ON ALPERIN'S LOWER BOUND FOR THE NUMBER OF BRAUER CHARACTERS

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To the memory of Jim Humphreys

ABSTRACT. We prove that the number of conjugacy classes of a finite group G consisting of elements of odd order, is larger than or equal to that number for the normaliser of a Sylow 2-subgroup of G. This is predicted by the Alperin Weight Conjecture.

1. INTRODUCTION

Let G be a finite group, and let p be a prime. The celebrated Alperin Weight Conjecture (AWC) [Al] states that the number $l(G)$ of conjugacy classes of G of elements of order not divisible by p equals the number of G-conjugacy classes of p-weights of G. (Recall a p-weight of G is a pair (Q, δ), where Q is a p-subgroup of G and δ is an irreducible complex character of $\mathbf{N}_G(Q)/Q$ such that $\delta(1)$ and $|\mathbf{N}_G(Q)/Q|$ have the same p-part.)

One of the consequences of AWC is the following, purely group-theoretic innocent-looking inequality.

CONJECTURE A (Alperin). *If G is a finite group, p is a prime and $P \in \mathrm{Syl}_p(G)$, then*

$$l(G) \geq l(\mathbf{N}_G(P))\,.$$

The Alperin Weight Conjecture was reduced to a problem on simple groups in [NT1]: if the non-abelian finite simple groups satisfy what is now called the *inductive AWC-condition*, then AWC is true for every finite group. (Its block-wise refinement was reduced in [Sp].) Despite the fact that certain families of simple groups have been shown to satisfy the inductive AWC condition (see [BS, FLZ, FM, Li, Ma], etc.), it is fair to say that a full proof of AWC seems yet, unfortunately, out of reach.

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The inductive AWC condition requires the existence of well-behaved bijections between the weights (R, γ) of a quasi-simple group S and the irreducible Brauer characters of S. Here, we propose to focus on the case where $R \in \mathrm{Syl}_p(S)$, and we prove that checking this (important) case is enough to verify, for instance, Conjecture A (see Theorem 2.2). The first case is, of course, $p = 2$, and this is what we prove in this paper. As a consequence, we obtain:

THEOREM B. *Assume that* $p = 2$. *Then* $l(G) \geq l(\mathbf{N}_G(P))$.

As noted in Remark 3.4, there are infinite families of simple and quasi-simple groups where $l(G) - l(\mathbf{N}_G(P)) = 1$. We suspect that the equality $l(G) = l(\mathbf{N}_G(P))$ can only hold when $\mathbf{N}_G(P) = G$, although this does not seem to be implied by AWC. We will comment on this in the final section of this paper, but see also Remark 2.3.

Perhaps it is of interest to remark that the following naive approach to proving Theorem B does not always work: if H is a 2-complement of P in $\mathbf{N}_G(P)$, then $l(\mathbf{N}_G(P)) = k(H)$, but it is not in general true that the different conjugacy classes of H fuse into distinct conjugacy classes of G, as shown by $G = \mathfrak{A}_5$.

Let us close this introduction with a couple of related observations. The other famous counting conjecture, the McKay conjecture, also carries with it some relationship between the global and the local numbers of conjugacy classes. If $k(G)$ denotes the number of conjugacy classes of G, it was already pointed out by W. Feit in [Fe] that the McKay conjecture implies that $k(G) \geq k(\mathbf{N}_G(P))$ whenever P is abelian, an inequality that has not yet been proved either. In fact, a combination of the Itô–Michler theorem and the McKay conjecture easily implies the following purely group-theoretical conjecture, which perhaps it is convenient to have written down.

CONJECTURE C. *Let G be a finite group, p a prime. Then*

$$k(G) \geq k(\mathbf{N}_G(P)/P'),$$

with equality if and only if $P \trianglelefteq G$ and P is abelian.

Of course, for $p = 2$, Conjecture C follows from the main result of [MS] and the Itô–Michler theorem.

Finally, it has been asked if there is some group-theoretical characterization of the Itô–Michler theorem for p-Brauer characters of p-solvable groups (outside p-solvable groups, the degrees of Brauer characters do not behave well). The p-solvable groups G such that q does not divide $\varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$, where q is a prime, were already studied by Manz–Wolf [MW1], and more recently by Lewis–Tong Viet [LT]. A natural characterization might be the following.

CONJECTURE D. *Let G be a finite p-solvable group, let q be a prime, and let $Q \in \mathrm{Syl}_q(G)$. Then q is coprime to the degree of every irreducible p-Brauer character of G if and only if $l(G) = l(\mathbf{N}_G(Q)/Q')$.*

By work in [NST], the inductive McKay-condition of [IMN] implies Conjecture D, see Remark 3.5.

2. Reduction to Simple Groups

We use the notation in [IMN], [Na] and [NT1]. The definition of when a non-abelian simple group satisfies the *inductive Sylow-AWC condition* will be given in the next section.

Theorem 2.1. *Let p be a prime. Suppose that K/Z is a direct product of isomorphic non-abelian simple groups of order divisible by p which satisfy the inductive Sylow-AWC condition, where $Z = \mathbf{Z}(K)$ is a cyclic p'-group. Let $Q \in \mathrm{Syl}_p(K)$ and $\lambda \in \mathrm{Irr}(Z)$ be faithful. Then there exists a subgroup H of K with $\mathbf{N}_K(Q) \leq H < K$ and an injection*

$$^* : \mathrm{IBr}(H|\lambda) \to \mathrm{IBr}(K|\lambda).$$

Now, suppose that K is a normal subgroup in some group G with with $Z \leq \mathbf{Z}(G)$, and write $N = \mathbf{N}_G(H)$. Then the following hold.

(1) *$KN = G$, $K \cap N = H$ and $\mathbf{N}_G(Q) \leq N < G$.*
(2) *The injection * can be chosen to be N-equivariant and*

$$|\mathrm{IBr}(G|\theta^*)| = |\mathrm{IBr}(N|\theta)|$$

for all $\theta \in \mathrm{IBr}(H|\lambda)$.

Proof. Follow the proofs of Theorem 13.1 of [IMN] and Theorem 3.2 of [NT1]. In each step of Theorem 3.2 of [NT1], the radical subgroups under consideration are simply the Sylow subgroups and the inductive Sylow-AWC condition is guaranteed by the inductive AWC condition. The "intermediate subgroup" H (which was not considered in [NT1]) is dealt with in exactly the same way as in Theorem 13.1 of [IMN]. □

We are now ready to give the main result in this section.

Theorem 2.2. *Let p be a prime, let G be a finite group, and let $P \in \mathrm{Syl}_p(G)$. Let $Z \trianglelefteq G$ and suppose that $\lambda \in \mathrm{IBr}(Z)$ is P-invariant. If all the simple groups involved in G/Z satisfy the inductive Sylow-AWC condition then*

$$|\mathrm{IBr}(G|\lambda)| \geq |\mathrm{IBr}(Z\mathbf{N}_G(P)|\lambda)|\,.$$

Proof. We argue by induction on $|G : Z|$. By using the Clifford correspondence for Brauer characters, we may assume that λ is G-invariant. By using modular character triples ([Na, Thm. 8.28]), we may assume that Z is a central cyclic p'-subgroup.

If G is p-solvable, in this case, the theorem follows from [MW2, Thm. 23.10]. Hence, we may assume that G is not p-solvable.

Let K/Z be a chief factor of G. Let Δ be a complete set of representatives for the $\mathbf{N}_G(P)$-action on the set $\mathrm{IBr}_P(K|\lambda)$ of P-invariant irreducible Brauer characters of K that lie over λ. Suppose that $\eta, \eta^g \in \Delta$ for some $g \in G$. Then $P, P^{g^{-1}} \leq G_\eta$, where G_η is the stabiliser of η in G. Thus there is $x \in G_\eta$ such that $xg \in \mathbf{N}_G(P)$. Now, $\eta, \eta^g = \eta^{xg} \in \Delta$, so $\eta = \eta^g$. Therefore, we can write

$$\mathrm{IBr}(G|\lambda) = \bigsqcup_{\eta \in \Delta} \mathrm{IBr}(G|\eta) \ \sqcup \ \Xi$$

as a disjoint union, where Ξ is the set of irreducible Brauer characters of G that lie over no element of Δ. Now,

$$\mathrm{IBr}(K\mathbf{N}_G(P)|\lambda) = \bigsqcup_{\eta\in\Delta} \mathrm{IBr}(K\mathbf{N}_G(P)|\eta)$$

is also a disjoint union. Hence, if KP is not normal in G, by induction, we have that

$$\begin{aligned}|\mathrm{IBr}(G|\lambda)| &\geq \sum_{\eta\in\Delta} |\mathrm{IBr}(G|\eta)| \\ &\geq \sum_{\eta\in\Delta} |\mathrm{IBr}(K\mathbf{N}_G(P)|\eta)| \\ &= |\mathrm{IBr}(K\mathbf{N}_G(P)|\lambda)| \\ &\geq |\mathrm{IBr}(Z\mathbf{N}_G(P)|\lambda)|\,.\end{aligned}$$

Therefore, we may assume that $KP \trianglelefteq G$.

If K/Z is p-solvable, then so is G which is not the case. So we have that K/Z is a direct product of isomorphic simple groups of order divisible by p.

Let $Q = P \cap K \in \mathrm{Syl}_p(K)$. Now, we use Theorem 2.1. We know that there is a subgroup $\mathbf{N}_K(Q) \leq H < K$, an injection

$$^* : \mathrm{IBr}(H|\lambda) \to \mathrm{IBr}(K|\lambda)$$

such that, if $N = \mathbf{N}_G(H)$, then $KN = G$, $K \cap N = H$, $\mathbf{N}_G(Q) \leq N < G$, the injection * is N-equivariant and

$$|\mathrm{IBr}(G|\theta^*)| = |\mathrm{IBr}(N|\theta)|$$

for all $\theta \in \mathrm{IBr}(H|\lambda)$. Since $P \cap K = Q$, we have that $\mathbf{N}_G(P) \leq \mathbf{N}_G(Q) \leq N$.

Since $N < G$, by induction we have that

$$|\mathrm{IBr}(N|\lambda)| \geq |\mathrm{IBr}(\mathbf{N}_G(P)|\lambda)|\,.$$

Let Ψ be a complete set of representatives for the N-action on $\mathrm{IBr}(H|\lambda)$. Then

$$\mathrm{IBr}(N|\lambda) = \bigsqcup_{\tau\in\Psi} \mathrm{IBr}(N|\tau)$$

and

$$\bigsqcup_{\tau\in\Psi} \mathrm{IBr}(G|\tau^*) \subseteq \mathrm{IBr}(G|\lambda)$$

are disjoint unions. By Theorem 2.1(2) and the Clifford correspondence, we have that

$$|\mathrm{IBr}(N|\lambda)| = \sum_{\tau\in\Psi} |\mathrm{IBr}(G|\tau^*)| \leq |\mathrm{IBr}(G|\lambda)|\,,$$

as wanted. □

Remark 2.3. In Theorem 2.2 equality can hold even when PZ is not normal; e.g., for $G = \mathrm{SL}_2(5)$ with $p = 3$ we have $|\mathrm{IBr}(G|\lambda)| = 3 = |\mathrm{IBr}(Z\mathbf{N}_G(P)|\lambda)|$ for the non-trivial character λ of $Z = \mathbf{Z}(G) = C_2$.

3. The inductive Sylow AWC-condition and simple Groups

In this section we show that all simple groups satisfy the inductive Sylow-AWC condition for $p = 2$, thereby completing the proof of Theorem B. We follow the definitions in [NT1, §3] and [IMN, §10], which we adapt to our situation. Suppose that X is a non-abelian simple group of order divisible by p. We say that X satisfies the *inductive Sylow-AWC condition* if the following two conditions hold for every choice of perfect group S such that $S/Z = X$, with $Z := \mathbf{Z}(S)$ cyclic of order not divisible by p. Fix $Q \in \mathrm{Syl}_p(S)$ and a faithful $\lambda \in \mathrm{Irr}(Z)$. Let A be the subgroup of $\mathrm{Aut}(S)$ consisting of all automorphisms that act trivially on Z and stabilise Q. We require that there exists a subgroup T stabilised by A, with $\mathbf{N}_S(Q) \leq T < S$ such that:

(1) There is an A-equivariant injection

$$^* : \mathrm{IBr}(T|\lambda) \to \mathrm{IBr}(S|\lambda).$$

(2) For each $\theta \in \mathrm{IBr}(T|\lambda)$, there is a group G satisfying the following conditions:
   (a) $S \trianglelefteq G$, $Z \leq \mathbf{Z}(G)$. In particular, if $N = \mathbf{N}_G(T)$, as in Lemma 10.1 of [IMN], we have that $G = SN$, $T = S \cap N$, and the injection * is N-equivariant.
   (b) The stabiliser B of θ in A is exactly the group of automorphisms of S induced by the conjugation action of the subgroup $M = \mathbf{N}_G(Q)$ on S.
   (c) The subgroup $C = \mathbf{C}_G(S)$ is abelian, and the set $\mathrm{IBr}(C|\lambda)$ contains a G-invariant character γ.
   (d) We have the equality

$$[\theta^* \cdot \gamma]_{G/SC} = [\theta \cdot \gamma]_{N/TC}\,,$$

   where these cohomology elements are defined as in [NT1, §3],

We remark that this condition is implied by the inductive AWC condition of [NT1]. Indeed, if we have that the simple group $S/Z = X$ satisfies the inductive AWC-condition (or, as it is called in [NT1], if X is *AWC-good*), then we let $T = \mathbf{N}_S(Q)$ and the injection * to be the inverse of the map $^{*(Q,\lambda)}$ onto its image. We notice that the set $\mathrm{dz}(\mathbf{N}_S(Q)/Q|\lambda)$ is simply the set $\mathrm{IBr}(\mathbf{N}_S(Q)|\lambda)$.

Finally, we remark that in this paper, that is for $p = 2$, our intermediate subgroup T will always be $\mathbf{N}_S(Q)$. For future applications (for p odd), though, it is convenient to allow for more generality.

Lemma 3.1. *Let $q = p^f$ be a power of the odd prime p, $r \in \mathbb{Z}_{\geq 1}$, and $\epsilon \in \{\pm\}$. Let C_ϵ denote the cyclic subgroup of order $q - \epsilon 1$ in $\overline{\mathbb{F}}_q^\times$ and $\tilde{C}_\epsilon$ its character group. If $\epsilon = +$, let $D := C_f \times C_2 = \langle \sigma, \tau \rangle$ act on $\tilde{C}_\epsilon$ via $\sigma(\mu) = \mu^p$ and $\tau(\mu) = \mu^{-1}$. If $\epsilon = -$, let $D := C_{2f} = \langle \sigma \rangle$ act on $\tilde{C}_\epsilon$ via $\sigma(\mu) = \mu^p$. Extend this action to*

$$\Omega(r) := \underbrace{\mathbf{O}_{2'}(\tilde{C}_\epsilon) \times \cdots \times \mathbf{O}_{2'}(\tilde{C}_\epsilon)}_{r}$$

component-wise, and for any element $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r) \in \Omega(r)$, let $J(\boldsymbol{\mu})$ consist of all $\gamma \in D$ such that

$$\gamma(\boldsymbol{\mu}) = \lambda\boldsymbol{\mu} := (\lambda\mu_1, \lambda\mu_2, \ldots, \lambda\mu_r) \qquad \textit{for some } \lambda = \lambda_\gamma \in \tilde{C}_\epsilon.$$

Then the following statements hold.

(a) *Suppose* $r > 1$ *and* $\boldsymbol{\mu} \neq (\mu_1, \mu_1, \ldots, \mu_1)$. *Then* $J(\boldsymbol{\mu})$ *is cyclic. Furthermore, the subgroup* $J(\overline{\boldsymbol{\mu}})$ *and the inertia subgroup* $\mathrm{Stab}_D(\overline{\boldsymbol{\mu}})$ *of*

$$\overline{\boldsymbol{\mu}} := \mu_1^{-1}\boldsymbol{\mu} = \big(1_{C_\epsilon}, \mu_1^{-1}\mu_2, \ldots, \mu_1^{-1}\mu_r\big)$$

in D *are both equal to* $J(\boldsymbol{\mu})$.

(b) *If* $r = 1$ *and* $\mu_1 \neq 1_{C_\epsilon}$ *then the inertia subgroup* $\mathrm{Stab}_D(\boldsymbol{\mu})$ *is cyclic.*

Proof. (a) It suffices to show that $\mathbf{O}_2(J(\boldsymbol{\mu}))$ is cyclic when $\epsilon = +$. Assume that $\mathbf{O}_2(J(\boldsymbol{\mu}))$ is non-cyclic. Then the abelian group $\mathbf{O}_2(J(\boldsymbol{\mu}))$ contains three distinct involutions $\alpha, \beta, \alpha\beta$. Note that at least one of them, say α, is outside of the cyclic group $\langle\sigma\rangle \cong C_f$. Now, if $\beta \notin \langle\sigma\rangle$ then $\alpha\beta \in \langle\sigma\rangle$. Replacing β by $\alpha\beta$, we may assume that $\beta = \sigma^{f/2}$ (and $2|f$). Write $\alpha = \sigma^j\tau$ with $0 \leq j < f$. Then $1 = \alpha^2 = \sigma^{2j}$, so $f|2j$ and $j = 0$ or $j = f/2$. Replacing α by $\alpha\beta$ in the latter case, we may assume that $\alpha = \tau$. It follows that there exists some $\lambda \in \tilde{C}_\epsilon$ such that $\lambda\mu_i = \alpha(\mu_i) = \mu_i^{-1}$, whence $\mu_i^2 = \lambda^{-1}$ and $(\mu_1^{-1}\mu_i)^2 = 1_{C_\epsilon}$ for all $1 \leq i \leq r$. Since $\mu_1^{-1}\mu_i$ has odd order, it follows that $\mu_1 = \ldots = \mu_r$, contrary to the assumption.

By the preceding result, we have that $J(\boldsymbol{\mu}) = \langle\delta\rangle$ for some $\delta \in J(\boldsymbol{\mu})$, and there exists some $\lambda \in \tilde{C}_\epsilon$ such that $\delta(\mu_i) = \lambda\mu_i$ for all i. It follows that $\delta(\mu_1^{-1}\mu_i) = \mu_1^{-1}\mu_i$ for all i, and thus $\delta \in \mathrm{Stab}_D(\overline{\boldsymbol{\mu}})$ and $J(\boldsymbol{\mu}) \leq \mathrm{Stab}_D(\overline{\boldsymbol{\mu}})$. Since $\mathrm{Stab}_D(\overline{\boldsymbol{\mu}}) \leq J(\overline{\boldsymbol{\mu}})$ and $J(\overline{\boldsymbol{\mu}}) = J(\boldsymbol{\mu})$ by definition, the statements follow.

(b) As in (a), it suffices to rule out the case $\epsilon = +$ and $\mathbf{O}_2(\mathrm{Stab}_D(\boldsymbol{\mu}))$ is not cyclic. Arguing as in (a), we see in such a case that $\mu_1 = \tau(\mu_1) = \mu_1^{-1}$, and so $\mu_1 = 1_{C_\epsilon}$. □

As a consequence of [NT1, Thm. C], we have:

Theorem 3.2. *Let* X *be a finite non-abelian simple group of Lie type in characteristic* p. *Then* X *satisfies the inductive Sylow-AWC condition for the prime* p.

The main result of this section is the following statement:

Theorem 3.3. *Let* X *be a finite non-abelian simple group. Then* X *satisfies the inductive Sylow-AWC condition for the prime* $p = 2$.

Proof. We first discuss two easy situations.

Case 1: Suppose that $S = X$ (which is the only possibility if $\mathrm{Mult}(X)_{p'} = 1$), and $\mathbf{N}_S(Q) = Q$.

Then $\lambda = 1$, $\mathrm{IBr}(\mathbf{N}_S(Q)|\lambda) = \{1_{\mathbf{N}_S(Q)}\}$, and the map $1_{\mathbf{N}_S(Q)} \mapsto 1_S$ clearly satisfies the required conditions.

Thus we may assume that either $S \neq X$ (and so $\mathrm{Mult}(X)_{p'} > 1$) or $\mathbf{N}_S(Q) > Q$.

Case 2: Suppose now that $\mathbf{N}_S(Q)/Q \cong \mathbf{Z}(S) \cong C_3$ and $\mathrm{Out}(S) \cong C_2$ acts non-trivially on $\mathbf{Z}(S)$. Then $\mathbf{N}_S(Q)/Q$ has two irreducible characters α_1, α_2 not having $\mathbf{Z}(S)$ in their kernel, interchanged by $\mathrm{Out}(S)$, and clearly all faithful irreducible 2-Brauer characters of S are not invariant under $\mathrm{Out}(S)$. Since there are at least two such, say β_1 and its image β_2 under $\mathrm{Out}(S)$, we are again done by sending $\alpha_i \mapsto \beta_i$ for $i = 1, 2$.

We now treat the various groups in turn, using the classification of finite simple groups.

Case 3: Suppose that $X = \mathfrak{A}_n$, $n \geq 6$.

If $n \geq 8$ then $S = X$ and $\mathbf{N}_S(Q) = Q$ (see e.g. [Ko, Cor.]), whence we are done by Case 1. If $n = 7$, we are in Case 2. If $n = 6$, then $S \cong 3.\mathfrak{A}_6$, $A \cong \mathbf{N}_{\mathfrak{A}_6.2_3}(Q) = Q \cdot 2_3$, and $\mathbf{N}_S(Q) = Q \times Z$. We take G to be the non-split extension $S \cdot 2_3$ in [GAP], and note that S has two faithful irreducible 2-Brauer characters $\beta_{1,2}$ of degree 9, which lie above the non-principal linear characters $\lambda_{1,2}$ of Z and which extend to G. Now the map * can be taken to be $\lambda_i \mapsto \beta_i$.

Case 4: Suppose that X is one of the 26 sporadic simple groups. For these the inductive AWC condition was verified in [AD]. (In fact, for most groups we are either in Case 1 or Case 2, and the few remaining cases can also be dealt with easily.)

We are left to deal with the simple groups X of Lie type. If X is in characteristic 2, we are done by Theorem 3.2, whence it remains to consider groups of Lie type in odd characteristic q_0 with $p = 2$. The ones with non-self-normalising Sylow 2-subgroups are listed in [Ko, Cor.].

Case 5: Suppose that $X = {}^2G_2(q^2)$ with $q^2 > 3$. Then we are done by [NT1, Prop. 8.4].

Case 6: Suppose that $X = \mathrm{PSL}_n(\epsilon q)$ with $n \geq 3$, $\epsilon \in \{\pm\}$, and $q = q_0^f$.

Here, as customary, we let $\mathrm{PSL}_n(\epsilon q)$ denote $\mathrm{PSL}_n(q)$ when $\epsilon = +$, respectively $\mathrm{PSU}_n(q)$ when $\epsilon = -$, and similarly for $\mathrm{SL}_n(\epsilon q)$ and $\mathrm{GL}_n(\epsilon q)$.

For the group $X \cong \mathrm{PSU}_4(3)$ we are in Case 1 or 2, and thus by [KL, Thm. 5.1.4] may assume that S is a central quotient of $\mathrm{SL}_n(\epsilon q)$. First assume that n is a power of 2. Then the Sylow 2-subgroups of X are self-normalising, see e.g. [Ko, Cor.]. As $\mathbf{Z}(\mathrm{SL}_n(\epsilon q)) \cong C_{\gcd(n,q-\epsilon 1)}$ is a 2-group in this case, we have that $S = X$, and so we are done by Case 1.

So now write $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r}$, where $r > 1$ and $n_1 > n_2 > \ldots > n_r \geq 0$. Relaxing the condition that λ is faithful, we may assume that $S = \mathrm{SL}_n(\epsilon q)$, and embed S into $H := \mathrm{GL}_n(\epsilon q)$. View $H = \mathbf{H}^F$ for $\mathbf{H} = \mathrm{GL}_n(\overline{\mathbb{F}_q})$ and a Steinberg endomorphism $F : \mathbf{H} \to \mathbf{H}$, and consider the F-fixed point subgroup of a Levi subgroup of $\mathbf{H}$, $L = H_1 \times \cdots \times H_r \leq H$ with $H_i = \mathrm{GL}_{2^{n_i}}(\epsilon q)$, and a Sylow 2-subgroup $P = P_1 \times \cdots \times P_r$ of H with $P_i \in \mathrm{Syl}_2(H_i)$. Then we can take $Q = P \cap S$, and have

$$\mathbf{N}_H(P) = \mathbf{N}_{H_1}(P_1) \times \cdots \times \mathbf{N}_{H_r}(P_r) = P \times N_1, \quad \mathbf{N}_S(Q) = Q \times (N_1 \cap S),$$

with

$$\mathbf{N}_{H_i}(P_i) = P_i \times \mathbf{O}_{2'}(\mathbf{Z}(H_i)),\ N_1 := \mathbf{O}_{2'}(\mathbf{N}_H(P)) = \mathbf{O}_{2'}(\mathbf{Z}(H_1)) \times \cdots \times \mathbf{O}_{2'}(\mathbf{Z}(H_r))$$

(see e.g. [NT3, (3.3)] and the first displayed formula in part (b) of the proof of [NT3, Thm. 4.3]).

As in the proof of [NT3, Thm. A] and Lemma 3.1, let C_ϵ denote the cyclic subgroup of order $q-\epsilon 1$ in $\overline{\mathbb{F}}_q^\times$, $\tilde{C}_\epsilon$ denote its character group, and set

$$\Omega^*(n) := \Omega(r) = \underbrace{\mathbf{O}_{2'}(\tilde{C}_\epsilon) \times \cdots \times \mathbf{O}_{2'}(\tilde{C}_\epsilon)}_{r};$$

in particular, $\mathrm{Irr}(N_1)$ can be canonically identified with $\Omega^*(n)$. Then the proof of [NT3, Thm. A] yields canonical bijections

$$\alpha : \mathcal{B} \to \Omega^*(n) \quad \text{and} \quad \beta : \mathrm{IBr}(\mathbf{N}_H(P)/P) \to \Omega^*(n)$$

where $\mathcal{B}$ is the set of odd-degree irreducible 2-Brauer characters of H. Moreover, for any Brauer character

$$\mu = \mu_1 \boxtimes \mu_2 \boxtimes \cdots \boxtimes \mu_r$$

with $\mu_i \in \mathrm{IBr}(\mathbf{O}_{2'}(\mathbf{Z}(H_i)))$, of N_1, we have $\beta(\mu) = (\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_r)$, and

$$\mu(h) = \prod_{j=1}^{r} \hat{t}_j(z^{2^{n_j}}) \quad \text{for all } h = zI_n \in \mathbf{O}_{2'}(\mathbf{Z}(H)). \tag{3.1}$$

Similarly, we claim that if $\alpha(\varphi) = (\hat{t}_1, \ldots, \hat{t}_r)$, then

$$\varphi(h) = \varphi(1) \prod_{j=1}^{r} \hat{t}_j(z^{2^{n_j}}) \quad \text{for all } h = zI_n \in \mathbf{O}_{2'}(\mathbf{Z}(H)). \tag{3.2}$$

Indeed, this is [NT3, (3.6)] when $\epsilon = +$. In general, φ is the restriction χ_s° to $2'$-elements of some semisimple character

$$\chi_s = \pm R_M^H(\psi) \tag{3.3}$$

of H that is Lusztig induced from a linear character ψ of a Levi subgroup

$$M = \mathbf{C}_H(s) = \mathrm{GL}_{k_1}(\epsilon q) \times \cdots \times \mathrm{GL}_{k_m}(\epsilon q)$$

of H, where m is the number of distinct elements among $\hat{t}_1, \ldots, \hat{t}_r$, and, for each j, k_j is the sum of the 2^{n_i} for which the $\hat{t}_i$ coincide, and

$$\psi(h) = \prod_{j=1}^{r} \hat{t}_j(z^{2^{n_j}}) \quad \text{for all } h = zI_n \in \mathbf{O}_{2'}(\mathbf{Z}(H)).$$

After identifying the dual group H^* with H, the label s is a semisimple $2'$-element in H, see the first displayed formula in the proof of [NT3, Thm. A]. Let St_H and St_M denote the Steinberg characters of H and of M. Then by [DM, Cor. 9.3 and Prop. 9.6] for all $h \in \mathbf{Z}(H)$ we have

$$|H|_{q_0} \chi_s(h) = \pm(\mathsf{St}_H \cdot \chi_s)(h) = \pm\mathrm{Ind}_M^H(\mathsf{St}_M \cdot \psi)(h) = \pm|H/M| \cdot |M|_{q_0} \psi(h), \tag{3.4}$$

and so $\chi_s(h) = \pm|H/M|_{q_0'} \psi(h) = \pm\chi_s(1)\psi(h)$. In particular, if h has odd order, then we get $\chi_s(h) = \chi_s(1)\psi(h)$, proving (3.2).

Thus, if $\alpha(\varphi) = \beta(\mu)$, then (3.1) and (3.2) show that χ and μ lie above the same central character of $\mathbf{O}_{2'}(\mathbf{Z}(H))$. As shown in the proof of [NT3, Thm. A], the map

$$\alpha^{-1}\beta : \mathrm{IBr}(\mathbf{N}_H(P)/P) \to \mathcal{B}$$

is a bijection which commutes with the action of the subgroup of $\mathrm{Aut}(H)$ that stabilises P, and with the multiplication by linear Brauer characters of H, whose set can be identified with $\mathrm{Irr}(H/\mathbf{O}^{2'}(H))$ and thus with $\mathrm{IBr}(\mathbf{N}_H(P)/\mathbf{N}_S(Q))$.

Since $\mathbf{N}_H(P)/P \cong N_1$ is an abelian $2'$-group, each such μ restricts irreducibly to $N_1 \cap S \cong \mathbf{N}_S(Q)/Q$, and conversely, each $\nu \in \mathrm{IBr}(\mathbf{N}_S(Q)/Q)$ extends to exactly $(q-\epsilon 1)_{2'}$ linear Brauer characters of $\mathbf{N}_H(P)/P$.

Next, recall that $H/S \cong C_{q-\epsilon 1}$. Since any $\varphi \in \mathcal{B}$ has odd degree, it is irreducible over $\mathbf{O}^{2'}(H)$. Hence, by Lemmas 3.2 and 3.3 of [KT], the number of irreducible constituents of $\varphi|_S$ is equal to the number of linear (Brauer) characters ξ of $H/\mathbf{O}^{2'}(H)$ such that $\varphi = \varphi\xi$. The multiplication of $\varphi = \chi_s^\circ$, see (3.3), by such a ξ has the effect of replacing the semisimple $2'$-element $s \in H$ by sz for some $z \in \mathbf{O}_{2'}(\mathbf{Z}(H))$, see e.g. [DM, Prop. 13.30]. The conjugacy class s^H labels a union of 2-blocks of H that contains φ by the main result of [BM]. Hence $\varphi = \varphi\xi$ implies that s and sz are conjugate in H. Now, since $k_1, \ldots, k_m$ are pairwise distinct, this can happen only when $z = 1$, i.e., if ξ is trivial. We have shown that every $\varphi \in \mathcal{B}$ is irreducible over S.

We can write $\mathrm{Aut}(X) = (H/\mathbf{Z}(H)) \rtimes D$ for a group D of outer automorphisms that fix Q, with $D = \langle\sigma\rangle \times \langle\tau\rangle \cong C_f \times C_2$ if $\epsilon = +$ and $D = \langle\sigma\rangle \cong C_{2f}$ if $\epsilon = -$ (where σ is induced by a standard q_0th Frobenius map F_{q_0}, and τ is the transpose-inverse). Now, arguing as in part (b) of the proof of [NT3, Thm. 4.3], we see that the assignment

$$\theta := \mu|_{\mathbf{N}_S(Q)} \mapsto \alpha^{-1}\beta(\mu)|_S = \varphi|_S =: \theta^* \tag{3.5}$$

yields a map $\mathrm{IBr}(\mathbf{N}_S(Q)|\lambda) \to \mathrm{IBr}(S|\lambda)$ (if $\varphi|_S$ lies over λ) that is D-equivariant; it trivially commutes with the action of $\mathbf{N}_H(Q)$. Note that if $\varphi|_S = \varphi'|_S$ for $\varphi, \varphi' \in \mathcal{B}$, then, as they are both irreducible over S, they differ by some linear $\xi \in \mathrm{IBr}(H/S)$. Again using (3.3) to locate the unions of 2-blocks that contain $\varphi = \chi_s^\circ$ and $\varphi\xi = \varphi' = \chi_{s'}^\circ$, we see that s' and s are H-conjugate, and so, after a suitable conjugation, we have $\mathbf{C}_H(s') = \mathbf{C}_H(s)$, and $s' = sz$ for some $z \in \mathbf{Z}(H)$ (if ξ corresponds to z). This shows that $\mu = \beta^{-1}\alpha(\varphi)$ and $\mu' := \beta^{-1}\alpha(\varphi')$ agree on $\mathbf{N}_S(Q)$, and thus the map defined in (3.5) is injective.

If $\theta \in \mathrm{IBr}(\mathbf{N}_S(Q))$ is trivial, then we can take $S = X$ and argue as in Case 1. Suppose $\theta \neq 1_{\mathbf{N}_S(Q)}$. In this case, θ, as an $N_1 \cap S$-character, is obtained by restricting some N_1-character $\mu = \mu_1 \boxtimes \mu_2 \boxtimes \cdots \boxtimes \mu_r$ with

$$\boldsymbol{\mu} := (\mu_1, \mu_2, \ldots, \mu_r) \neq (\mu_1, \mu_1, \ldots, \mu_1).$$

The action of σ, and τ when $\epsilon = +$, on $\Omega^*(n)$ is described in the displayed formula after [GKNT, (5.2)], which agrees with the action prescribed in Lemma 3.1, and α and β both commute with the action of D. Now, $\delta \in D$ fixes θ if and only if $\delta(\mu)$ agrees with μ on $N_1 \cap S$, i.e., if δ belongs to the subgroup $J := J(\boldsymbol{\mu})$ of Lemma 3.1. Hence J is cyclic, and, replacing $\boldsymbol{\mu}$ by $\overline{\boldsymbol{\mu}}$, we have that J is the same as $\mathrm{Stab}_D(\mu)$, on the local side. On the global side, the arguments with $\theta^* = \varphi|_S$ right after (3.5) show that $\delta \in D$ fixes θ if and only if $\delta(\varphi)$ agrees with φ on S, i.e., if $\delta \in J$, and, again by Lemma 3.1, after replacing $\boldsymbol{\mu}$ by $\overline{\boldsymbol{\mu}}$, which amounts to multiplying φ by a linear character, we also have that J is the same as $\mathrm{Stab}_D(\varphi)$.

Again relaxing the faithfulness of λ, we can take $G = H \rtimes J$. As the H-character φ is J-invariant, φ is G-invariant; also J fixes Q, θ, and θ^* as mentioned above.

Furthermore, $G = \mathbf{N}_G(Q)S$ by the Frattini argument. Hence, the stabiliser B of Z, Q, and θ in $\mathrm{Aut}(S)$ induces the action of $\mathbf{N}_G(Q)$ on S, as required in part (2)(b) of the inductive Sylow-AWC condition. Furthermore, in the notation of [NT1, §3.3], we have $C = \mathbf{C}_G(S) = \mathbf{Z}(H)$ and γ is the restriction of φ to C. Since J is cyclic and both μ and φ are J-invariant, μ extends to $\mathbf{N}_G(Q)$ and φ extends to G. Thus both cocycles $[\theta \cdot \gamma]_{\mathbf{N}_G(Q)/\mathbf{N}_S(Q)C}$ and $[\theta^* \cdot \gamma]_{G/SC}$ are trivial, and we are done.

Case 7: Suppose that $X = E_6(\epsilon q)$ with $\epsilon \in \{\pm\}$ and $2 \nmid q = q_0^f$.
(Again, we let $E_6(\epsilon q)$ denote $E_6(q)$ when $\epsilon = +$ and ${}^2E_6(q)$ when $\epsilon = -$.)

Here, $\mathrm{Mult}(X) = C_{\gcd(3,q-\epsilon 1)}$, so we may assume that $S = \mathbf{G}^F$, where $\mathbf{G}$ is a simple, simply connected algebraic group of type E_6 and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism. According to [GLS, Table 4.5.2], S has a unique conjugacy class t^S of 2-central involutions whose centraliser $L = \mathbf{C}_S(t)$ has a component $L_1 = \mathrm{Spin}_{10}^{\epsilon}(q)$; in fact, t is the unique central involution in both L and L_1. We can certainly assume $t \in Q \in \mathrm{Syl}_2(L)$. Let $\mathbf{L} = \mathbf{C}_{\mathbf{G}}(t)$, an F-stable Levi subgroup with $L = \mathbf{L}^F$. Then $L_1 = [\mathbf{L}, \mathbf{L}]^F$ and $\mathbf{T}_1 := \mathbf{Z}(\mathbf{L})^\circ$ is a 1-dimensional F-stable torus of $\mathbf{G}$ with $\mathbf{T}_1^F \cong C_{q-\epsilon 1}$ (as $\mathbf{C}_L(L_1) \cong C_{q-\epsilon 1}$, see [GLS, Table 4.5.2]). So

$$A_1 := \mathbf{O}_{2'}(\mathbf{T}_1^F) \cong C_{(q-\epsilon 1)_{2'}}$$

centralises Q. Since $N_X(Q/\mathbf{Z}(S)) \cong C_{(q-\epsilon 1)_{2'}/\gcd(3,q-\epsilon 1)}$ by [Ko, Cor.] we see that

$$\mathbf{N}_S(Q) = Q \times A_1. \tag{3.6}$$

As shown in the proof of [NT2, Prop. 4.3], $\mathbf{L}$ and hence also $\mathbf{T}_1 = \mathbf{Z}(\mathbf{L})^\circ$ and A_1 can be chosen to be D-invariant, where $D = \langle\sigma\rangle \cong C_{2f}$ if $\epsilon = -$ and $D = \langle\sigma\rangle \times \langle\tau\rangle \cong C_f \times C_2$ if $\epsilon = +$, and σ and τ act on $\mathrm{Irr}(A_1)$ as prescribed in Lemma 3.1. Note that $L = \mathbf{C}_S(s)$ for $1 \neq s \in A_1 \smallsetminus \mathbf{Z}(S)$ (indeed, $\mathbf{L}$ centralises s, but $s \notin \mathbf{Z}(\mathbf{G})$, so the claim follows by the maximality of the proper Levi subgroup $\mathbf{L}$). Furthermore, two such elements s, s' are S-conjugate only when $s' = s$ (indeed, if $s' = s^x$ for some $x \in S$, then $t^x = t$ for the unique central involution in $L = \mathbf{C}_S(s) = \mathbf{C}_S(s')$, whence $x \in \mathbf{C}_S(t) = L$, and so $s^x = s'$.) Since $[L : L_1\mathbf{C}_L(L_1)] = \gcd(4, q-\epsilon 1)$ and $|L/L_1| = q - \epsilon 1$, we have

$$L = \mathbf{O}^{2'}(L) \times A_1 \tag{3.7}$$

and can canonically identify the set of linear 2-Brauer characters of L with $\mathrm{Irr}(A_1)$. Arguing as in Case 1, we may, and will in the subsequent analysis, assume that $\theta \in \mathrm{IBr}(\mathbf{N}_S(Q))$ is non-trivial at A_1, and define G and $\theta^* \in \mathrm{IBr}(S)$.

First consider the case $3 \nmid (q - \epsilon 1)$. Then $S = X$, and it can also be identified with the dual group $S^* = \mathbf{G}^{*F}$; in particular, $\mathbf{Z}(S) = 1$. Similarly, L can be identified with the centraliser of any $1 \neq s \in A_1$ in $\mathbf{G}^{*F}$. Here, $\mathrm{Out}(S) = D$, and $J := \mathrm{Stab}_D(\theta)$ is cyclic by Lemma 3.1. Now we can take $G = S \rtimes J$, and define θ^* to be the restriction χ° to $2'$-elements of the character $\chi = \pm R_L^S(\theta)$, with $\theta \in \mathrm{IBr}(A_1)$ viewed as a linear character of $L/\mathbf{O}^{2'}(L)$. At the same time, this χ is the semisimple character labelled by the semisimple $2'$-element s corresponding to θ, cf. [DM, Thm. 13.25]. The conjugacy class s^S labels the union of 2-blocks that contain χ°, and, as mentioned above, $1 \neq s, s' \in A_1$ are S-conjugate only when $s' = s$. Using this, we see that

$\mathrm{Stab}_D(\theta^*) = J$, and the map $\theta \mapsto \theta^*$ is a D-equivariant injection. As J is cyclic, both θ and θ^* extend to J, and so we are done.

In the remaining case $3|(q-\epsilon 1)$, we need to use a regular embedding. In this case, F acts trivially on $Z := \mathbf{Z}(\mathbf{G}) \cong C_3$ which is contained in $\mathbf{T}_1$. Fix a 1-dimensional torus $\mathbf{T}_2$ and a D-equivariant isomorphism $\iota : \mathbf{T}_1 \to \mathbf{T}_2$, and let

$$\mathbf{H} := \big(\mathbf{G} \times \mathbf{T}_2\big)/Z', \quad H := \mathbf{H}^F,$$

where $Z' := \{(y, \iota(y^{-1})) \mid y \in Z\}$. Then $g \mapsto (g,1)Z'$ gives a D-equivariant regular embedding $\mathbf{G} \to \mathbf{H}$; correspondingly, $S = \mathbf{G}^F$ embeds in H as a normal subgroup with cyclic quotient $H/S \cong C_{q-\epsilon 1}$. We also embed $\mathbf{T}_2$ in $\mathbf{H}$ via $x \mapsto (1,x)Z'$. By [Ca, Prop. 3.6.8], $\mathbf{Z}(H) = \mathbf{Z}(\mathbf{H})^F$, and so it is equal to $\mathbf{T}_2^F \cong C_{q-\epsilon 1}$. Fix an element $z \in \mathbf{T}_1$ of order $3(q-\epsilon 1)$, so that $F(z)z^{-1}$ generates Z, and let $\mathbf{z} := (z, \iota(z^{-1}))$, so that $Z' := \langle \mathbf{z}^{q-\epsilon 1}\rangle$. Now we consider the torus $\mathbf{T} := (\mathbf{T}_1 \times \mathbf{T}_2)/Z'$ of $\mathbf{H}$. Then

$$\mathbf{T}^F = \{(z^i, \iota(z^{-j})) \mid i,j \in \mathbb{Z}/3(q-\epsilon 1)\mathbb{Z}, i+j \in 3\mathbb{Z}/3(q-\epsilon 1)\mathbb{Z}\}$$

is homocyclic of order $(q-\epsilon 1)^2$. Indeed, $\mathbf{T}^F = B_1 \times B_2$, where

$$B_1 := \langle (z, \iota(z^{-1}))Z'\rangle \cong C_{q-\epsilon 1},\ B_2 := \langle (z^{-2}, \iota(z^{-1}))Z'\rangle \cong C_{q-\epsilon 1}.$$

Fixing the isomorphism $\iota' : (z, \iota(z^{-1}))Z' \mapsto (z^{-2}, \iota(z^{-1}))Z'$ between B_1 and B_2, we have that the subgroup $\{b\iota'(b^{-1}) \mid b \in B_1\}$ is precisely $\langle (z^3,1)Z'\rangle = \mathbf{T}_1^F = \mathbf{C}_L(L_1)$ (under the embedding $\mathbf{G} \to \mathbf{H}$); in particular, it contains the subgroup A_1 in (3.6). Viewing $\mathbf{T}^F = \{(b_1,b_2) := b_1\iota'(b_2) \mid b_1, b_2 \in B_1\}$, we see that the irreducible characters $\mu \boxtimes \mu$ with $\mu \in \mathrm{IBr}(B_1)$ of $\mathbf{T}^F$ are precisely the ones that are trivial at A_1. Hence, by Lemma 3.1 with $r = 2$, if $\theta \in \mathrm{IBr}(A_1)$ is non-trivial, then it admits an extension $\tilde\theta$ to $\mathbf{T}^F$ such that

$$\mathrm{Stab}_D(\theta) = \mathrm{Stab}_D(\tilde\theta) =: J \quad \text{is cyclic.} \tag{3.8}$$

Since $\mathbf{Z}(H) \cap S = Z$, $P := Q \times \mathbf{O}_2(\mathbf{Z}(H))$ is a Sylow 2-subgroup of H. Note that

$$\mathbf{N}_H(P) = P \times A, \tag{3.9}$$

where $A := \mathbf{O}_{2'}(\mathbf{T}^F) = \mathbf{O}_{2'}(B_1) \times \mathbf{O}_{2'}(B_2)$. (Indeed, A centralises Q, and so P as well, and $P \cap A = 1$. Next, $\mathbf{N}_H(P) \le \mathbf{N}_H(Q)$, and $\mathbf{N}_S(Q)$ has index $q - \epsilon 1 = |H/S|$ in $\mathbf{N}_H(Q)$ by the Frattini argument. Hence,

$$|\mathbf{N}_H(P)| \le (q-\epsilon 1)|\mathbf{N}_S(Q)| = (q-\epsilon 1)(q-\epsilon 1)_{2'}|Q| = (q-\epsilon 1)^2_{2'}|P| = |P \times A|,$$

and (3.9) follows.)

On the global side, since $\mathbf{G}^* \cong \mathbf{G}/Z$, we may take $\mathbf{H}^* = \mathbf{H}$, with the surjection $\mathbf{H} \to \mathbf{G}^*$ given by $(g,t)Z' \mapsto gZ$. (This can be seen by a direct calculation with root data to determine the dual, or using the classification results in [Ta].) Hence we can identify the dual group H^* with H. Next we consider the Levi subgroup $M := \mathbf{M}^F$, with $\mathbf{M} := (\mathbf{L} \times \mathbf{T}_2)/Z' = \mathbf{C}_{\mathbf{H}}(\mathbf{T}_1) = \mathbf{C}_{\mathbf{H}}(t)$ the centraliser of the involution $t \in \mathbf{Z}(P)$. It follows from (3.7) that $\mathbf{O}^{2'}(M) \le PL = P(\mathbf{O}^{2'}(L) \times A_1)$, i.e. $\mathbf{O}^{2'}(M) \le \mathbf{O}^{2'}(L)P$. As $\mathbf{O}^{2'}(M)$ contains both P and $\mathbf{O}^{2'}(L)$, we must have that $\mathbf{O}^{2'}(M) = \mathbf{O}^{2'}(L)P$.

Now $P \cap \mathbf{O}^{2'}(L) = P \cap L = Q$, so

$$|\mathbf{O}^{2'}(L)P| = \frac{|P| \cdot |\mathbf{O}^{2'}(L)|}{|Q|} = \frac{(q-\epsilon 1)_2 \cdot |Q| \cdot |L|/(q-\epsilon 1)_{2'}}{|Q|}$$
$$= \frac{(q-\epsilon 1)_2 \cdot |M|}{(q-\epsilon 1)_{2'} \cdot (q-\epsilon 1)} = \frac{|M|}{|A|}.$$

We can also check, using the fact that F acts trivially on Z', that $A \cap L \leq \mathbf{T}^F \cap \mathbf{L}^F = \mathbf{T}_1^F$, so $|AL/L| = |A/(A \cap L)| = |A|/|A_1| = (q-\epsilon 1)_{2'}$. Hence $\mathbf{O}^2(M/L) = AL/L$, $M = ALP = A\mathbf{O}^{2'}(L)P = A\mathbf{O}^{2'}(M)$, and thus

$$M = \mathbf{O}^{2'}(M) \times A; \tag{3.10}$$

in particular, we can canonically identify the set of linear 2-Brauer characters of M with $\mathrm{Irr}(A)$.

We can view M as the centraliser of any $s \in A \smallsetminus \mathbf{Z}(H)$ in the dual group. It follows from [DM, Thm. 1.3.25] that the map $\varepsilon_{\mathbf{H}}\varepsilon_{\mathbf{M}}R_M^H$ gives a bijection between the rational Lusztig series $\mathcal{E}(M,(s))$ and $\mathcal{E}(H,(s))$. We also have that $L \lhd M$ with $M/L \cong \mathbf{T}_2^F \cong C_{q-\epsilon 1}$, and so $\mathbf{O}^{2'}(M/L) = PL/L$. Here, $\mathrm{Out}(S) = (H/\mathbf{Z}(H)) \rtimes D$, and $J = \mathrm{Stab}_D(\theta)$ is cyclic by (3.8). Now we can take $G = H \rtimes J$ (so that $\mathbf{C}_G(S) = \mathbf{Z}(H)$), and define θ^* to be the restriction to S of the Brauer character χ°, where $\chi = \pm R_M^H(\tilde{\theta}) \in \mathrm{Irr}(H)$, with $\tilde{\theta} \in \mathrm{IBr}(\mathbf{T}^F) = \mathrm{IBr}(A)$ viewed as a linear character of $M/\mathbf{O}^{2'}(M)$ by (3.10). At the same time, this χ is the semisimple character χ_s labelled by the semisimple $2'$-element s corresponding to $\tilde{\theta}$. The conjugacy class s^S labels the union of 2-blocks that contain χ°. Note that two such elements s, s' are H-conjugate only when $s' = s$. (Indeed, as mentioned above, $M = \mathbf{C}_H(s) = \mathbf{C}_H(s') = \mathbf{C}_H(t)$. Now if $s' = s^x$ for some $x \in H$, then $t^x = t$ for the unique central involution in M that belongs to L, whence $x \in \mathbf{C}_H(t) = M$, and so $s^x = s'$.)

Note that $H/S \cong C_{q-\epsilon 1}$ has exactly $(q-\epsilon 1)_{2'}$ linear Brauer characters λ_v labelled by the elements $v \in \mathbf{O}_{2'}(\mathbf{Z}(H)) = \mathbf{O}_{2'}(\mathbf{T}_2^F)$, cf. [DM, Prop. 13.30]. Now, multiplying χ_s by λ_v amounts to replacing s by sv. Hence, $\chi_s = \chi_s\lambda_v$ only when s and sv are H-conjugate, i.e., if $v = 1$, as explained above. Since H/S is cyclic and $2 \nmid \chi(1)$, by Lemmas 3.2 and 3.3 of [KT], this implies that $\chi^\circ|_S$ is irreducible, justifying the definition $\theta^* = \chi^\circ|_S$. Next, for any two such s, s', χ_s° and $\chi_{s'}^\circ$ can agree on S only when they differ by some λ_v. Given the canonical isomorphisms $M/\mathbf{O}^{2'}(M) \cong A \cong \mathbf{N}_H(P)/P$ of (3.9) and (3.10), M also has exactly $(q-\epsilon 1)_{2'}$ linear Brauer characters that are trivial at $A_1 < S$, which can be identified with the restrictions to M of the λ_v's (when we identify M with M^*). So, multiplying χ_s by λ_v amounts to multiplying $\tilde{\theta}$ by a character of A that is trivial on A_1 (see [DM, Prop. 12.2] for the values of R_M^H), and the latter operation does not change $\theta = \tilde{\theta}|_{A_1}$. Thus the map $\theta \mapsto \theta^*$ is injective. Next, $\delta \in D$ fixes θ^* only when χ_s° and $\delta(\chi_s)^\circ$ agree on S, and the previous arguments then show that δ fixes θ, that is, $\mathrm{Stab}_D(\theta^*) = J$. The map $\theta \mapsto \theta^*$ is D-equivariant; it is certainly $\mathbf{N}_H(Q)$-equivariant. The calculation (3.4) can be repeated to show that θ and θ^* lie above the same character of Z. As J is cyclic, both θ and θ^* extend to J, and so we are done.

Case 8: Suppose that $X = \mathrm{PSp}_{2n}(q)$ with $2 \nmid q$ and $n \geq 1$.

Here, the inductive AWC condition was shown in [FM], implying the inductive Sylow-AWC condition.

Aside from groups already considered in Cases 5–8 above, and $X = G_2(3)$ and $\Omega_7(3)$ (for which we may have $S = 3X$) we have $S = X$ by [KL, Thm. 5.1.4], and $\mathbf{N}_S(Q) = Q$ by [Ko, Cor.], hence we are done. For the groups $S = 3.G_2(3)$, $3.\Omega_7(3)$, $3_1.\,\mathrm{PSU}_4(3)$ and $3_2.\,\mathrm{PSU}_4(3)$ we are in Case 2. □

Theorem B now follows by combining Theorem 3.3 with Theorem 2.2.

As we have mentioned in the Introduction, we suspect that the equality $l(G) = l(\mathbf{N}_G(P))$ is only possible when $\mathbf{N}_G(P) = G$. This is at least the case for p-solvable groups, as shown in ([MW2, Thm. 23.12]). In the general case, Alperin's Weight Conjecture, in its block-wise form, implies (as pointed out by Alperin) that if B and b are Brauer correspondent p-blocks of a finite group, then $l(B) \geq l(b)$, where $l(B)$ is the number of irreducible Brauer characters in the block B. Hence, if $l(G) = l(\mathbf{N}_G(P))$, AWC implies that G has only p-blocks of maximal defect, and that $l(B) = l(b)$ in each case. By [Wi], we see that G is not a simple group either.

Remark 3.4. There are infinitely many simple and quasi-simple groups G with $l(G) = l(\mathbf{N}_G(P)) + 1$. For example, $G = \mathrm{SL}_2(q)$ with $p|q$ has

$$l(G) = q,\ l(\mathbf{N}_G(P)) = q - 1.$$

Further examples are given by $\mathrm{PSL}_3(3)$ and M_{11} with $p = 3$.

Remark 3.5. In Theorem C of [NST], it is proved that if G is p-solvable, q is any prime, $Q \in \mathrm{Syl}_q(G)$, and all the simple groups of order divisible by q involved in G satisfy the inductive McKay condition of [IMN] with respect to q, then

$$|\,\mathrm{IBr}_{q'}(G)| = |\,\mathrm{IBr}_{q'}(\mathbf{N}_G(Q))|\,,$$

where $\mathrm{IBr}_{q'}(G)$ is the set of p-Brauer characters of G degree not divisible by q. Using the p-solvability of G (in particular, [Na, Thm. 8.30]), we see that

$$\mathrm{IBr}_{q'}(\mathbf{N}_G(Q)) = \mathrm{IBr}(\mathbf{N}_G(Q)/Q').$$

Therefore, we have that $q \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$ if and only if $l(G) = l(\mathbf{N}_G(Q)/Q')$. This shows that the inductive McKay condition implies Conjecture D.

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