60 = 1^2 + 3^2 + 3^2 + 4^2 + 5^2\,.$$

The proof of the important fact that the degrees $\chi(1)$ divide $|G|$ uses a bit of elementary number theory and algebraic integers. To relate all this with a prime p and the representations over fields of characteristic p was essentially the work of Brauer and his modular representation theory, developed from the 1940's.

Most of the global/local conjectures which we will discuss here deal with character degrees from the perspective of a prime p. If $\chi \in \mathrm{Irr}(G)$, we are interested in $\chi(1)_p$, the largest power of p that divides $\chi(1)$. The two extremes, when p does not divide

$\chi(1)$ (we say that χ has p'-*degree*) or when $\chi(1)_p = |G|_p$ are particularly interesting. Brauer already realised the importance of the latter, the *p-defect zero characters*. McKay's conjecture deals with the former. Although the Height Zero Conjecture precedes McKay's chronologically, its simple beautiful statement makes it irresistible not to state it as soon as possible. If G is a finite group and p is a prime, we let

$$\mathrm{Irr}_{p'}(G) := \{\ \chi \in \mathrm{Irr}(G) \mid p \text{ does not divide } \chi(1)\}.$$

The McKay Conjecture. *If G is a finite group and $P \in \mathrm{Syl}_p(G)$, then*

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|\,.$$

Perhaps we should have stated this as a theorem, since M. Cabanes and B. Späth have announced a proof of this ([24]).

If we come back to the $\mathfrak{A}_5$ example, recall that if $p = 2$ then $\mathbf{N}_G(P) = \mathfrak{A}_4$ has irreducible character degrees $1, 1, 1$ and 3. If $p = 3$, then $\mathbf{N}_G(P) = \mathfrak{S}_3$ is the symmetric group on three letters which has degrees $1, 1$ and 2. If $p = 5$, then $\mathbf{N}_G(P) = \mathsf{D}_{10}$ is the dihedral group of order 10 with degrees $1, 1, 2$ and 2. We invite the reader to check these numbers against the character degrees of $\mathfrak{A}_5$, as McKay proposed.

In order to introduce the Height Zero Conjecture, we need to understand a bit of Modular Representation Theory. The first consideration, perhaps, is that Brauer's aim was not only to study representations over an algebraically closed field $\mathbb{F}$ of characteristic p (work that was initiated by Dickson and Speiser), but to relate them to the complex representations. A good start is to point out that in order to develop the character theory over $\mathbb{C}$ it is enough to work in the much smaller field $\overline{\mathbb{Q}}$, the algebraic closure of the rational numbers. That is, Wedderburn's theory over $\overline{\mathbb{Q}}$ allows us to prove that every character χ of G can in fact be afforded by a representation $G \to \mathrm{GL}_m(\overline{\mathbb{Q}})$. Furthermore, using some ring theory, it is possible to do even better and show that every character can be afforded by a representation $G \to \mathrm{GL}_m(\mathcal{O})$, where $\mathcal{O}$ is the localisation of the ring of algebraic integers $\mathbf{R}$ of $\overline{\mathbb{Q}}$ at some fixed maximal ideal M of $\mathbf{R}$ containing p. Recall that the values of the complex characters are sums of roots of unity, and therefore elements of $\mathbf{R}$. The gain now is fundamental. The ring $\mathcal{O} = \{\alpha/\beta \mid \alpha \in \mathbf{R}, \beta \in \mathbf{R} - M\}$ has a unique maximal ideal $\mathcal{J}$, and $\mathbb{F} := \mathcal{O}/\mathcal{J}$ is an algebraic closure of the field of p elements. The canonical map

$$^* : \mathcal{O} \to \mathbb{F}$$

is the bridge from characteristic 0 to characteristic p; from $\mathrm{GL}_m(\mathcal{O})$ to $\mathrm{GL}_m(\mathbb{F})$. Furthermore, if $\mathcal{O}G$ is the free $\mathcal{O}$-algebra with basis G, we have a natural map $^* : \mathcal{O}G \to \mathbb{F}G$ given by

$$\sum_{g\in G} r_g g \mapsto \sum_{g\in G} (r_g)^* g\,.$$

In summary, the ring $\mathcal{O}$ allows us to relate complex representations to representations over a field $\mathbb{F}$ of characteristic p. More will be said later (in §4) about this. For now, if we write

$$\mathcal{O}G = B_1 \oplus \cdots \oplus B_t$$

as a direct sum of indecomposable two-sided ideals, then $B_1, \ldots, B_t$ are the Brauer *p-blocks* of G. The set is denoted by $\mathrm{Bl}(G)$, and it induces naturally a partition of

$\mathrm{Irr}(G)$, as we shall see. When studying χ from the point of view of p, only the block in which χ lives is going to be important. For certain properties of χ, even the group is irrelevant! If we write

$$1 = f_1 + \cdots + f_t\,,$$

where $f_i \in B_i$, we have that $f_i f_j = 0$ if $i \neq j$ and $f_i^2 = f_i$, using that the B_i's are two-sided ideals and that the sum is direct. (The f_i are called the *central primitive idempotents*, or the *block idempotents*.) Notice too that B_i is an $\mathcal{O}$-algebra with identity f_i. Now, if $\chi \in \mathrm{Irr}(G)$ is afforded by a representation $\mathcal{X} : G \to \mathrm{GL}_m(\mathcal{O})$, then the $\mathcal{O}G$-module $V = \mathcal{O}^m$, defined by $vg := v\mathcal{X}(g)$ for $v \in V$ and $g \in G$, decomposes as

$$V = Vf_1 \oplus \cdots \oplus Vf_t\,,$$

and this yields χ as a sum of characters. The conclusion is that there is a unique i such that $V = Vf_i \neq 0$. We say that χ *belongs* to B_i and write $\chi \in \mathrm{Irr}(B_i)$.

As shown by Brauer, in order to effectively partition $\mathrm{Irr}(G)$ into blocks one only needs to know the values of the characters: it is not necessary to work in the algebra $\mathcal{O}G$, or to find indecomposable ideals; a task that might seem daunting. While we defer an explanation of this for later on, let us mention now that for $p = 2$, the alternating group $\mathfrak{A}_5$ has two p-blocks whose irreducible characters have degrees $\{1, 3, 3, 5\}$ and $\{4\}$. The character χ of degree $4 = 2^2$ lives alone in what we call a block of "defect zero," and we notice that 4 is the largest power of 2 that divides $60 = |\mathfrak{A}_5|$. We also notice that the degrees of the characters of the other block, the "principal block," are not divisible by 2.

A single object made block theory amazingly deep and interesting. To each p-block B of a finite group G, Brauer associated a conjugacy class of p-subgroups which he called *defect groups*. As we shall see, the way to associate them is not particularly complicated. But these subgroups will govern the complexity of B. To get a taste of what defect groups mean for the theory, let us start with what Brauer called the First Main Theorem: if D is any p-subgroup of G and $\mathrm{Bl}(G|D)$ is the set of p-blocks of G with defect group D, then there is a natural bijection $b \mapsto b^G$ from $\mathrm{Bl}(\mathbf{N}_G(D)|D)$ onto $\mathrm{Bl}(G|D)$. This Brauer's First Main Theorem is probably the first global/local theorem in the modular representation theory of finite groups. Recall that b and b^G are $\mathcal{O}$-algebras. A central theme is to understand the exact relation between these two.

Brauer soon proved that if B has defect group D, $|G|_p = p^a$, and $|D| = p^d$, then

$$p^{a-d} = \min\{\chi(1)_p \mid \chi \in \mathrm{Irr}(B)\}\,.$$

In particular, whenever $\chi \in \mathrm{Irr}(B)$, then we can write

$$\chi(1)_p = p^{a-d+h}\,,$$

where $h = h(\chi)$ is a non-negative integer called the *height* of χ. We have now all the ingredients to state Brauer's conjecture.

Brauer's Height Zero Conjecture. *Let B be a p-block of a finite group G with defect group D. Then D is abelian if and only if $h(\chi) = 0$ for all $\chi \in \mathrm{Irr}(B)$.*

Formulated in 1955 ([14]), it has taken almost 70 years and efforts by many mathematicians around the world to complete a proof of it ([53, 94, 66]).

The concerted efforts to prove these conjectures have also given all kinds of impetus to the further development of the character theory and block theory of simple groups, in particular the alternating groups and the finite groups of Lie type, and have driven and guided the research in those areas.

Our aim in this paper is to discuss how both conjectures have finally been proven, the implications that this has in our field, and the conjectures and challenges that remain or may open up ahead of us. As one may expect, it would be impossible to encompass in this survey all the exciting developments that occurred recently in our field.

2. The McKay Conjecture

The history of what we now know as the McKay conjecture starts with a conjecture of John McKay (1939–2022) about odd-degree irreducible characters of simple groups in 1971. (A few years later, McKay would startle the mathematicians with another observation about the smallest non-trivial character degree of the Monster sporadic simple group, which led to the Monstrous Moonshine, and some wonderful mathematics.) It seems that McKay had a predilection for simple groups. Since non-abelian simple groups have even order by the Feit–Thompson theorem, odd-degree characters had the appropriate tension to be considered.

Simple groups and their classification have an important role in this paper. Perhaps some readers would not mind to be reminded that a subgroup H of a finite group G is normal in G if $gH = Hg$ for all $g \in G$. We write $H \trianglelefteq G$ in this case. Normal subgroups go back to Galois. They are important because if $H \trianglelefteq G$, then the set $G/H = \{gH \mid g \in G\}$ consisting of the *cosets* $gH = \{gh \mid h \in H\}$, with the set multiplication $(g_1H)(g_2H) = (g_1g_2)H$, forms another group, of order $|G|/|H|$, called the *factor group*. If G has a normal subgroup $1 < H \trianglelefteq G$, information about the two groups H and G/H might ideally be put together to yield information on G. Therefore a central aim of finite group theory should be to classify the *simple* groups, that is, the groups with no non-trivial proper normal subgroups. The story of the Classification of Finite Simple Groups (CFSG) has been told in other places, and we will not attempt to repeat it here. Let us only say that the CFSG establishes that a finite group S is simple if $|S|$ is a prime; $S = \mathfrak{A}_n$ is an alternating group, $n \geq 5$; $S = G(q)$ is a group of Lie type over a field of q elements; or S is one of the 26 *sporadic* groups.

Coming back to characters: if G is a finite group and p is a prime, let $m_p(G) = |\mathrm{Irr}_{p'}(G)|$ be the number of irreducible characters of G of degree not divisible by p. In [71, 72] McKay made the following conjecture:

Conjecture 2.1 (McKay, 1971)**.** *If S is a simple group and $P \in \mathrm{Syl}_2(S)$, then $m_2(S) = m_2(\mathbf{N}_S(P))$.*

This was astonishing. How was it possible that two such very different groups, S and $\mathbf{N}_S(P)$, could share such a fundamental invariant? How can the tiny (in comparison) $\mathbf{N}_S(P)$ influence the degrees of the irreducible characters of the big group S? Honestly, at the beginning, one would not have been too surprised if a counterexample had been found sooner than later. In the two-page paper [72], after proposing his conjecture,

McKay proved the same statement for the symmetric group $\mathfrak{S}_n$ (which is not a simple group when $n > 2$). It is therefore possible that he was thinking that the equality $m_2(G) = m_2(\mathbf{N}_G(P))$ might hold for more classes of groups. We do not know for sure. (It is also true that the contents of [71] and [72] differ slightly. In the seven-line announcement in [71] McKay also mentions blocks and it is not clear to which kind of groups he is referring in his predictions. In the paper [72], he does only refer to simple groups and $p = 2$.)

In any case, the first place where other primes and more classes of groups are treated is in 1973 by I. M. Isaacs ([45]). He proves that $m_2(G) = m_2(\mathbf{N}_G(P))$ for G solvable. Solvable groups, groups with many normal subgroups, are at the opposite extreme from simple groups, and yet they satisfy the McKay equation. Isaacs also proved that $m_p(G) = m_p(\mathbf{N}_G(P))$ for any group G of odd order, for any prime p. He did not make any conjecture on this, though.

Our next stop is at the University of Chicago, in the group theory seminar, in 1974. Recall that if p is a prime and G is a finite group, then the set $\mathrm{Irr}(G)$ is partitioned into Brauer p-blocks. As we have mentioned, if B is a p-block with defect group D, then $\min\{\chi(1)_p \mid \chi \in \mathrm{Irr}(B)\} = |G : D|_p$. The subset

$$\mathrm{Irr}_0(B) = \{\chi \in \mathrm{Irr}(B) \mid \chi(1)_p = |G : D|_p\}$$

is the set of height zero characters of B. Notice that if $\chi \in \mathrm{Irr}_{p'}(G)$ (has degree not divisible by p) lives in a block B with defect group D, then $|G : D|_p \leq \chi(1)_p = 1$, and we conclude that $D \in \mathrm{Syl}_p(G)$. Therefore, if $P \in \mathrm{Syl}_p(G)$, we can partition

$$\mathrm{Irr}_{p'}(G) = \bigcup_{B \in \mathrm{Bl}(G|P)} \mathrm{Irr}_0(B)\,.$$

By the same argument,

$$\mathrm{Irr}_{p'}(\mathbf{N}_G(P)) = \bigcup_{b \in \mathrm{Bl}(\mathbf{N}_G(P)|P)} \mathrm{Irr}_0(b)\,.$$

Since Brauer's First Main Theorem establishes a bijection $b \mapsto b^G$ from $\mathrm{Bl}(\mathbf{N}_G(P)|P)$ to $\mathrm{Bl}(G|P)$, could it be that the equation

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|$$

holds only because the equation

$$|\mathrm{Irr}_0(b^G)| = |\mathrm{Irr}_0(b)|$$

holds for corresponding blocks? In the Chicago seminar, and afterwards in [1], J. L. Alperin proposed the first strengthening of the (not officially announced) McKay conjecture.

Conjecture 2.2 (Alperin–McKay Conjecture, 1974)**.** *Let D be any p-subgroup of a finite group G. Let $B \in \mathrm{Bl}(G|D)$ and $b \in \mathrm{Bl}(\mathbf{N}_G(D)|D)$ be Brauer correspondents. Then*

$$|\mathrm{Irr}_0(B)| = |\mathrm{Irr}_0(b)|\,.$$

Why bother to strengthen an unproved conjecture? This is a legitimate question: it is reasonable to think that the stronger and more specific a conjecture is, the better it points towards where to find a proof of it. Now that blocks were involved in the McKay

conjecture, perhaps the conjecture was a consequence of a deeper more structural statement about blocks or algebras.

From 1974 on, different families of groups were going to be checked to either satisfy the McKay or the Alperin–McKay conjecture. In 1975, J. B. Olsson proves the Alperin–McKay conjecture for symmetric groups and the McKay conjecture for the general linear groups $\mathrm{GL}_n(q)$, for p not dividing q. He writes: "*Opinions are divided as to whether (I)* [McKay] *and (II)* [Alperin–McKay] *are reasonable conjectures...*"([84]). In 1978, T. R. Wolf proves the McKay conjecture for solvable groups, for any prime p [111], and Michler and Olsson, Alperin–McKay for general linear groups and unitary groups ([73]). The sporadic simple groups are checked much later in 1997 by R. A. Wilson, who writes about his work: "[it] *was done well over 10 years ago, but was not written up at the time partly because of difficulties completing one or two tricky cases*" [110]. By quoting Wilson, we stress that McKay's would not be the first conjecture that turns out to be false because a single sporadic group fails it! Each family of groups is checked with its own techniques, which cannot be generalised to other classes of groups, and with the same concluding result: no counterexample is found.

Over the next years, several strengthenings of the conjectures are proposed (on which we will write later), again clearly pointing into some directions: to work over the field of p-adic numbers, to use isometries, chains of p-subgroups, derived equivalences, algebraic topology, K-theory, etc. However none of these have helped, so far, to find a proof of the McKay conjecture.

The idea of using the CFSG to prove the McKay conjecture was in the air since the 1990's, perhaps before. The proof of the Classification was already considered to be definite. Around that time, E. C. Dade ([27, 28]), after work of R. Knörr and G. R. Robinson [57], had proposed several conjectures which simultaneously generalised McKay, Alperin–McKay and the other notorious conjecture proposed in 1987 by J. Alperin: the Alperin Weight Conjecture ([3], more on this later in Section 5). Dade's insight seems to have been that the ultimate version of his conjectures, if checked for simple groups, should imply all the previous conjectures. He wrote [28, p. 99] "*... the forms of our conjecture discussed here are weaker than the final form which will be needed to reduce that conjecture to the case where G is simple and non-abelian,*" and believed [27, p. 188] "*... the final and most delicate form of the conjecture only has to be verified directly for simple groups of Lie type.*" However, Dade's reduction never appeared in print.

In 2004 M. Isaacs, the first and the second author met in Valencia to try to decide if there existed a specific condition on simple groups which if satisfied, would imply the McKay conjecture. In 1980 W. Feit in his paper on the problems after the Classification ([32]) already wrote that the McKay conjecture appeared not to be affected by the CFSG. He was right in the sense that to have a proof of the McKay conjecture for simple groups does not imply that the McKay conjecture is true for every finite group. We had to demand something far stronger. To explain what exactly, we need to tell the reader a bit more of basic character theory. And this is what we do next.

If $\chi \in \mathrm{Irr}(G)$ and $H \leq G$ is a subgroup, by restricting to H the representation that affords χ, we see that the restricted function χ_H is a character of H. Very little can be said about χ_H except the trivial fact that it is a sum of irreducible characters of H.

Restriction of characters has an adjoint: the *induction* of characters. If $\theta \in \mathrm{Irr}(H)$, then the *induced character* θ^G is the unique character of G that satisfies

$$[\theta^G, \chi] = [\theta, \chi_H] \qquad \text{for all } \chi \in \mathrm{Irr}(G).$$

(The last equation is called *Frobenius reciprocity*). A formula for the values of θ^G can easily be given. The control on the restricted character χ_H dramatically improves if we further assume that $H \trianglelefteq G$. Indeed, Clifford's theorem asserts that if $\chi \in \mathrm{Irr}(G)$, then χ_H is a multiple of the sum of the G-orbit of a single irreducible character $\theta \in \mathrm{Irr}(H)$. That is

$$\chi_H = e(\theta^{g_1} + \cdots + \theta^{g_t}),$$

where the θ^{g_i} are the different G-conjugates of θ, and e is a positive integer. (The character θ^g is easily defined using that conjugation by g defines an automorphism of H: that is, $\theta^g(h) := \theta(ghg^{-1})$.) We write $\mathrm{Irr}(G|\theta)$ to denote the irreducible characters of G whose restriction to H contains θ as a constituent. The second part of Clifford theory is the *Clifford correspondence* which asserts that if G_θ is the stabiliser of θ in G, then induction $\psi \mapsto \psi^G$ defines a bijection

$$\mathrm{Irr}(G_\theta|\theta) \to \mathrm{Irr}(G|\theta).$$

The Clifford correspondence usually enables us to assume, when working by induction, that $G_\theta = G$; that is, that θ is *G-invariant*. In this case, we say that (G, H, θ) is a *character triple*, a term coined by Marty Isaacs. (It is a fact of fundamental importance for the theorems that we shall be describing, that given a character triple, it is possible to construct a *projective representation* $\mathcal{P}$ of G such that $\mathcal{P}_H$ affords θ. In the case that θ extends to a character of G, $\mathcal{P}$ can be chosen to be an ordinary representation of G. See [46, Thm. 11.2].)

Continue to assume that $H \trianglelefteq G$, $\theta \in \mathrm{Irr}(H)$ and $\chi \in \mathrm{Irr}(G|\theta)$. If $P \in \mathrm{Syl}_p(G)$ is a Sylow p-subgroup of G, then P acts on the G-orbit of θ by conjugation, and thereby partitions the G-orbit into different P-orbits of size a power of p. If χ has degree not divisible by p, it follows that χ_H necessarily has a P-invariant constituent (a P-orbit of size 1). Using Sylow theory in G_θ, we easily prove that any two P-invariant constituents of χ_H are conjugate in $\mathbf{N}_G(P)$. This is quite elementary but it gives a first connection between two of the actors in the McKay conjecture: $\mathrm{Irr}_{p'}(G)$ and $\mathbf{N}_G(P)$. Hence, if we choose representatives Δ of the action of $\mathbf{N}_G(P)$ on the P-invariant characters of H, we can partition

$$\mathrm{Irr}_{p'}(G) = \bigcup_{\theta \in \Delta} \mathrm{Irr}_{p'}(G|\theta). \tag{1}$$

Suppose now that we are trying to prove McKay by induction on $|G|$ and we are willing to use the CFSG. Definitely we have to assume that we know how to prove the McKay conjecture for simple groups, no question about that. But surely we will need more. Assume then that G is not simple, and let K be a proper normal subgroup of G. Then $Q = P \cap K \in \mathrm{Syl}_p(K)$ and by induction, we have a McKay bijection

$$\Omega : \mathrm{Irr}_{p'}(K) \to \mathrm{Irr}_{p'}(\mathbf{N}_K(Q)).$$

By the so-called Frattini argument (elementary Sylow theory), we have that $G = K\mathbf{N}_G(Q)$. Notice too that $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(Q)$, since $K \trianglelefteq G$. So if McKay is true for G,

we should have that

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(Q))|$$

too. Now, $\mathbf{N}_K(Q) \trianglelefteq \mathbf{N}_G(Q)$, and by the same argument to get equation (1), we have

$$\mathrm{Irr}_{p'}(\mathbf{N}_G(Q)) = \bigcup_{\theta^* \in \Delta^*} \mathrm{Irr}_{p'}(\mathbf{N}_G(Q)|\theta^*)\,, \tag{2}$$

where Δ^* is a complete set of representatives for the orbits of $\mathbf{N}_G(P)$ on the P-invariant characters of $\mathbf{N}_K(Q)$. The key conclusion is that the only "reasonable" way to have that $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(Q))|$, using (1) and (2), is that we impose the bijection Ω to commute with the $\mathbf{N}_G(Q)$-action in such a way that

$$|\mathrm{Irr}_{p'}(G|\theta)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(Q)|\Omega(\theta))|$$

for every $\theta \in \mathrm{Irr}_{p'}(K)$. So we shall need a "good" McKay type bijection $\mathrm{Irr}_{p'}(K) \to \mathrm{Irr}_{p'}(\mathbf{N}_K(Q))$ that somehow controls what is happening "above." If $\mathbf{N}_G(Q) < G$, then we could apply induction in $\mathbf{N}_G(Q)$, to obtain

$$|\mathrm{Irr}_{p'}(\mathbf{N}_G(Q))| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|$$

and McKay would follow. This is a very good idea, but there might be some troubling cases. Suppose that we are really unfortunate and all proper normal subgroups of G have order not divisible by p (or have a normal Sylow p-subgroup). In this case $\mathbf{N}_G(Q) = G$ all the time and we cannot apply induction. Our group will have a tight structure; for instance, we might have that $G/\mathbf{O}_{p'}(G)$ is simple, where $\mathbf{O}_{p'}(G)$ is the largest normal subgroup of G of order not divisible by p. In this extreme case, our idea above is useless. But at least the idea has pointed in some interesting directions which we will later use: it is telling us to look at characters of p'-degree over a P-invariant character of a normal subgroup, and then to control the number of characters of G over those of p'-degree of the normal subgroup. At this stage, the only way out was to make a bolder conjecture. If we want simple groups to show up in a problem, a good idea is to find a more general statement of the problem relative to a normal subgroup N, such that when $N = 1$, you recover the original problem. We have been lucky that this has worked for McKay (and, it seems, it might for many others of our counting conjectures).

Conjecture 2.3 (Relative McKay)**.** *Let G be a finite group, $P \in \mathrm{Syl}_p(G)$. Suppose that $N \trianglelefteq G$ and let $\theta \in \mathrm{Irr}(N)$ be P-invariant. Then*

$$|\mathrm{Irr}_{p'}(G|\theta)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P)N|\theta)|\,.$$

Clearly McKay follows by the special case $N = 1$. Now, we start a proof of Conjecture 2.3 by induction on $|G : N|$. Using the Clifford correspondence, we may assume that θ is G-invariant. That is (G, N, θ) is a character triple. A well-known technique in character theory developed by Schur tells us that we can replace (G, N, θ) with another triple (G^*, N^*, θ^*), where N^* is central in G^*, and G/N and G^*/N^* are isomorphic. In other words, in Conjecture 2.3, we can assume that N is central in G. Since we are dealing with characters of p'-degree over θ, standard character theory allows us to assume that N has order not divisible by p.

The next step is to take a non-trivial normal subgroup K/N of G/N as small as possible. This is called a *minimal normal* subgroup of G/N. Elementary group theory tells us that they come in two types: either K/N is an abelian group of order a power of a prime q, or K/N is a direct product of non-abelian simple groups. Using that $|G:K| < |G:N|$, and induction, one easily can assume that $KP \trianglelefteq G$. Hence the group G/K has a very particular structure. If K/N is abelian and K/N is a power of p, then elementary group theory tells us that $P \trianglelefteq G$, and in this case there is nothing to prove. The case where K/N has order not divisible by p is solved by using the so-called *Glauberman correspondence*: this is a natural bijection $\theta \mapsto \hat{\theta}$ between the P-invariant characters of a group K and the characters of the fixed point subgroup $\mathbf{C}_K(P)$ (see [46, Chap. 13] or [77, Chap. 2]). For each P-invariant $\theta \in \mathrm{Irr}(K)$, it is possible to construct a bijection above $\mathrm{Irr}(G|\theta) \to \mathrm{Irr}(\mathbf{N}_G(P)|\hat{\theta})$ with good behaviour (see [108]); exactly as we described before with the correspondence that we called Ω. This is a good place to remind the reader that a finite group G is *p-solvable* if it has a chain of normal subgroups such that the respective factor groups are either p-groups or groups of order not divisible by p. The Glauberman correspondence hence is the key tool to prove the McKay conjecture in p-solvable groups. And the general case is inspired by it.

We are left with the most difficult case, where K/N is a direct product of non-abelian simple groups of order divisible by p. The above discussion with $K \trianglelefteq G$ is indication to us what we should demand now from simple groups, or more accurately, from *quasi-simple* groups (which are the non-abelian groups S all of whose proper normal subgroups lie inside $\mathbf{Z}(S)$): we want a McKay bijection in S that has good behaviour above S, no matter where S lives as a "subnormal" subgroup. (Recall that simple groups show up in general finite groups as minimal normal subgroups, and that these are direct products of simple groups. Hence a simple group S usually appears in G as a normal subgroup of a normal subgroup.) Is there a "good" group Γ that we can choose which determines the behaviour of the character theory above S no matter where the group S lives as a subnormal subgroup? This is one of the *miracles* of the "inductive McKay condition": there is one. (This important fact is nowadays better explained by the so-called "Butterfly Lemma", by B. Späth; see, for instance, [77, Thm 10.18].)

Suppose that S is a non-abelian simple group and X is the universal covering of S. (Finite group theory ensures that given a non-abelian simple group S there is a unique quasi-simple group X such that $X/\mathbf{Z}(X) \cong S$ and whenever Y is perfect, that is $Y = [Y,Y]$, the group generated by the commutators $[y_1,y_2] = y_1^{-1}y_2^{-1}y_1y_2$ of elements $y_1, y_2 \in Y$, and $Y/\mathbf{Z}(Y) \cong S$, then $Y = X/Z$ for some $Z \leq \mathbf{Z}(X)$. The group X is called the *universal covering group* of S. For instance, the universal covering group of the simple group $\mathrm{PSL}_n(q)$ is $\mathrm{SL}_n(q)$ except when $(n,q) = (2,4), (2,9), (3,2)$ or $(4,2)$.) The "good" group Γ is the semidirect product $X \rtimes \mathrm{Aut}(X)_R$, where $R \in \mathrm{Syl}_p(X)$, and $\mathrm{Aut}(X)_R$ is the group of automorphisms α of X such that $\alpha(R) = R$.

Definition 2.4 (*Inductive McKay Condition, iMcK*)**.** Suppose that S is a non-abelian simple group and let X be its universal covering. Let $R \in \mathrm{Syl}_p(X)$ and let $A =$

$\mathrm{Aut}(X)_R$. We say that S satisfies the *inductive McKay condition* if there is an A-invariant proper subgroup $M < X$ containing $\mathbf{N}_X(R)$ and an A-equivariant bijection

$$\Omega : \mathrm{Irr}_{p'}(X) \to \mathrm{Irr}_{p'}(M)$$

such that

$$(\Gamma_\theta, X, \theta) \geq_c (\mathbf{N}_\Gamma(M)_{\Omega(\theta)}, M, \Omega(\theta)) \qquad \text{for every } \theta \in \mathrm{Irr}_{p'}(X).$$

We shall not explain here in full detail the $\geq_c$ relation (introduced in [79]) between character triples, which involves the so-called projective representations. But essentially it is the exact way of saying that the character theory of Γ above θ is "equivalent" to the character theory of $\mathbf{N}_\Gamma(M)$ above $\Omega(\theta)$, as we have discussed above. Many times the group M in the Inductive McKay Condition is $\mathbf{N}_X(R)$ (as the McKay conjecture demands); but allowing for a possibly different M gives more freedom to the simple group theorists to find an intermediate group with which it is more convenient to work. The Inductive McKay condition as stated here is due mainly to B. Späth and it is equivalent to the original one given by Isaacs, Malle, and Navarro [47].

The main result now is:

Theorem 2.5 (Isaacs–Malle–Navarro, 2007)**.** *Suppose that all the non-abelian finite simple groups satisfy the Inductive McKay Condition. Then the McKay conjecture is true.*

It is worth mentioning that the iMcK can be stated for any finite group and that one can prove that iMcK for quasi-simple groups implies iMcK for any finite group. (See Theorem 2.10 in [24] and [91]). So, for McKay at least, Dade's dream of a conjecture reducing to the same statement on (close to) simple groups is reality.

Later on, B. Späth modified the Inductive McKay Condition, in order to accomodate for Brauer blocks [99]. She proved the analogue of Theorem 2.5 for the Alperin–McKay conjecture. Perhaps it was surprising that she and the second author later proved that the validity of the Inductive Alperin–McKay Condition implied Brauer's Height Zero conjecture ([79], cf. Theorem 5.6)! The main result up to date on this is L. Ruhstorfer's proof of the Alperin–McKay condition for $p = 2$. This, therefore, proved the Alperin–McKay conjecture for $p = 2$ and at the same time Brauer's Height Zero conjecture for $p = 2$ ([94]). The proof of Brauer's Height Zero conjecture for odd primes turned out to be a totally different story, see §5.

3. Proving the inductive McKay condition for simple groups

The first simple groups for which the Inductive McKay Condition was shown were the alternating and sporadic groups [63]. Then Späth [98] could settle the case of groups of Lie type in their defining characteristic, extending work of Brunat [20] and building on a result of Maslowski [70] which uses results of Steinberg on semisimple elements in linear algebraic groups. The main difficulty turned out to occur for the simple groups of Lie type at primes p *different* from their defining characteristic. For many purposes, these are best studied as finite versions of connected reductive linear algebraic groups. The investigation of the Inductive McKay Condition led to several interesting and difficult problems on the representation theory of such groups.

If G is the universal covering group of a simple group of Lie type, then by a result of Steinberg, up to finitely many exceptions (see e.g. [69, Tab. 24.3]) there exists a simple linear algebraic group $\mathbf{G}$ of simply connected type over the algebraic closure of a finite field $\mathbb{F}_q$ and a Steinberg endomorphism $F : \mathbf{G} \to \mathbf{G}$ such that $G = \mathbf{G}^F$ is the finite group of fixed points in $\mathbf{G}$ under F, a so-called *finite reductive group*. (We refer to [69, Chap. 3] for a first introduction).

Now there are sixteen infinite families of simple groups of Lie type, some of them depending on two parameters like the projective special linear groups $\mathrm{PSL}_n(q)$ for $n \geq 2$ an integer and q any prime power. For any fixed prime p their Sylow p-structures behave quite erratically when q varies; it is therefore of fundamental importance to devise a *generic* approach if one wants to prove something like the inductive conditions for all of them simultaneously.

Example 3.1. Consider the general linear groups $G = \mathrm{GL}_n(q)$ for prime powers q, and for primes $p > 2$ dividing $q-1$. Let $T \leq \mathrm{GL}_n(q)$ be the *maximal torus* of diagonal matrices of G, isomorphic to $(\mathbb{F}_q^\times)^n$ and N the subgroup of monomial matrices (with a single non-zero entry in any row and column), so $N \cong T \rtimes \mathfrak{S}_n$ is an extension of T by the *Weyl group* $W = \mathfrak{S}_n$ of G. For large enough q we actually have that $N = \mathbf{N}_G(T)$.

It is easily seen that N contains the normaliser of a Sylow p-subgroup of G, so it can be chosen as the subgroup M in the inductive condition from Definition 2.4. Using the work of Green [43] one can show that $\mathrm{Irr}_{p'}(G)$ consists precisely of the constituents of all induced characters

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta) := \tilde{\theta}^G := (\mathrm{Infl}_T^B(\theta))^G \qquad \text{for } \theta \in \mathrm{Irr}(T),$$

where θ is considered as a (linear) character $\tilde{\theta}$ of the *Borel subgroup* B of upper triangular matrices, via the natural map $B \to T$. (This process is called *Harish-Chandra induction*.) It is well-known that the constituents of $R_{\mathbf{T}}^{\mathbf{G}}(1_T)$, for example, are in bijection with the irreducible characters of W, and hence parametrised independently of p and q (subject to $p|(q-1)$), and by Clifford theory, the p'-characters of N above 1_T are also parametrised by $\mathrm{Irr}(W)$. Thus, we obtain a natural bijection Ω above M as desired.

The proper setting to generalise the above example to all primes and all types turns out to be the one of Sylow d-theory, first introduced in [17]. The basic objects are Sylow d-tori of $\mathbf{G}$. Here an F-stable torus $\mathbf{T} \leq \mathbf{G}$ is called a *d-torus* if it splits over $\mathbb{F}_{q^d}$ and no F-stable proper subtorus of $\mathbf{T}$ splits over any smaller field. A d-torus of maximal possible dimension in $\mathbf{G}$ is called a *Sylow d-torus*. Such Sylow d-tori are unique up to G-conjugacy and satisfy an analogue of the Sylow theorems (see [17]). In Example 3.1, the torus T is the F-fixed points of a Sylow 1-torus of GL_n. The automiser $W_d := \mathbf{N}_G(\mathbf{T}_d)/\mathbf{C}_G(\mathbf{T}_d)$ of a Sylow d-torus $\mathbf{T}_d$ is called a *relative Weyl group*. These are always finite complex reflection groups, and they do not depend on the underlying field size q of G. In Example 3.1 we have $W_1 = \mathfrak{S}_n$.

We relate this to primes as follows. For a prime p not dividing the underlying field size q of G, let

$$d = d_p(q) := \text{multiplicative order of } q \begin{cases} \text{modulo } p & \text{if } p \text{ is odd,} \\ \text{modulo } 4 & \text{if } p = 2. \end{cases}$$

It is this parameter which governs the Sylow structure (and in fact, as we will see, also the block structure) of G with respect to p; indeed, the following result [62, Thms. 5.14 and 5.19] shows that in most cases we may choose $M := \mathbf{N}_G(\mathbf{T}_d)$ as the intermediary subgroup occurring in the Inductive McKay Condition (Definition 2.4):

Theorem 3.2 (Malle (2007))**.** *Let $\mathbf{G}$ be simple, defined over $\mathbb{F}_q$ with corresponding Frobenius map $F : \mathbf{G} \to \mathbf{G}$ and let $G := \mathbf{G}^F$. Let $p \nmid q$ be a prime divisor of $|G|$, and set $d = d_p(q)$. Then the normaliser $\mathbf{N}_G(\mathbf{T}_d)$ of a Sylow d-torus $\mathbf{T}_d$ of G contains the normaliser of a Sylow p-subgroup of G unless one of the following holds:*

(a) *$p = 3$, and $G = \mathrm{SL}_3(q)$ with $q \equiv 4, 7 \pmod 9$, $G = \mathrm{SU}_3(q)$ with $q \equiv 2, 5 \pmod 9$, or $G = G_2(q)$ with $q \equiv 2, 4, 5, 7 \pmod 9$; or*

(b) *$p = 2$, and $G = \mathrm{Sp}_{2n}(q)$ with $n \geq 1$ and $q \equiv 3, 5 \pmod 8$.*

In particular, with this choice M only depends on d, but not on the precise structure of a Sylow p-subgroup or the Sylow p-normaliser, which makes a uniform, generic argument feasible. The four exceptional series in Theorem 3.2 can be dealt with separately (see [64]), and we will ignore them in the sequel.

We also need to understand $\mathrm{Irr}_{p'}(G)$. In his monumental work [60] George Lusztig has obtained a parametrisation of the irreducible characters of G in terms of combinatorial data related to a Langlands dual group $\mathbf{G}^*$. His Jordan decomposition of characters provides a reduction of many questions to the so-called *unipotent characters* of G (and of groups of strictly smaller ranks) and in particular gives a degree formula. From this, it should in principle be possible to describe $\mathrm{Irr}_{p'}(G)$.

Adapted to the Sylow d-theory, M. Broué, J. Michel and the first author [18] introduced a *d-Harish-Chandra theory*, further developed by the first author [61, 62], which not only is fundamental in the understanding of block theory (see Section 5), but also turns out to give perfect control of p-parts of character degrees. Using this, [62] gives a parametrisation of $\mathrm{Irr}_{p'}(G)$ in terms of combinatorial data related to the relative Weyl group W_d of a Sylow d-torus $\mathbf{T}_d$ of G. In a delicate Clifford theoretic analysis Späth [96, 97] has shown that $\mathrm{Irr}_{p'}(\mathbf{N}_G(\mathbf{T}_d))$ can be parametrised by the same combinatorial objects as for $\mathrm{Irr}_{p'}(G)$, thus completing the proof of:

Theorem 3.3 (Malle (2007) and Späth (2010))**.** *Let $\mathbf{G}$ be simple, defined over $\mathbb{F}_q$ with corresponding Frobenius map $F : \mathbf{G} \to \mathbf{G}$ and let $G := \mathbf{G}^F$. Let $p \nmid q$ be a prime divisor of $|G|$, $d = d_p(q)$, and assume that we are not in one of the exceptions (a) or (b) of Theorem 3.2. Then there is a bijection*

$$\Omega : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(\mathbf{T}_d)) \text{ with } \Omega(\chi)(1) \equiv \pm\chi(1) \pmod p \text{ for } \chi \in \mathrm{Irr}_{p'}(G).$$

So, moreover, we obtain a bijection satisfying degree congruences as predicted by Isaacs and the second author in [49].

Now, what's missing for proving the inductive McKay condition? Equivariance and Clifford theory! Both are much easier to obtain when the outer automorphism group of the simple group is cyclic, as is the case, for example, when $G = \mathrm{Sp}_{2n}(q)$ is a symplectic group with q even, but also when $G = E_8(q)$. (Recall that if G is a group, then $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$, where $\mathrm{Inn}(G)$ are the automorphisms of G induced by conjugation.) Consequently these were the first families that could be handled in non-defining characteristic (see [22]):

Theorem 3.4 (Cabanes–Späth (2013)). *Let S be simple of Lie type such that* $\mathrm{Out}(S)$ *is cyclic. Then the bijection in Theorem 3.3 can be made equivariant. In particular, S then satisfies the Inductive McKay Condition.*

In general, we need to solve the following hard problem: For G quasi-simple of Lie type, determine the action of $\mathrm{Aut}(G)$ on $\mathrm{Irr}(G)$. Recall that for such G, $\mathrm{Out}(G)$ is made up of diagonal, graph and field automorphisms (see e.g.[69, Thm. 24.24]):

(1) diagonal automorphisms are induced, e.g., by the embedding of $\mathrm{SL}_n(q)$ into $\mathrm{GL}_n(q)$,
(2) graph automorphisms come from the Dynkin diagram (e.g., the transpose-inverse automorphism for $\mathrm{SL}_n(q)$, $n \geq 3$, or the action of $\mathrm{GO}^+_{2n}(q)$ on $\mathrm{SO}^+_{2n}(q)$),
(3) field automorphisms come from automorphisms of the field $\mathbb{F}_q$ over which G is defined.

The worst case, in the sense that the structure of the outer automorphism group is most complicated, occurs for $G = \mathrm{Spin}^+_8(q)$ with q odd; here $\mathrm{Out}(G) \cong \mathfrak{S}_4 \times C_f$, where $q = r^f$ for r the characteristic of $\mathbb{F}_q$.

To deal with the Clifford theory when outer automorphism groups are non-cyclic, Späth developed the following approach. Let $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a *regular embedding*, that is, $\tilde{\mathbf{G}}$ is a connected reductive group with a connected centre and the same derived subgroup as $\mathbf{G}$. For example, the inclusion of SL_n in GL_n is a regular embedding. The Frobenius endomorphism F extends to a Frobenius morphism, also denoted F, of $\tilde{\mathbf{G}}$. By definition, $\tilde{G} = \tilde{\mathbf{G}}^F$ induces all inner-diagonal automorphisms of G. Denote by D the group of graph and field automorphisms of $\tilde{G}$. Späth [98] showed the following, where P denotes a Sylow p-subgroup of G, $M \geq P$ is as constructed in Theorem 3.2 and $\tilde{M} := M\mathbf{N}_{\tilde{G}}(P)$:

Theorem 3.5 (Criterion of Späth). *Assume there is an* $\mathrm{Aut}(G)_P$*-equivariant bijection* $\tilde{\Omega} : \mathrm{Irr}_{p'}(\tilde{G}) \to \mathrm{Irr}_{p'}(\tilde{M})$ *compatible with multiplication by* $\mathrm{Irr}(\tilde{G}/G)$*. If*

- *for every $\tilde{\chi} \in \mathrm{Irr}_{p'}(\tilde{G})$ there is $\chi \in \mathrm{Irr}_{p'}(G|\tilde{\chi})$ with*

$$(\tilde{G} \rtimes D)_\chi = \tilde{G}_\chi \rtimes D_\chi$$

  and χ extends to $(G \rtimes D)_\chi$, and
- *the analogous condition holds on the local side, that is, for $\tilde{M}$,*

then $G/\mathbf{Z}(G)$ satisfies the Inductive McKay Condition for p.

Here, the index χ denotes the stabiliser of χ in the ambient group. So, for $G = \mathrm{SL}_n(q)$, for example, one uses the bijection for $\tilde{G} = \mathrm{GL}_n(q)$ (see Example 3.1), then has to check the stabiliser condition and finally prove extendibility. It turns out that the situation for the prime 2 is special: here except in types A_n and C_n, any character $\chi \in \mathrm{Irr}_{2'}(G)$ is a constituent of a Harish-Chandra induced character $R^{\mathbf{G}}_{\mathbf{T}}(\theta) = (\mathrm{Infl}^B_T(\theta))^G$ as in Example 3.1, whose decomposition is controlled by the Iwahori–Hecke algebra of the relative Weyl group $W(\theta) = N_G(T, \theta)/T$. This does allow one to deduce the action of $\mathrm{Aut}(G)$ on $\mathrm{Irr}_{2'}(G)$; further considerations are needed on the local side, and extendibility has to be guaranteed. This lead to the solution for the prime $p = 2$, that is, for McKay's original first conjecture, in [68]:

Theorem 3.6 (Malle–Späth (2016)). *The McKay Conjecture 2.1 holds for the prime $p = 2$.*

By far the most complicated case turned out, not surprisingly, to be the groups of type D_n. The reasons for this are many-fold. These are the only infinite series for which the universal covering groups can have non-cyclic centre. These universal covering groups are the so-called spin groups for which no convenient natural matrix representation exists. Last but not least, the outer automorphism groups are far from being cyclic with all three types of automorphisms being present. It already constituted the technically by far most difficult case in Lusztig's construction of the irreducible characters.

In a veritable tour-de-force, Späth [102, 103] and Cabanes–Späth [23, 24] finally managed to settle this case:

Theorem 3.7 (Cabanes–Späth (2024)). *The McKay conjecture holds for all finite groups and all primes.*

The proof uses delicate properties of Lusztig's parametrisation of characters. Moreover, it replaces the group M discussed above, in some cases, by an even larger, more manageable subgroup which lends itself more easily to induction. A detailed description of the necessary steps and encountered difficulties is given in the introduction to [24].

4. Block Theory

4.1. **Blocks.** If G is a finite group, $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_k\}$ is the set of irreducible characters of G, and $\mathrm{Cl}(G) = \{K_1, \dots, K_k\}$ is the set of conjugacy classes of G, we form the $k \times k$ matrix $X(G) = (\chi_i(x_j))$, where $x_j \in K_j$. This is called the *character table* of G. Of course, this complex regular matrix is unique only up to permutations of rows and columns. The following is the character table of $\mathfrak{A}_5$. (Above it, we write representatives of the conjugacy classes of the group. We always write first the *trivial* character $1_G : G \to \mathbb{C}^\times$, defined by $g \mapsto 1$.)

	(1)	(12)(34)	(123)	(12345)	(13524)
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

Brauer wanted to study to what amount the character values determine the structure or at least important features of the group. For instance, the sum of the squares of the first column in the character table is the order of the group, as we have mentioned. More generally,

$$\sum_{\chi \in \mathrm{Irr}(G)} |\chi(x)|^2 = |\mathbf{C}_G(x)|$$

for all $x \in G$, where $\mathbf{C}_G(x)$ is the subgroup of the elements of G that commute with x. (This follows from the so called Second Orthogonality Relation.) Therefore the sizes of the conjugacy classes of G, the numbers $|G|/|\mathbf{C}_G(x)|$, are found in the character table. Since it is true that given a representation $\mathcal{X}$ of G and $g \in G$, we have that $\mathcal{X}(g)$ is the identity matrix if and only if $\chi(g) = \chi(1)$, it is not difficult to show that the character table determines the lattice of normal subgroups together with their orders. In the character table of $\mathfrak{A}_5$, we do not see $1 \neq g \in G$ such that $\chi(g) = \chi(1)$ because $\mathfrak{A}_5$ is simple (i.e., it does not have proper non-trivial normal subgroups). The character table of a group easily determines many global properties of the group: whether it is simple, solvable, nilpotent, etc. A more delicate issue is whether the character table determines local properties.

Brauer wrote in 1963 a famous article ([15]) in which he listed questions and open problems in the Representation Theory of Finite Groups. In many instances, this paper has served as a guide in our subject. In Problem 12, for instance, he asked if the character table determines if the Sylow p-subgroups are abelian or not. Problem 23 (the Height Zero Conjecture) would give a very elegant solution to this problem, as we will see later.

As we have said in the Introduction, ordinary character theory is developed using Wedderburn's theory of finite dimensional algebras over $\mathbb{C}$. If G is a finite group and R is any commutative ring, the R-algebra RG is the R-module of formal sums over G with coefficients in R with the multiplication of G extended linearly. The centre $\mathbf{Z}(RG)$ of RG, the elements of RG that commute with all elements of RG, is easily seen to be the R-linear combinations of the elements

$$\hat{K} = \sum_{x \in K} x \,,$$

where K is a conjugacy class of G. If we write

$$\mathbb{C}G = I_1 \oplus \cdots \oplus I_k \tag{3}$$

as the sum of two-sided indecomposable ideals, then Wedderburn's theory tells us that $I_i \cong \mathrm{Mat}_{n_i}(\mathbb{C})$, where the n_i's are the degrees of the irreducible characters of G, hence the formula

$$|G| = \sum_{i=1}^{k} \chi_i(1)^2$$

which we have already mentioned. Also, we see, by calculating centres in Equation (3), that $k = |\mathrm{Cl}(G)|$. Furthermore, if we write

$$1 = e_1 + \cdots + e_k \,,$$

where $e_i \in I_i$, then it can be proved that

$$e_i = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \chi_i(g^{-1}) g \qquad \text{for } 1 \leq i \leq k,$$

are the primitive central idempotents of $\mathbb{C}G$, where $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$. Sometimes we write $e_i = e_{\chi_i}$. The orthogonality relations (and more) easily follow from the

equation

$$e_i e_j = \delta_{i,j} e_i \qquad \text{for } 1 \le i, j \le k.$$

Things become more subtle if we substitute $\mathbb{C}$ with other fields or other rings, which is what Brauer did.

Let us fix a prime p for the rest of the section. Recall that if $\mathbf{R}$ is the ring of algebraic integers in $\mathbb{C}$ (that is, the roots of the monic polynomials with integer coefficients), we have chosen a maximal ideal M of $\mathbf{R}$ containing p, and we work in the ring

$$\mathcal{O} = \{\alpha/\beta \mid \alpha \in \mathbf{R}, \beta \in \mathbf{R} - M\},$$

of M-local integers. The ring $\mathcal{O}$ has a unique maximal ideal

$$\mathcal{J} = \{\alpha/\beta \mid \alpha \in M, \beta \in \mathbf{R} - M\},$$

and $\mathbb{F} = \mathcal{O}/\mathcal{J}$ is an algebraic closure of the field of p elements. In particular, the natural projection $\alpha \mapsto \alpha^* := \alpha + \mathcal{J}$ gives a ring homomorphism $^* : \mathcal{O} \to \mathbb{F}$, and a bridge from characteristic 0 to characteristic p. The ring $\mathcal{O}$ is rich enough so that complex representations can be realised over $\mathcal{O}$ — this is proved using some ring theory — and at the same time it can be used to relate them to characteristic p representations. We have a ring homomorphism

$$^* : \mathcal{O}G \to \mathbb{F}G, \qquad \sum_{g \in G} \alpha_g g \mapsto \sum_{g \in G} (\alpha_g)^* g\,.$$

In the Introduction, we decomposed

$$\mathcal{O}G = B_1 \oplus \cdots \oplus B_t \tag{4}$$

as a direct sum of indecomposable two-sided ideals, and defined $\mathrm{Bl}(G) = \{B_1, \ldots, B_t\}$ as the set of Brauer p-blocks of G. If we write

$$1 = f_{B_1} + \cdots + f_{B_t}\,,$$

with $f_{B_i} \in B_i$, we call the f_{B_i} the *block idempotents* (in $\mathcal{O}G$). This decomposition canonically partitions the complex characters

$$\mathrm{Irr}(G) = \bigcup_{B \in \mathrm{Bl}(G)} \mathrm{Irr}(B)\,.$$

In fact,

$$f_{B_i} = \sum_{\chi \in \mathrm{Irr}(B_i)} e_\chi\,.$$

The *principal block* is the block that contains the trivial character 1_G. If we apply the map * to equation (4), we also have that

$$\mathbb{F}G = (B_1)^* \oplus \cdots \oplus (B_t)^* \tag{5}$$

is a sum of indecomposable two-sided ideals. (With some abuse of language, we also say that $(B_1)^*, \ldots, (B_t)^*$ are the p-blocks of G.) The

$$e_B := (f_B)^* \in \mathbf{Z}(\mathbb{F}G), \qquad \text{for } B \in \mathrm{Bl}(G),$$

are the primitive central idempotents of $\mathbb{F}G$. The idempotent e_B is also the identity of the $\mathbb{F}$-algebra B^*. Using (5), it is easy to check that there are exactly t algebra homomorphisms

$$\lambda_{B_i} : \mathbf{Z}(\mathbb{F}G) \to \mathbb{F}$$

and that these satisfy

$$\lambda_{B_i}(e_{B_j}) = \delta_{i,j} \qquad \text{for } 1 \leq i, j \leq t.$$

So far we have seen that p-blocks are the two-sided ideals into which the algebra $\mathbb{F}G$ (or the ring $\mathcal{O}G$) decomposes. These are determined by the primitive central idempotents of $\mathbb{F}G$ (or $\mathcal{O}G$) since

$$B_i = \mathcal{O}G f_{B_i} \quad \text{and} \quad (B_i)^* = \mathbb{F}G e_{B_i}\,.$$

However, with this definition, how do we distribute in practice the irreducible characters of $\mathfrak{A}_5$ into, say, 2-blocks? Do we have to calculate ideals? How does the computational group theory software GAP [38], say, compute the blocks from the character tables so easily? We need to do some more work in order to answer this.

First of all, if R is a commutative ring and G is a finite group, it is convenient to R-linearly extend any R-representation $\mathcal{X} : G \to \mathrm{GL}_m(R)$ to

$$\mathcal{X} : RG \to \mathrm{Mat}_m(R)$$

so that we can use the theory of R-algebras. If $R = \mathbb{C}$, it is a fact that $\mathcal{X}$ affords an irreducible character if and only if $\mathcal{X}$ is surjective. Therefore if $\mathcal{X}$ affords $\chi \in \mathrm{Irr}(G)$ and $K \in \mathrm{Cl}(G)$, we have that $\mathcal{X}(\hat{K})$ commutes with all the matrices. Hence

$$\mathcal{X}(\hat{K}) = \omega_\chi(\hat{K}) I$$

is a scalar matrix. From this, it easily follows that $\omega_\chi : \mathbf{Z}(\mathbb{C}G) \to \mathbb{C}$ is an algebra homomorphism. Taking traces, we find the values of these scalars:

$$\omega_\chi(\hat{K}) = \frac{\chi(x)|K|}{\chi(1)} \qquad \text{for any } x \in K.$$

Using that $\hat{K}\hat{L}$ is an integer linear combination of the basis $\{\hat{M} \mid M \in \mathrm{Cl}(G)\}$ for $K, L \in \mathrm{Cl}(G)$, it is not difficult to obtain the fundamental fact that

$$\omega_\chi(\hat{K}) \in \mathbf{R}$$

is an algebraic integer. Hence, the $\mathbb{F}$-linear map

$$\lambda_\chi : \mathbf{Z}(\mathbb{F}G) \to \mathbb{F}, \qquad \hat{K} \mapsto \omega_\chi(\hat{K})^*,$$

defines an algebra homomorphism. Thus λ_χ should be one of the λ_{B_i}'s that appeared before. Indeed Brauer proved that $\chi, \psi \in \mathrm{Irr}(G)$ lie in the same p-block if and only if $\lambda_\chi = \lambda_\psi$. And we write $\lambda_B = \lambda_\chi$, where $\chi \in \mathrm{Irr}(B)$.

Let us reveal now how we distribute the characters of $\mathfrak{A}_5$ into 2-blocks, in a practical way. We do not have to dig much into the theory to show that in order to have that $\lambda_\chi = \lambda_\psi$, it is enough to check that $\lambda_\chi(\hat{K}) = \lambda_\psi(\hat{K})$ for all conjugacy classes K of G

consisting of elements of order not divisible by p. These are called the *p-regular* classes of G. Hence, we have to decide if

$$\left(\frac{\chi(x)|K|}{\chi(1)}\right) \equiv \left(\frac{\psi(x)|K|}{\psi(1)}\right) \pmod{M},$$

where M is the chosen maximal ideal of $\mathbf{R}$ containing p, for elements $x \in G$ of order prime to p. Now since $\chi(x)$ and $\psi(x)$ are sums of $o(x)$-th roots of unity, where $o(x)$ is the order of x, notice that the above congruence is realised in the $o(x)$-th cyclotomic field. Now we use some number theory on the ramification of prime ideals, to deduce that we only need to decide whether

$$\left(\frac{\chi(x)|K|}{\chi(1)} - \frac{\psi(x)|K|}{\psi(1)}\right) \in p\mathbf{R}.$$

And this can easily be checked (especially with the help of a computer!). Going back to the character table of $\mathfrak{A}_5$, we readily find that $\{\chi_1, \chi_2, \chi_3, \chi_5\}$ form a 2-block (the *principal* block) and that $\{\chi_4\}$ is another 2-block.

There is yet another characterisation of blocks which does not leave the character table and that does not lie too deep. If G^0 denotes the set of p-regular elements of G and $\chi, \psi \in \mathrm{Irr}(G)$, we define the linking $\chi \leftrightarrow \psi$ if and only if

$$\sum_{x \in G^0} \chi(x)\psi(x^{-1}) \neq 0.$$

This is a "truncated" scalar product on the p-regular elements. It can be proved that the connected components under the $\leftrightarrow$ linking are exactly the p-blocks. This is the easiest "definition" of blocks, if we do not want to leave the character table.

4.2. **Brauer Characters.** So far we have avoided talking about representations of groups over fields of characteristic p, which, of course, are behind the scenes, and the Brauer blocks introduced in the previous section are one way to manifest the relationships between them and complex representations.

Recall that we choose a maximal ideal M of the ring of algebraic integers $\mathbf{R}$, $\mathcal{O}$ is the localisation of $\mathbf{R}$ at M, $^* : \mathcal{O} \to \mathbb{F} = \mathcal{O}/\mathcal{J}$ is the canonical ring epimorphism, and $\mathbb{F}$ is an algebraic closure of the field of p elements. If $\chi \in \mathrm{Irr}(G)$, as we have mentioned, then χ can be afforded by some $\mathcal{O}$-representation, and therefore $\chi^* : G \to \mathbb{F}$ is the trace of some $\mathbb{F}$-representation. But this is not the best idea. In a field of characteristic p, like $\mathbb{F}$, there are no non-trivial p-power order roots of unity. Hence, if $\epsilon \in \mathbb{C}$ is such that $\epsilon^p = 1$, then $\epsilon^* = 1$. In particular, if G is a p-group and $\chi(1) > 1$, then $\chi^* = 0$. And we do not want the zero function to be a character! (For the more interested reader, there are also examples of non-abelian simple groups with this pathology of having complex characters such that $\chi^* = 0$.) Brauer, who else!, had an idea to overcome this problem. First of all, when dealing with characteristic p representations, he only considered G^0, the set of p-regular elements of the group G. If U is the subgroup of $\mathbb{C}^\times$ consisting of the roots of unity of order not divisible by p, then one can check that U is isomorphic to $\mathbb{F}^\times$ via *. Therefore, given an $\mathbb{F}$-representation $\mathcal{X}$ of G of degree m and given an element $g \in G$ of order not divisible by p, we can find a unique subset

$\{\epsilon_1, \dots, \epsilon_m\} \subseteq U$ such that $(\epsilon_1)^*, \dots, (\epsilon_m)^*$ are the eigenvalues of the matrix $\mathcal{X}(g)$. The complex function

$$\varphi : G^0 \to \mathbb{C}, \qquad g \mapsto \varphi(g) := \epsilon_1 + \cdots + \epsilon_m,$$

is the *Brauer character* of G afforded by $\mathcal{X}$. A representation

$$\mathcal{X} : G \to \mathrm{GL}_m(\mathbb{F})$$

is *irreducible* if its image is not similar to matrices of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Contrary to the case where the characteristic of the field is zero, an $\mathbb{F}$-representation is not necessarily a direct sum of irreducible representations (it is not *completely reducible*), and therefore Brauer characters do not uniquely determine non-irreducible representations. However, two irreducible $\mathbb{F}$-representations with the same Brauer character are similar. The set $\mathrm{IBr}(G)$ of the Brauer characters afforded by $\mathbb{F}$-irreducible representations plays the role of $\mathrm{Irr}(G)$ with respect to complex representations. Indeed, Brauer proved that the number of non-similar irreducible representations of G over $\mathbb{F}$ is $l(G)$, the number of conjugacy classes of G consisting of p-regular elements. This is the analogous result to $|\mathrm{Irr}(G)| = k(G)$. Not every result has an analogue: for instance, the degrees of the irreducible Brauer characters do not necessarily divide the order of the group. Not only that: for instance $\mathfrak{A}_7$ has an irreducible 3-Brauer character of degree 13, a prime which does not even divide the order of the group. What is true is that $\mathrm{IBr}(G)$ is a basis of the space of complex valued class functions $G^0 \to \mathbb{C}$. We see that if $\chi \in \mathrm{Irr}(G)$, then the restriction of χ to G^0 can uniquely be written as

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi\,,$$

where the non-negative integers are the famous p-modular *decomposition numbers*. (Calculating the decomposition numbers of groups of Lie type or symmetric groups is a monumental ongoing problem.)

If B is a p-block, then $\mathrm{IBr}(B)$ are the irreducible Brauer characters φ of G for which there is some $\chi \in \mathrm{Irr}(B)$ with $d_{\chi\varphi} \neq 0$. If $k(B)$ is the number of ordinary irreducible characters in B, then $l(B)$ is the number of irreducible Brauer characters in B.

It is not often the case that χ^0 is an irreducible Brauer character of G (i.e., that χ modulo p affords an irreducible $\mathbb{F}$-representation). As proved by Brauer, an exception is when χ has *p-defect zero*, which we recall is when $\chi(1)_p = |G|_p$, cf. Theorem 4.5. These characters will play an important role in subsequent discussions, in particular in §6.1.

4.3. **Defect Groups.** Our next objective is to associate to any p-block B of G a conjugacy class of p-subgroups of G called the *defect groups*. This is the first time that we go from global to local in block theory. And the tool that we use is quite elementary. If P is any p-subgroup of G (that is a subgroup of order a power of p), using that $\mathbb{F}$ has characteristic p, it is easy to prove that the map

$$\mathrm{Br}_P : \mathbf{Z}(\mathbb{F}G) \to \mathbf{Z}(\mathbb{F}C)$$

which sends $\hat{K} \mapsto \widehat{K \cap C}$ defines an algebra homomorphism, where $C = \mathbf{C}_G(P)$, $K \in \mathrm{Cl}(G)$, and $\widehat{K \cap C} := 0$ if $K \cap C$ is empty. (If $X \subseteq G$, we use $\hat{X} = \sum_{x \in X} x$.) This follows from a counting argument of the action of P on K. If B is a p-block, Brauer proved that there is a unique conjugacy class of p-subgroups D of G such that $\mathrm{Br}_P(e_B) \neq 0$ if and only if P is contained in some G-conjugate of D. The G-conjugates of D are called the *defect groups of* B, and if $|D| = p^d$, then d is called the *defect* of the block. The set of blocks of G with defect group D is denoted $\mathrm{Bl}(G|D)$. In a sense, the defect of the block is a measure for its complexity and of how "far" the Brauer characters in the block are from the block's complex characters. In particular, a block of defect zero consists of exactly one complex character and one Brauer character (see Theorem 4.5 below).

Once defect groups were discovered, Brauer started to find deep connections between the characters in the block and the defect group. Brauer's first main theorem is probably the first global/local theorem in finite group representation theory. (See, for instance, [75, Thm 4.12].)

Theorem 4.1 (First Main)**.** *If D is any p-subgroup of G, then there is a natural bijection $b \mapsto b^G$ from $\mathrm{Bl}(\mathbf{N}_G(D)|D)$ to $\mathrm{Bl}(G|D)$.*

To find the exact relationship between the two $\mathbb{F}$-algebras b and b^G remains an open problem. If D is abelian, M. Broué conjectured in 1988 ([16]) that b and b^G are equivalent as derived categories. (And this would explain most of the global/local conjectures, but only in the D abelian case. If D is not abelian, the relationship between b and b^G remains a mystery.)

The p-parts of the degrees of the characters in p-blocks are also related to the defect groups via the following theorem. (See, for instance, [75, Thm 6] with [75, Def. 3.15].)

Theorem 4.2. *Let $B \in \mathrm{Bl}(G|D)$. Then $\min\{\chi(1)_p \mid \chi \in \mathrm{Irr}(B)\} = |G : D|_p$.*

This allows us to define the heights of the characters in a p-block of defect group D: if $|D| = p^d$, $|G|_p = p^a$, and $\chi \in \mathrm{Irr}(B)$, then

$$\chi(1)_p = p^{a-d+h}\,,$$

where $h = h(\chi) \geq 0$ is the *height* of χ. We see now that the Height Zero Conjecture is central to the theory.

Conjecture 4.3 (Brauer's Height Zero Conjecture, BHZC)**.** *Let $B \in \mathrm{Bl}(G|D)$. Then $h(\chi) = 0$ for all $\chi \in \mathrm{Irr}(B)$ if and only if D is abelian.*

If B_0 is the principal block of G, then $1_G \in \mathrm{Irr}(B_0)$, and we have $\min\{\chi(1)_p \mid \chi \in \mathrm{Irr}(B_0)\} = 1$. Hence, the defect groups of B_0 are the Sylow p-subgroups of G. (As we mentioned in the Introduction, this happens in every block that contains an irreducible character of degree not divisible by p.) Since the distribution of characters in blocks can be detected in the character table, the Height Zero Conjecture provided an elegant solution to Brauer's Problem 12: the character table determines if Sylow p-subgroups are abelian. (The Height Zero Conjecture for principal blocks was proven in [65].)

One of the deepest results of Brauer on Block Theory is his so-called Second Main Theorem. (See, for instance, [75, Thm 5.2].) This theorem allows us to find many zeros

in the character table, in a precise way related to defect groups, which we describe now. Recall that if $g \in G$, it is elementary to show that g can be written in a unique way as $g = xy$, where $xy = yx$, $o(x)$ is a power of p, and p does not divide $o(y)$. We write $x = g_p$, as the *p-part of* g. For a proof of the following theorem of Brauer see, for instance, [75, Cor 5.9].

Theorem 4.4. *Let $B \in \mathrm{Bl}(G|D)$ and $\chi \in \mathrm{Irr}(B)$. Then $\chi(g) = 0$ if g_p does not lie in any G-conjugate of D.*

The case where $D = 1$, that is, blocks of defect zero, constitutes a singularity of the theory, and has a great impact. Notice that the First Main Theorem 4.1 says nothing in this case, so blocks of defect zero have to be studied by other means. For the time being, we recall the following characterisation due to Brauer. (See, for instance, [75, Thm 3.18].)

Theorem 4.5. *Let $B \in \mathrm{Bl}(G|D)$ and $\chi \in \mathrm{Irr}(B)$. Then the following are equivalent:*

(i) $\mathrm{Irr}(B) = \{\chi\}$.
(ii) $D = 1$.
(iii) $\chi(1)_p = |G|_p$.
(iv) $\chi(x) = 0$ *for all non-trivial p-elements $x \in G$.*
(v) $\chi(x) = 0$ *for all $x \in G$ with non-trivial x_p.*

We understand better now why $\{\chi_4\}$ in $\mathfrak{A}_5$ constitutes a 2-block!

Perhaps this is the right time to state Brauer's other famous conjecture (Problem 20 in [15]) which has a different flavour from the central conjectures in this survey.

Conjecture 4.6 (Brauer's $k(B)$)**.** *If B is a block with defect group D then $|\mathrm{Irr}(B)| \leq |D|$.*

Although Brauer and Feit managed to prove that $|\mathrm{Irr}(B)| \leq \frac{1}{4}|D|^2 + 1$ (see [75, Thm 3.27]), Brauer's proposed inequality is still wide open. In the "easier" case where G is p-solvable, the conjecture was finally solved in [41] after monumental effort by many mathematicians.

Around 1960, J. A. Green joined modular representation theory with a fundamental paper which had important consequences ([44]). Working with a "better" ring $\mathcal{O}$, which allowed him to have a Krull–Schmidt theorem for $\mathcal{O}G$-modules (the ring of integers of the $|G|$-th cyclotomic extension over the p-adics, for instance), he associated to every indecomposable $\mathcal{O}G$-module a *vertex* (a uniquely defined conjugacy class of p-subgroups of G) and a *source*. In particular, every complex irreducible character, when afforded by an $\mathcal{O}$-representation, has a uniquely defined vertex. After work by J. G. Thompson ([106]), this was extensively used by E. C. Dade in his celebrated theory of blocks with cyclic defect groups ([26]). All of these results are used in the proof of Brauer's Height Zero Conjecture.

5. Brauer's Height Zero Conjecture

Brauer's main purpose in founding Modular Representation Theory was to obtain a deeper understanding of ordinary complex characters, which in turn, he thought,

would help in the main problem of group theory at that time: the Classification of Finite Simple Groups. As Berger and Knörr wrote in 1988 when they reduced the "if" implication of the Height Zero Conjecture: "*We should mention that, ironically, this way of proving the "if" part of Brauer's conjecture runs contrary to Brauer's intentions. He seems to have hoped to somehow structure the theory of simple groups using the conjecture; now a proof will be given using the knowledge of simple groups.*" ([8]). This is what has happened 36 years later. Our aim in this section is to tell how.

Neither of the implications in Brauer's Height Zero conjecture (BHZC from now on) is easy. And both have been proved using the CFSG. However, this is not the case for the p-solvable case of BHZC. After convenient standard reductions, the typical case of a p-block of a p-solvable group is when $\mathrm{Irr}(B) = \mathrm{Irr}(G|\theta)$, where $\theta \in \mathrm{Irr}(Z)$, $Z = \mathbf{O}_{p'}(G) \subseteq \mathbf{Z}(G)$. In this situation, the defect group of B is $P \in \mathrm{Syl}_p(G)$. If we assume that P is abelian, then some finite group theory easily implies that $P \trianglelefteq G$. A theorem of Itô [46, Thm 6.15] asserts that if $\chi \in \mathrm{Irr}(G)$ and $A \trianglelefteq G$ is abelian, then $\chi(1)$ divides $|G : A|$. Hence, if a Sylow p-subgroup is normal and abelian, $\chi(1)$ is prime to p for every $\chi \in \mathrm{Irr}(G)$, and therefore, BHZC holds for B. (The converse of this result is the Itô–Michler theorem, whose proof requires the CFSG.) To prove that having all irreducible characters in $\mathrm{Irr}(G|\theta)$ of p'-degree implies that P is abelian is quite another story. If G is p-solvable, this is the celebrated Gluck–Wolf theorem [42].

All further substantial progress on this question was made using the classification of finite simple groups. A first significant result of this sort concerned 2-blocks whose defect groups are Sylow 2-subgroups [82]:

Theorem 5.1 (Navarro–Tiep (2012))**.** *Let B be a 2-block of a finite group of maximal defect. Then Brauer's Height Zero Conjecture 4.3 holds for B.*

A crucial ingredient of Navarro and Tiep's proof of Theorem 5.1 was the theorem of Gluck and Wolf for the prime 2. The missing odd-p analogue of this seemed to constitute a major obstacle towards establishing the remaining, "only if" direction of the Height Zero Conjecture. Using the CFSG, Navarro and Tiep [83] have obtained a proof of this result, and this is essentially the proof of the Height Zero Conjecture in the case where the *generalised Fitting subgroup* $\mathbf{F}^*(G)$ of G is nilpotent:

Theorem 5.2 (Navarro–Tiep (2013))**.** *Let $N \trianglelefteq G$ be finite groups, p a prime, and $\theta \in \mathrm{Irr}(N)$ a G-invariant character. If $\chi(1)/\theta(1)$ is prime to p for all $\chi \in \mathrm{Irr}(G|\theta)$ then G/N has abelian Sylow p-subgroups.*

In the general case of arbitrary blocks and primes, Berger and Knörr, in their earlier work [8] cited above, showed the following optimal reduction of the "if" direction to the same statement for blocks of quasi-simple groups (recall that a finite group G is *quasi-simple* if G is perfect and $G/\mathbf{Z}(G)$ is simple):

Theorem 5.3 (Berger–Knörr (1988))**.** *The "if"-direction of Brauer's Height Zero Conjecture 4.3 holds for the p-blocks of all finite groups, if it holds for the p-blocks of all quasi-simple groups.*

The verification of the assumption of this reduction theorem for groups of Lie type in their defining characteristic is easy, as defect groups are either Sylow p-subgroups or trivial, and Sylow p-subgroups are non-abelian unless we are in the case of $\mathrm{PSL}_2(q)$.

To deal with all other cases, it is necessary to have a good understanding of the intricate block structure. Simple groups are mainly divided into alternating groups and groups of Lie type. Blocks of alternating groups are easily understood from the blocks of symmetric groups; after all, $\mathfrak{A}_n$ is a normal subgroup of $\mathfrak{S}_n$ of index 2, with a bit of Clifford theory for blocks. The representation theory of $\mathfrak{S}_n$ is naturally described in combinatorial terms. The complex irreducible characters of $\mathfrak{S}_n$ are labelled by the partitions of n, and two partitions belong to the same p-block if and only if they have the same *p-core* (where the latter is obtained by successively removing all the rim p-hooks from the Young diagram of the partition, see [51, §2.7]). This was a conjecture by Nakayama proved by Brauer and G. de B. Robinson. Using this, Olsson [85] managed to verify the assumption of Theorem 5.3 for all p-blocks of covering groups of alternating groups.

The study of blocks of groups of Lie type in non-defining characteristic was initiated by Fong and Srinivasan in [35] (see also [37]) who described the blocks of the finite classical groups at odd primes. Their parametrisation in [36] was already in terms of a d-Harish-Chandra theory, which was then extended to all types in the work of Broué–Malle–Michel [18] for large primes and unipotent blocks. M. Cabanes and M. Enguehard [21] generalised this to arbitrary blocks and primes $p \geq 5$, Enguehard [30] to unipotent blocks for all primes, and the final parts of the classification were achieved by Kessar–Malle [53, 54, 56].

By a careful analysis of the p-block structure of special linear and unitary groups, Blau and Ellers [9] obtained the following important result:

Theorem 5.4 (Blau–Ellers (1999))**.** *Brauer's Height Zero Conjecture 4.3 holds for all blocks of quasi-simple central factor groups of* $\mathrm{SL}_n(q)$ *and* $\mathrm{SU}_n(q)$*.*

We also mention [39, Thm 1.4] which establishes the BHZC for all finite groups of Lie type over a large enough field $\mathbb{F}_q$ and for large enough p. In general, the case of the other quasi-simple groups of Lie type could only be settled after having obtained a full parametrisation of their p-blocks, the completion of which very much relies on the results of Bonnafé, Dat, and Rouquier [12], [11]. R. Kessar and the first author [53] used this classification to complete the proof of the "if" direction of Conjecture 4.3, relying on the Berger–Knörr reduction (Theorem 5.3) and on the Blau–Ellers result (Theorem 5.4), thus offering further proof of the viability of the reduction approach to global/local conjectures:

Theorem 5.5 (Kessar–Malle (2013))**.** *Let B be a p-block of a finite group. If B has abelian defect groups, then all irreducible characters in B have height zero.*

So it remains "only" to prove the other, "only if" direction of Conjecture 4.3! Building on Theorem 5.2, Navarro and Späth [79] succeeded in proving the following reduction theorem for this direction of the conjecture:

Theorem 5.6 (Navarro–Späth (2014))**.** *The "only if"-direction of Brauer's Height Zero Conjecture 4.3 holds for all finite groups at the prime p, if*

(1) *it holds for all p-blocks of all quasi-simple groups, and*
(2) *all simple groups satisfy the inductive Alperin–McKay condition at p.*

To prove this result, they introduce and study the new notion $\geq_c$ of *central block isomorphic character triples*, which subsequently allowed Späth to reformulate the Inductive McKay Condition (see Definition 2.4), and which is also of importance in the current approach to some other conjectures in the field, including the Galois–McKay conjecture (see §6.3) and the Feit conjecture (see §6.4).

The first assumption of Theorem 5.6 was shown to hold by Kessar–Malle [55], again building on the classification of blocks of finite reductive groups described before. Thus, Brauer's height zero conjecture will follow once the inductive Alperin–McKay condition has been verified for all simple groups. This again underlines the central importance of the Alperin–McKay conjecture 2.2 in the representation theory of finite groups. As mentioned above, in the case $p = 2$ the second assumption of Theorem 5.6 has recently been established by Ruhstorfer [94], leading to the proof of the Alperin–McKay conjecture and (the "only if" direction of) BHZC, both for the prime 2.

The recent proof [66] of the "only if" direction of Conjecture 4.3 for *odd* primes, by Malle, Navarro, Schaeffer Fry, and Tiep, follows a different strategy, which we now explain. As is customary in group-theoretic proofs, let us consider a minimal counterexample G (more precisely, a counterexample G, first with $|G/\mathbf{Z}(G)|$ smallest possible, then with $|G|$ smallest possible) to the "only if" direction of the conjecture. So G has a p-block B where all characters in $\mathrm{Irr}(B)$ have height zero, but the defect groups D of B are non-abelian.

By the main result of [42], we may assume that G is not p-solvable. Using the minimality of G, we can show that $\mathbf{O}^{p'}(G) = G$, where $\mathbf{O}^{p'}(G)$ is the subgroup of G generated by its p-elements, $\mathbf{O}_p(G)$ is abelian, and $\mathbf{Z}(G) = \mathbf{O}_{p'}(G)$ is cyclic. Now, Theorem 5.2 allows us to deal with the case where $\mathbf{F}^*(G)$ is nilpotent. Hence, we may assume that G has a *component*, i.e., a quasi-simple subnormal subgroup S, and moreover $|S|$ is divisible by p. Therefore, G admits a normal subgroup $N = S_1 * S_2 * \cdots * S_n$, where $S_1 \cong S_2 \cong \ldots \cong S_n \cong S$ for some $n \geq 1$, and the conjugation action of G on N induces a transitive action on $\Omega := \{S_1, S_2, \ldots, S_n\}$.

Using induction, we may also assume that the block B *covers* a unique block e of N (that is, every irreducible constituent of the restriction to N of any $\psi \in \mathrm{Irr}(B)$ belongs to e), and then e covers a unique block b_1 of S_1. The next part of the proof is to rule out the case $n > 1$ (which usually necessitates the study of delicate equivariant and cohomological conditions in other approaches). For this one, we show that if $n > 1$ then we can find a suitable character $\theta \in \mathrm{Irr}(e)$ such that some character $\chi \in \mathrm{Irr}(B|\theta)$ has positive defect, violating the height 0 assumption. To do this, we need to find $\theta = \alpha_1 \boxtimes \alpha_2 \boxtimes \cdots \boxtimes \alpha_n \in \mathrm{Irr}(N)$ such that the inertia subgroup $I_G(\theta)$ has index divisible by p in G. To guarantee the existence of such a θ, we apply the following result, which is a consequence of [40, Thm 2]:

Theorem 5.7 ([66, Thm 2.1])**.** *Let $p > 2$ be a prime, and let $G < \mathrm{Sym}(\Omega) \cong \mathfrak{S}_n$ be a subgroup such that $G = \mathbf{O}^{p'}(G) \neq 1$. Then there is a partition $\Omega = \Delta_1 \sqcup \Delta_2 \sqcup \Delta_3$ with $|\Delta_1|, |\Delta_2| > 0$ and $|\Delta_3| \geq 0$ such that the index of $\cap_{i=1}^3 \mathrm{Stab}_G(\Delta_i)$ in G is divisible by p.*

Now, fix the partition $\Omega = \Delta_1 \sqcup \Delta_2 \sqcup \Delta_3$ obtained by this result. We can pick three characters $\beta_i \in \mathrm{Irr}(b_1)$, $1 \le i \le 3$, and then define $\alpha_j = \beta_i$ whenever $S_j \in \Delta_i$. To ensure that $I_G(\theta)$ fixes this partition (and hence p divides $|G : I_G(\theta)|$), we need the characters $\beta_1, \beta_2, \beta_3$ to belong to pairwise distinct $\mathrm{Aut}(S_1)$-blocks. This can be achieved by using the following theorem, which we believe will be useful in other situations as well:

Theorem 5.8 ([66, Thm B])**.** *Let p be an odd prime, S a quasi-simple group, and b a p-block of S with non-cyclic defect groups. Then at least one of the following holds.*
(1) $\mathrm{Irr}(b)$ *contains characters from at least three different* $\mathrm{Aut}(S)$*-orbits; or*
(2) *all ordinary characters in* $\mathrm{Irr}(b)$ *have the same degree.*

Thus we can rule out the possibility $n > 1$, as long as b_1 has non-cyclic defect groups and $\mathrm{Irr}(b_1)$ contains characters of different degrees. If $\mathrm{Irr}(b_1)$ consists only of characters of the same degree, then b_1 is nilpotent, in which case we complete the proof by using results of Broué, Külshammer and Puig ([19, 59]). If, on the other hand, b_1 has cyclic defect groups, then we argue by using a result of Koshitani and Späth [58] and [79, Thm 6.1(b)].

We have shown that all components of G are normal. Now the main result of Kessar and Malle [55] shows that the *layer* $\mathbf{E}(G)$ (i.e., the product of all components of G) is proper in G. By the choice of G, $D \cap \mathbf{E}(G)$ is abelian. Using Theorem 5.2 again, we then show that $\mathbf{O}_p(G) = 1$ and $[D, D] \le D \cap \mathbf{E}(G)$. The proof of the desired contradiction that $[D, D] = 1$ utilises the following result which controls the action of certain automorphisms on a defect group of a block in quasi-simple groups:

Theorem 5.9 ([66, Thm C])**.** *Let p be an odd prime, S a quasi-simple group such that $\mathbf{Z}(S)$ is a cyclic p'-group. Let b be a p-block of S with abelian defect group D. Suppose that σ is an automorphism of S of p-power order that fixes all the characters in* $\mathrm{Irr}(b)$*, normalises D, and stabilises a block b_D of $C_S(D)$ that induces b. Then σ acts trivially on D.*

As one may expect, the proofs of both Theorems 5.8 and 5.9 rely heavily on the Deligne–Lusztig theory of characters of finite reductive groups, and of course, on the CFSG. Is there any CFSG-independent proof of the McKay conjecture or the Brauer height zero conjecture? If it exists, such a proof will need totally new methods and ideas.

6. Future

6.1. **Alperin Weight Conjecture.** After Brauer, Finite Group Representation Theory owes a great deal to the visionary ideas of J. L. Alperin. His "Lie principle" (the idea that general finite groups can be interpreted as some kind of Lie groups [4]) surely led him to the celebrated *Alperin Weight Conjecture* (AWC), in which he borrowed the concept of weights from the representation theory of semisimple algebraic groups for an abstract finite group. Alperin's analogue of Borel subgroups were the Sylow normalisers (which as we know by now are central in the McKay conjecture). His analogue of the parabolic subgroups are the so-called "p-radical subgroups", and they are central in AWC. There is a general agreement that Alperin's weight conjecture is,

together with the Alperin–McKay conjecture, the most important open problem in our field at this time.

If G is a finite group, recall that we use $l(G)$ for the number of p-regular conjugacy classes of G. Also, we let $k_0(G)$ be the number of p-defect zero characters of G. This number can very well be 0, since p-defect zero characters are not that common. For instance, if $\mathbf{O}_p(G) > 1$ (that is, if G has non-trivial normal p-subgroups), then it is easy to prove that $k_0(G) = 0$.

Conjecture 6.1 (Alperin Weight Conjecture, 1987, [3])**.** *If G is a finite group, then*

$$l(G) = \sum_Q k_0(\mathbf{N}_G(Q)/Q)\,,$$

where Q runs over the set of G-conjugacy classes of p-subgroups of G.

If Q is a p-subgroup of G and $k_0(\mathbf{N}_G(Q)/Q) > 0$, then $\mathbf{O}_p(\mathbf{N}_G(Q)/Q) = 1$, as we said. This means that $Q = \mathbf{O}_p(\mathbf{N}_G(Q))$ is the largest normal p-subgroup of $\mathbf{N}_G(Q)$. The p-subgroups Q satisfying this condition are the *p-radical subgroups* of G. Sylow subgroups and defect groups, two of the actors in the previous conjectures, are easily proven to be p-radical.

At first sight it might look as if AWC is relating a group theoretical invariant $l(G)$ to some representation theoretical invariant $k_0(X)$, and this might be regarded as strange. But this is not exactly like this. As we know by now, AWC is really telling us that the number of $\mathbb{F}$-irreducible representations which do not come from defect zero representations, that is

$$l(G) - k_0(G) = \sum_{Q>1} k_0(\mathbf{N}_G(Q)/Q)$$

can be calculated from local subgroups. There is a wealth of reformulations of AWC, including one [52] for fusion systems (without groups!), giving an idea of its richness. P. Webb [109] called AWC a "*manifestation of something quite profound in representation theory, perhaps suggesting structure that has not yet been identified.*"

The story of AWC follows the same pattern as McKay. The blockwise version of AWC, sometimes referred to as BAWC, proposed also by Alperin in [3], asserts that, if B is a p-block, then $l(B)$ counts the weights in blocks that "induce" B. Alperin and Fong proved BAWC for symmetric groups in [5]; J. An checked many sporadic groups and classical groups [6]. It is quite a hard problem to pin down the p-radical subgroups in a given finite group. The p-solvable case of AWC is credited to Okuyama although a full first proof appeared in [48]. Following the ideas of [47], the second and third authors [81] reduced AWC to the so-called *Inductive AWC Condition*. If S is a quasi-simple group, it proposes that the set of Brauer characters $\mathrm{IBr}(S)$ is partitioned as

$$\mathrm{IBr}(S) = \bigcup_Q \mathrm{IBr}(S|Q),$$

where Q runs over the p-radical subgroups of S up to G-conjugacy, and that there are special bijections * between $\mathrm{IBr}(S|Q)$ and the *weights* (Q, γ), where $\gamma \in \mathrm{Irr}(\mathbf{N}_G(Q)/Q)$ has defect zero.

The reduction of the block version of AWC was again achieved shortly afterwards by Späth [100]. Many families of groups have been proved by now to satisfy the inductive BAWC condition (see [33] for certain groups of Lie type). We refer the reader to [34] for recent developments in verifying the so-called *Inductive BAWC Condition*. The problems that arise when checking AWC and BAWC are the classification of p-radical subgroups of quasi-simple groups together with our not complete knowledge of Brauer characters. (The knowledge of ordinary characters is of course much more advanced.)

6.2. **Dade's conjecture.** In the 1970s K. S. Brown and D. Quillen were already using simplicial complexes on the poset of p-subgroups of a finite group. Using those, R. Knörr and G. Robinson [57] came up with an exciting reformulation of AWC. If G is a finite group, let $\mathcal{C}(G)$ be the set of chains

$$C := 1 = P_0 < P_1 < \cdots < P_n\,,$$

where P_i is a p-subgroup of G. The *length* $|C|$ of the chain is n, and

$$G_C := \bigcap_i \mathbf{N}_G(P_i)$$

is the stabiliser of the chain. Notice that the stabiliser of the chain is an intersection of local subgroups, which, with some abuse of language, we can again call local. The following is the Knörr–Robinson reformulation of AWC.

Conjecture 6.2 (AWC, Knörr–Robinson reformulation)**.** *If G is a finite group, then*

$$k_0(G) = \sum_C (-1)^{|C|} k(G_C)\,,$$

where C runs over representatives of G-conjugacy classes of chains of p-subgroups of G.

It is not difficult to prove that Conjecture 6.2 holds for a finite group G if and only if AWC holds (see e.g. [77, Thm 9.17]). Notice that by considering the chain $C = 1$, the KR-reformulation is telling us that $k(G) - k_0(G)$ is "local", that is, it can be calculated from local subgroups. In fact, in [57], Knörr and Robinson also proved that $k(G) - l(G)$ was "local" for every finite group. Considering now k, k_0, l etc. as integer valued functions defined on finite groups, since

$$k - k_0 = (k - l) + (l - k_0)\,,$$

and $k - l$ is "local", we can understand the equivalence between AWC and the KR reformulation. A similar statement with blocks (although much trickier) was proven in [57] to be equivalent to the blockwise AWC.

Inspired by the KR-reformulation Dade made a bold move. Recall that the McKay conjecture only involved irreducible characters of degree not divisible by p. Can something be said about irreducible characters of arbitrary p-part? For d any non-negative integer and G a finite group, Dade defined

$$k_d(G) = |\{\chi \in \mathrm{Irr}(G) \mid \frac{|G|_p}{\chi(1)_p} = p^d\}|\,.$$

Hence, we can write

$$k - k_0 = \sum_{d>0} k_d \,.$$

Dade's conjecture posits that k_d is "local" if $d > 0$.

Conjecture 6.3. (Dade, Ordinary, Block Free Form, cf. [27, Conj. 6.3]) *If G is a finite group and $d > 0$, then*

$$0 = \sum_C (-1)^{|C|} k_d(G_C)\,,$$

where C runs over representatives of G-conjugacy classes of chains of p-subgroups of G.

The stunning result was that Dade's conjecture implied McKay and AWC at the same time! (To ease the narrative we do not incorporate the original block version, and the subsequent extensions, which are more complicated to state.) Dade's conjecture has been verified for many groups, although the calculations can be exhausting. The symmetric groups were checked by Olsson, Uno and An in [7] and [86]. The case of p-solvable groups was proved in [90]. To check a single sporadic group (J. An did many cases) often takes a whole paper. The completion for the finite groups of Lie type is open, with a recent approach by D. Rossi ([92]). Dade's projective conjecture has been reduced to a question on simple groups by Späth [101]. As in the other conjectures, she has put forward the *Character Triple Conjecture*, which if true, would imply Dade's conjecture. Unfortunately, there are many technical difficulties to check Dade's conjecture, and even more for this inductive version.

Using chains, there is an appealing way to define local functions (following ideas in [50] and [105]). Suppose that f is an integer-valued function defined on all finite groups. Let's say that $f(G)$ is the number of conjugacy classes of p-elements of the finite group G, for instance. When do we say that f is local? Do we have an "official" definition? We do. We say that f is *local* if

$$0 = \sum_C (-1)^{|C|} f(G_C)\,,$$

where C runs over representatives of G-conjugacy classes of chains of p-subgroups of G, for every finite group G. It can be shown that the McKay conjecture is equivalent to the function m_p being local, where $m_p(G)$ counts the number of irreducible complex characters of degree not divisible by p (see [50]). The KR-reformulation easily implies that AWC is equivalent to $k - k_0$ being local; while Dade's conjecture posits directly that k_d is local. It can be shown that the function that counts the number of conjugacy classes of p-elements is local (see [50] for this and more examples). Although much has been searched, no further character-theoretical local functions have been found (or conjectured to be) after Dade's function k_d. The function k_0 which is certainly not local was characterised by group theoretical means in a paper by Robinson ([89]) (solving Brauer's Problem 19).

6.3. **Coming back to Alperin–McKay.** As we have previously written, all these conjectures (now unified in Dade's conjecture) might be the manifestation of a deep hidden structure... which nobody has yet found.

If B is a block with abelian defect group D, M. Broué proposed a deep conjecture which would explain Alperin–McKay and AWC in a structural way. If A and B are Brauer first main correspondents over $\mathcal{O}$, Broué's conjecture ([16]) asserts that the derived categories $D^b(\text{mod-}A)$ and $D^b(\text{mod-}B)$ of bounded complexes of finitely generated modules for A and B are equivalent as triangulated categories. The most spectacular result on this was the proof of the conjecture for blocks of symmetric groups by Chuang and Rouquier using categorification in their seminal paper [25]. (See also the survey [93] for the connections with AWC.) However, as we have mentioned, it remains to find the right structure which would explain, say McKay or AWC, for non-abelian Sylow p-subgroups. Furthermore, Broué's conjecture is still open.

At the beginning of the 2000s, two extensions of the counting conjectures were introduced. McKay "simply" proposed that there is a bijection

$$^* : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))\,.$$

In [49], M. Isaacs and the second author proposed that there should exist a bijection such that $\chi(1) \equiv \pm\chi^*(1) \bmod p$ for all $\chi \in \mathrm{Irr}_{p'}(G)$. (This congruence conjecture seems to have been proved in [24] except for a few cases.) A fundamental player in character theory is the action of the absolute Galois group $\mathcal{G} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the irreducible complex characters. It is not true that there exists a McKay bijection * which commutes with the $\mathcal{G}$-action, since this would imply the number of rational valued characters in $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ agree. And this is false (as shown for instance by $\mathrm{GL}_2(3)$ for $p = 3$). However, the Galois–McKay conjecture ([76]) proposes that there is a bijection * which commutes with the action of $\mathcal{G}_{\mathcal{P}}$, which is the subgroup of $\mathcal{G}$ fixing the prime ideals containing p in the ring of algebraic integers. This amounts to saying that the fields generated by the values of χ and χ^* over the p-adics $\mathbb{Q}_p$ coincide for every $\chi \in \mathrm{Irr}_{p'}(G)$. Turull goes even further and conjectures that the Schur indices of χ and χ^* over $\mathbb{Q}_p$ are the same ([107]). The Galois–McKay extension has many consequences, especially predicting that new group theoretical properties can be detected in the character table (see for instance [78] and the references there). This extension of the McKay conjecture was reduced to simple groups in [80] to the so-called *inductive Galois–McKay Condition*, and it has been recently proved for the prime 2 [95]. There is a lot of work to do in this version of the McKay conjecture because the action of $\mathcal{G}$, or of the absolute Galois group, on the characters of quasi-simple groups is still not perfectly understood. The congruences of character degrees and the Galois action can be combined and applied to Alperin–McKay, AWC and Dade's conjectures.

6.4. **Miscellanea.** There are more global/local conjectures in the representation theory of finite group which we have not touched upon, and which are wide open. Morita equivalences between algebras has influenced part of our research. In general, Brauer first main correspondent blocks B and b are not Morita equivalent. (If this would be the case, $k(B) = k(b)$, $l(B) = l(b)$ and the decomposition matrices of B and b would be the same. And none of these three equalities usually holds.) However, in certain key situations the existence of blocks which are Morita equivalent has been of great utility. Donovan's conjecture (Conjecture M in [2]), which asserts that given a p-subgroup D there are only finitely many Morita equivalence classes of blocks with defect group D is a lively area of research. (In the case where D is abelian, this has been reduced to

quasi-simple groups in [29]. See also [87] and [88] for two seminal references on this subject.) This is much related to the question whether the decomposition numbers are bounded by a function of the defect group. (For instance, it is not known if the decomposition numbers $d_{\chi\varphi}$ are bounded by $|D|$. See also Conjecture I of [2].) Another well-known old conjecture, which can be seen as global/local is Feit's conjecture [32]. If $\chi \in \mathrm{Irr}(G)$, then the conductor $c(\chi)$ is the smallest positive integer n such that the values of χ are contained in the nth cyclotomic field $\mathbb{Q}_n$. Feit's conjecture, which generalised some other question by Brauer, asserts that there is always $x \in G$ such that $o(x) = c(\chi)$. This has now been reduced in [10] to a new condition on quasi-simple groups, which is related to Galois–McKay.

There are also several block inequalities which need to be clarified, aside of the $k(B)$-conjecture (Conjecture 4.6). For instance, Malle and Robinson [67] conjecture that $l(B)$ is less than or equal to $p^{s(B)}$, where $s(B)$ is the sectional rank of a defect group of B. An exciting extension of Brauer's Height Zero Conjecture was proposed by C. Eaton and A. Moretó ([31]) and it is open: if B and b are Brauer first main correspondents, then the minimum non-zero height of the characters of B and b coincide. Another global/local problem is the Glauberman correspondence for Brauer characters: if A acts coprimely on G, is the number of irreducible p-Brauer characters fixed by A the same as the number of Brauer characters of the fixed point subgroup $\mathbf{C}_G(A)$? (See Problem 5 in [74] and [104].) As we are finishing the writing of this survey, A. Moretó and N. Rizo have conjectured exciting new character correspondences that also relate character values, not just degrees.

There is not enough space to credit all the mathematicians that have been contributing to all these problems in a fundamental way. But our point here is that there is plenty of meaningful (hard) work to be expected in the future.

Of course, this story has been told from our perspective. We acknowledge and sometimes even embrace the idea that we rely on the Classification of Finite Simple Groups to prove all these conjectures. We recognise that this may not be the most elegant way to prove or even to fully understand all these problems. Some of us hope that someday, someone will achieve a breakthrough and uncover a brilliant explanation that illuminates all these conjectures. This certainly would not be the first time: more than 50 years passed in the time between when Burnside conjectured that non-abelian simple groups have even order and when the Feit–Thompson theorem was proved. While we wait for a "John Thompson" to discover such an approach, we remain committed to proving the conjectures in the only way currently available to us.

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