FINITE GROUPS WITH MINIMAL 1-PIM

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Abstract. Let $\mathbb{F}$ be a field of characteristic $\ell > 0$ and let G be a finite group. It is well-known that the dimension of the minimal projective cover Φ_1^G (the so-called 1-PIM) of the trivial left $\mathbb{F}[G]$-module is a multiple of the ℓ-part $|G|_\ell$ of the order of G. In this note we study finite groups G satisfying $\dim_{\mathbb{F}}(\Phi_1^G) = |G|_\ell$. In particular, we classify the non-abelian finite simple groups G and primes ℓ satisfying this identity (Thm. A). As a consequence we show that finite soluble groups are precisely those finite groups which satisfy this identity for all prime numbers ℓ (Cor.B). Another consequence is the fact that the validity of this identity for a finite group G and for a small prime number $\ell \in \{2, 3, 5\}$ implies the existence of an ℓ'-Hall subgroup for G (Thm. C). An important tool in our proofs is the super-multiplicativity of the dimension of the 1-PIM over short exact sequences (Prop. 2.2).

1. Introduction

Let G be a finite group, let ℓ be a prime number and let $\mathbb{F}$ be a field of characteristic ℓ. We denote by Φ_1^G, respectively Φ_1 if the ambient group is clear from the context, the projective cover of the trivial (left) $\mathbb{F}[G]$-module $\mathbb{F}$, that is the unique (up to isomorphism) projective indecomposable $\mathbb{F}[G]$-module with socle (and head) isomorphic to the trivial $\mathbb{F}[G]$-module $\mathbb{F}$. This module is also called the 1-PIM of $\mathbb{F}[G]$. Obviously, $\Phi_1^G(\mathbb{F}) = \Phi_1^G(\mathbb{F}_\ell) \otimes_{\mathbb{F}_\ell} \mathbb{F}$. It is well-known that the $\mathbb{F}$-dimension of Φ_1 is divisible by $|G|_\ell$, the ℓ-part of $|G|$. Following [16] we define

$$c_\ell(G) := \dim_{\mathbb{F}}(\Phi_1^G)/|G|_\ell.$$

The main purpose of this paper is to study pairs (G, ℓ) for which $c_\ell(G) = 1$, i.e., finite groups G and prime numbers ℓ for which Φ_1 has minimal $\mathbb{F}$-dimension. Let G be a finite group containing an ℓ'-Hall subgroup H. By Maschke's theorem, the trivial $\mathbb{F}[H]$-module $\mathbb{F}$ is projective. As ind_H^G is mapping projective $\mathbb{F}[H]$-modules to projective $\mathbb{F}[G]$-modules, $\Phi_1^G \simeq \mathrm{ind}_H^G(\mathbb{F})$. In particular, $c_\ell(G) = 1$. An ℓ-soluble group G has a unique G-conjugacy class of ℓ'-Hall subgroups, and thus $c_\ell(G) = 1$. For non-abelian finite simple groups we will show the following:

Theorem A. *Let G be a finite non-abelian simple group with $c_\ell(G) = 1$ for a prime divisor ℓ of $|G|$. Then one of the following holds:*

(a) $G = \mathfrak{A}_\ell$, $\ell \geq 5$,
(b) $G = \mathrm{L}_2(\ell)$, $\ell \geq 5$,
(c) $G = \mathrm{L}_n(q)$, $(q^n - 1)/(q - 1) = \ell^f$ *is a prime power,*
(d) $G = M_{11}$, $\ell = 11$, *or*
(e) $G = M_{23}$, $\ell = 23$.

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This is proved in Theorems 3.2, 4.1, 4.8, 5.8 and 6.1 using the classification of finite simple groups. As an immediate consequence of our result one obtains the following characterization of soluble groups:

Corollary B. *Let G be a finite group. Then G is soluble if and only if $c_\ell(G) = 1$ for all prime numbers ℓ.*

This follows from the fact that for all the exceptions (G, ℓ) in Theorem A there exists another prime divisor $p \neq \ell$ of $|G|$ with $c_p(G) > 1$, and the super-multiplicativity of c_ℓ over short exact sequences proved in Proposition 2.2 below.

From Theorem A one concludes easily that for every prime number ℓ there exists a finite group G satisfying $c_\ell(G) = 1$ which is not ℓ-soluble. Moreover, for small primes ℓ — which are usually the "bad primes" — one has the following phenomenon (cf. §7).

Theorem C. *Let $\ell \in \{2, 3, 5\}$ and let G be a finite group.*

(a) *G contains an ℓ'-Hall subgroup if and only if $c_\ell(G) = 1$.*

(b) *If G contains an ℓ'-Hall subgroup, then it contains a unique G-conjugacy class of ℓ'-Hall subgroups.*

Note that part (a) of Theorem C does not hold for primes $\ell \geq 7$ (cf. Remark 7.2). However, it is less clear whether (b) fails for all primes $\ell \geq 7$, although it certainly fails for some primes $\ell \geq 7$.

2. General statements

In this section ℓ will denote a prime number, and G will denote a finite group.

2.1. **The lift of Φ_1 to characteristic 0.** It is well-known that Φ_1 lifts to characteristic 0, that is, there exist a $\mathbb{Q}_\ell[G]$-module $\tilde{\Phi}_1$ whose ℓ-modular reduction equals Φ_1. Let $\mathrm{ch}(\tilde{\Phi}_1)$ denote the complex character associated to $\tilde{\Phi}_1$. In particular,

$$\dim_{\mathbb{F}}(\Phi_1) = \mathrm{ch}(\tilde{\Phi}_1)(1). \tag{2.1}$$

It is possible to describe the irreducible constituents of $\mathrm{ch}(\tilde{\Phi}_1)$ by Brauer reciprocity, i.e., we have

$$\langle \mathrm{ch}(\tilde{\Phi}_1), \chi \rangle_G = \langle 1_G, \bar{\chi} \rangle \qquad \text{for any } \chi \in \mathrm{Irr}(G), \tag{2.2}$$

where $\bar{\chi}$ denotes the ℓ-modular reduction of the complex character χ, and the right hand side is the number of trivial composition factors of $\bar{\chi}$.

2.2. **Properties of $c_\ell(G)$.** We have the following weak relation between $c_\ell(G)$ and $c_\ell(H)$ for subgroups $H \leq G$:

Lemma 2.1. *Let $H \leq G$. Then $c_\ell(G) \geq c_\ell(H)|H|_\ell/|G|_\ell$.*

Proof. Indeed, the restriction of Φ_1^G from G to H is still projective, and has the trivial module in the socle, so it contains Φ_1^H. Thus

$$c_\ell(G) = \mathrm{ch}(\tilde{\Phi}_1^G)(1)/|G|_\ell \geq \mathrm{ch}(\tilde{\Phi}_1^H)(1)/|G|_\ell = c_\ell(H)|H|_\ell/|G|_\ell \tag{2.3}$$

as claimed. □

Much more can be said in the case of normal subgroups:

Proposition 2.2. *Let N be normal in G. Then*

$$c_\ell(G) \geq c_\ell(N)\, c_\ell(G/N). \tag{2.4}$$

Furthermore, equality holds in the following cases:

(a) *If N is ℓ-soluble, then $c_\ell(G) = c_\ell(G/N)$.*
(b) *If G/N is soluble, then $c_\ell(G) = c_\ell(N)$.*
(c) *If $G = N \times H$, then $c_\ell(G) = c_\ell(N)c_\ell(H)$.*

Proof. Let L be a Sylow ℓ-subgroup of G. Then $L_N \colon = L \cap N$ is a Sylow ℓ-subgroup of N and LN/N is a Sylow ℓ-subgroup of G/N.

For a left $\mathbb{F}[G]$-module M the co-invariants with respect to N are given by

$$M_N \colon = \mathbb{F}[G/N] \otimes_{\mathbb{F}[G]} M. \tag{2.5}$$

By construction, M_N is an $\mathbb{F}[G/N]$-module. Since $\mathbb{F}[G]_N = \mathbb{F}[G/N]$, R_N is a projective left $\mathbb{F}[G]$-module for any projective left $\mathbb{F}[G]$-module R.

The head of $(\Phi_1^G)_N$ is a homomorphic image of Φ_1^G. Thus $(\Phi_1^G)_N = \Phi_1^{G/N}$, and

$$\dim_{\mathbb{F}}((\Phi_1^G)_N) = c_\ell(G/N) \cdot |L/L_N|. \tag{2.6}$$

Let $\_^* \colon = \mathrm{Hom}_{\mathbb{F}}(\_, \mathbb{F})$. Then $(\Phi_1^G)^*$ — considered as left $\mathbb{F}[G]$-module — is isomorphic to Φ_1^G. Moreover, as $((\Phi_1^G)^*)_N \simeq ((\Phi_1^G)^N)^*$, where $\_^N \colon = \mathrm{Hom}_N(\mathbb{F}, \mathrm{res}_N^G(\_))$ denote the N-invariants, one has

$$(\Phi_1^G)^N \simeq \mathbb{F}^r, \tag{2.7}$$

where $r \colon = c_\ell(G/N) \cdot |L/L_N|$, and the isomorphism is an isomorphism of left $\mathbb{F}[N]$-modules. Restriction is mapping projectives to projectives. As

$$(\Phi_1^G)^N \leq \mathrm{soc}(\mathrm{res}_N^G(\Phi_1^G)), \tag{2.8}$$

(2.7) implies that $\mathrm{res}_N^G(\Phi_1^G)$ contains an $\mathbb{F}[N]$-submodule isomorphic to $(\Phi_1^N)^r$. Thus

$$c_\ell(G) \cdot |L| = \dim_{\mathbb{F}}(\Phi_1^G) \geq c_\ell(G/N) \cdot |L/L_N| \cdot \dim_{\mathbb{F}}(\Phi_1^N) = c_\ell(G/N) \cdot c_\ell(N) \cdot |L| \tag{2.9}$$

as claimed.

Part (a) is [16, Prop. 2.3(b)]. For part (b) we may assume by induction that G/N is an abelian group of prime power order. If G/N is an elementary abelian ℓ-group, Φ_1^G is an $\mathbb{F}_\ell$-submodule of $\mathrm{ind}_N^G(\Phi_1^N)$. This implies $c_\ell(G) \leq c_\ell(N)$ and thus (2.4) yields the claim.

Assume that G/N is abelian of order prime to ℓ and that $\mathbb{F}$ is algebraically closed. Since $\mathbb{F}$ is a submodule of Φ_1^N and as G/N is an ℓ'-group, $\mathrm{ind}_N^G(\Phi_1^N)$ contains the semi-simple left $\mathbb{F}[G]$-module $\mathrm{ind}_N^G(\mathbb{F})$. Thus $\mathrm{ind}_N^G(\Phi_1^N)$ contains also a projective summand $\Phi_1^G \otimes S$ for every 1-dimensional $\mathbb{F}[G/N]$-module S. Comparing dimensions one obtains

$$|G/N| \cdot \dim_{\mathbb{F}}(\Phi_1^N) = \dim_{\mathbb{F}}(\mathrm{ind}_N^G(\Phi_1^N)) \geq |G/N| \cdot \dim_{\mathbb{F}}(\Phi_1^G). \tag{2.10}$$

Since clearly $\dim_{\mathbb{F}}(\Phi_1^G) \geq \dim_{\mathbb{F}}(\Phi_1^N)$, equality holds in (2.10) and this yields the claim. The third part is [16, Prop. 2.1(b)]. □

We are not aware of any example where the inequality in Proposition 2.2 is strict.

2.3. **Groups with $c_\ell(G) = 1$.** We now collect some properties of groups with $c_\ell(G) = 1$:

Lemma 2.3. *Assume that $c_\ell(G) = 1$ for some finite group G and some prime ℓ dividing $|G|$. Then there exists $1 \neq \chi \in \mathrm{Irr}(G)$ in the principal ℓ-block of G with $\chi(1) < |G|_\ell$.*

Proof. Since ℓ divides $|G|$, the principal block contains more than one character, hence $\mathrm{ch}(\tilde{\Phi}_1)$ contains some non-trivial irreducible constituent. Since $c_\ell(G) = 1$ if and only if $\mathrm{ch}(\tilde{\Phi}_1)(1) = |G|_\ell$, the claim follows. □

Lemma 2.4. *Let G be a finite group satisfying $c_\ell(G) = 1$.*

(a) *If $H \leq G$ is of ℓ'-index, then $c_\ell(H) = 1$.*
(b) *If $N \lhd G$ is normal, then $c_\ell(G/N) = c_\ell(N) = 1$.*

Proof. Part (a) is a special case of Lemma 2.1, (b) follows immediately from Proposition 2.2. □

Example 2.5. Note that the assumption that H has ℓ'-index in G is necessary in part (a) above: $\mathfrak{A}_5$ is a subgroup of $\mathrm{L}_2(31)$ of index $2^3.31$, but

$$c_2(\mathfrak{A}_5) = 3 > c_2(\mathrm{L}_2(31)) = 1$$

by Theorem 5.2 and Proposition 4.3. By Proposition 2.2, there are no such examples if H is normal.

Finite groups for which the 1-PIM has the smallest possible dimension have the following elementary property:

Lemma 2.6. *Let G be a finite group, ℓ a prime with $c_\ell(G) = 1$ and $L \in \mathrm{Syl}_\ell(G)$. Then any ℓ'-subgroup of G which is normalized by L is contained in $O_{\ell'}(G)$.*

Proof. Let $S \leq G$ be of ℓ'-order such that $L \leq N_G(S)$. Let $H\colon = S.L$. Since H is of ℓ'-index in G, $\Phi_1^H = \mathrm{res}_H^G(\Phi_1^G)$. However, since S is normal in H and $L \simeq H/S$, one has an isomorphism $\Phi_1^H \simeq \mathrm{ind}_S^H(\mathbb{F}_\ell)$. Hence S lies in the kernel of the representation afforded by Φ_1^H. So S lies in the kernel of the representation afforded by Φ_1^G, and thus in $O_{\ell'}(G)$. □

Proposition 2.2 and Lemma 2.6 show the following:

Corollary 2.7. *Let G be a finite group satisfying $c_\ell(G) = 1$, and let $\sigma \in \mathrm{Aut}(G)$ be of ℓ' order. Then σ does not centralize any Sylow ℓ-subgroup of G.*

2.4. **Divisors of cyclotomic polynomials.** We need the following well-known statement, where Φ_d denotes the dth cyclotomic polynomial (see [11, Lemma 5.2], for example).

Lemma 2.8. *Let $q \geq 1$, ℓ a prime not dividing q, and d the multiplicative order of q mod ℓ.*

(a) *We have $\ell | \Phi_f(q)$ if and only if $f = d\ell^i$ for some $i \geq 0$.*
(b) *If $\ell^2 | \Phi_f(q)$ then $f = d$, or $\ell = f = 2$.*

Let q and d be positive integers, $q, d \geq 2$. A prime number r is called a *primitive prime divisor* of $q^d - 1$, if r divides $q^d - 1$ but does not divide $q^k - 1$ for $k < d$. Clearly, a primitive prime divisor of $q^d - 1$ is also a divisor of $\Phi_d(q)$. Moreover, for such a divisor q has multiplicative d modulo r. The following statement is known as Zsigmondy's theorem.

Lemma 2.9. *Let q and d be positive integers $q \geq 2$, $d \geq 3$.*

(a) *If $(q, d) \neq (2, 6)$, then $q^d - 1$ has a primitive prime divisor r.*
(b) *For such a divisor one has $r \equiv 1 \mod d$.*

Proof. For (a) see [17]. Part (b) is a direct consequence of Fermat's theorem. □

3. Alternating groups

In this section we determine the alternating and symmetric groups with $c_\ell(G) = 1$. The following is well-known, but we have not been able to find a suitable reference:

Lemma 3.1. *Let $\ell \geq 5$. Then $c_\ell(\mathfrak{A}_{2\ell-1}) = (\binom{2\ell-1}{\ell-2} + 1)/\ell$.*

Proof. We first investigate the 1-PIM of $\mathfrak{S}_{2\ell-2}$. Since this group has cyclic Sylow ℓ-subgroup, it suffices to see which ordinary irreducible character is connected to the trivial character on the Brauer tree. For this, we describe the decomposition of the permutation character of the Young subgroup $H := \mathfrak{S}_{\ell-1} \times \mathfrak{S}_{\ell-1}$ of $\mathfrak{S}_{2\ell-2}$. It decomposes as $\psi_1 := \chi_{2\ell-2} + \chi_{(\ell-1)^2}$ plus irreducibles not lying in the principal block, by the Murnaghan-Nakayama rule. Since the order of H is prime to ℓ, ψ_1 is the character of a projective module. Again by the Murnaghan-Nakayama rule, the induction of ψ_1 to $\mathfrak{S}_{2\ell-1}$ decomposes as $\psi_2 := \chi_{2\ell-1} + \chi_{(\ell-1)^2 1}$ plus irreducibles not lying in the principal block. So ψ_2 is projective, and hence the character of the 1-PIM. The hook formula then gives the claim, using that both constituents restrict irreducibly to $\mathfrak{A}_{2\ell-1}$. □

Theorem 3.2. *Let $G = \mathfrak{A}_n$, $n \geq 5$, be a simple alternating group and $\ell \leq n$. Then $c_\ell(G) > 1$ unless $n = \ell$.*

Proof. First assume that $\ell \geq 5$. Since $\mathfrak{A}_\ell$ has the ℓ'-Hall subgroup $\mathfrak{A}_{\ell-1}$, this is an example for ℓ. On the other hand, the smallest non-trivial character degree of $\mathfrak{A}_{\ell+1}$ is ℓ, so we don't get an example there by Lemma 2.3. The group $\mathfrak{A}_n$ with $\ell+1 \leq n \leq 2\ell-1$ contains the ℓ'-index subgroup $\mathfrak{A}_{\ell+1}$, so does not give an example by Lemma 2.4(a).

Next, $c_\ell(\mathfrak{A}_{2\ell-1}) = (\binom{2\ell-1}{\ell-2} + 1)/\ell > 3\ell$ by Lemma 3.1. By Lemma 2.1 this shows that $c_\ell(\mathfrak{A}_{2\ell}) > 3$, so $\mathfrak{A}_{2\ell}$ is not an example.

For $2\ell < n < \ell^2$, $\mathfrak{A}_n$ contains the ℓ'-index subgroup $\mathfrak{A}_{2\ell} \times C_\ell^k$, where $k := \lfloor n/\ell \rfloor - 2$, so this isn't an example either.

For $n = n_0 + \ell^2$, $0 \leq n_0 < \ell$, $G := \mathfrak{A}_n$ contains the subgroup $H := \mathfrak{A}_{2\ell} \times \ldots \times \mathfrak{A}_{2\ell}$ ($m := \lfloor \ell/2 \rfloor$ factors). Since $c_\ell(\mathfrak{A}_{2\ell}) \geq 2$ we conclude that $c_\ell(H) \geq 2^m$ by Proposition 2.2(c). But $|G|_\ell/|H|_\ell = \ell < c_\ell(H)$, which implies that $\mathfrak{A}_n$ is not an example, by Lemma 2.1. Finally, for $n \geq \ell^2 + \ell$, $\mathfrak{A}_n$ contains the ℓ'-index subgroup $C_\ell \wr \mathfrak{A}_m$ with $m := \lfloor n/\ell \rfloor \geq \ell+1$, hence is not an example by our preceding discussion.

Now let $\ell = 3$. Since $\mathfrak{A}_3$ and $\mathfrak{A}_4$ are soluble, we may take $n \geq 5$. $\mathfrak{A}_5$ and $\mathfrak{A}_6$ are no examples by [16, §3]. From the explicit knowledge of decomposition matrices [9] it follows that $\mathfrak{A}_n$ is not an example for $7 \leq n \leq 9$. We may now argue as before to exclude $n \geq 10$.

Finally, let $\ell = 2$. Again by [16, §3], $\mathfrak{A}_5$ and $\mathfrak{A}_6$ are no examples. In fact, the tables in [9] show that $c_2(\mathfrak{A}_n) \geq 3$ for $5 \leq n \leq 9$, so these are no examples. For $n \geq 10$, $\mathfrak{A}_n$ contains the subgroup $C_2 \wr \mathfrak{A}_m$, $m = \lfloor n/2 \rfloor \geq 5$, of $2'$-index, hence again $c_2(\mathfrak{A}_n) \geq 3$. □

With Proposition 2.2 we conclude:

Corollary 3.3. *Let $G = \mathfrak{S}_n$, $n \geq 5$, and $\ell \leq n$. Then $c_\ell(G) > 1$ unless $n = \ell$.*

4. Groups of Lie type in non-defining characteristic

We now consider groups of Lie type for primes ℓ different from their defining characteristic. The case of defining characteristic will be considered in the next section. We start with exceptional groups of Lie type. Using Lusztig's classification of complex irreducible characters, Lübeck [10] has determined the smallest non-trivial character degrees of the simple exceptional groups of Lie type. With this result we obtain:

Theorem 4.1. *Let G be a simple exceptional group of Lie type, and ℓ a prime divisor of $|G|$ different from the defining characteristic of G. Then $c_\ell(G) > 1$.*

Proof. Let $\mathbf{G}$ denote a simple simply-connected algebraic group of exceptional type, and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism with group of fixed points $\mathbf{G}^F$ such that $G = \mathbf{G}^F/Z(\mathbf{G}^F)$. (Such $\mathbf{G}$ exists unless $G = {}^2F_4(2)'$, the Tits group. For the latter, the assertion can be checked directly from the known decomposition numbers [9].) Let L denote a Sylow ℓ-subgroup of G. Since L is nilpotent, and ℓ is not the defining characteristic of G, L is contained in the normalizer of some maximal torus T of G, by [13, Cor. II.5.19(a)]. Thus, $|L| \leq |T|_\ell\,|W|_\ell$, where W denotes the Weyl group of $\mathbf{G}$. Now $|T|$ is a product of cyclotomic polynomials in q, the order of the underlying finite field. Thus, if r denotes the rank of $\mathbf{G}$, we clearly have $|T| \leq (q+1)^r$ and hence $|L| \leq (q+1)^r|W|_\ell$.

Comparing this bound with the lower bound for minimal non-trivial character degrees in [10] and using Lemma 2.3, we immediately find that the assertion holds if $q \geq 4$. Thus, there remain only finitely many cases to check. For those, we compute the precise value of $|G|_\ell$, and check that $|G|_\ell - 1$ is not a non-negative integral linear combination of the non-trivial character degrees $\chi(1) < |G|_\ell$.

We demonstrate the argument on the case of $G = F_4(q)$. The order of a Sylow ℓ-subgroup is bounded above by $(q+1)^4.|W|_\ell$, where $|W| = 2^7.3^2$. The smallest degree of a non-trivial complex character equals $\frac{1}{2}q(q^3-1)^2(q^4+1)$ if q is even, respectively q^8+q^4+1 if q is odd. It follows that $q \leq 3$ if $c_\ell(G) = 1$. For $q = 3$, if $\ell \geq 5$ then $|G|_\ell \leq (q+1)^4 = 256$, which is too small. For $\ell = 2$, $|G|_2 = (q+1)^4.2^7 = 4^4.2^7 = 32768$. The only non-trivial degree below this bound is 6643, which does not divide 32768-1. For $q = 2$ we have $\ell \neq 2$, so $|G|_\ell \leq (q+1)^4.3^2 = 3^6 = 729$, smaller than 833, the smallest non-trivial character degree. The other families of groups can be dealt with similarly. □

Let's turn to the classical groups of Lie type.

Also, we make use of the following consequence of Lemma 2.6:

Lemma 4.2. *Let $\mathbf{G}^F$ be a simple algebraic group of adjoint type defined over a field of characteristic p, $F : \mathbf{G} \to \mathbf{G}$ the corresponding Frobenius morphism and $G := \mathbf{G}^F$. Let $\ell \neq p$ be a prime number and $L \in \mathrm{Syl}_\ell(G)$. Assume that $c_\ell(G) = 1$ and that $[G, G]$ is non-abelian simple. Then there exists an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ such that $\mathbf{T}^F$ is an ℓ-group and $L \leq N_G(\mathbf{T}^F)$.*

Proof. As we assumed that $[G, G]$ is non-abelian simple, we have $O_{\ell'}(G) = 1$. By [13, Cor. II.5.19(a)], there exists an F-stable maximal torus $\mathbf{T}$ such that $L \leq N_G(\mathbf{T}^F)$. Then $L \leq N_G(O_{\ell'}(\mathbf{T}^F))$, and thus by Lemma 2.6, $O_{\ell'}(\mathbf{T}^F) = 1$. Hence $\mathbf{T}^F$ is an ℓ-group. □

Note further that by Proposition 2.2, if G is a finite group with a unique non-abelian simple composition factor S, then $c_\ell(G) = c_\ell(S)$.

The groups $\mathrm{L}_2(q)$ were handled in [16, §3]. Since this result is important as an induction base, we repeat it here:

Proposition 4.3. *Let $G = \mathrm{L}_2(q)$, $q \geq 4$. Then $c_\ell(G) = 1$ for a prime divisor ℓ of $|G|$, $\ell \nmid q$, if and only if $q + 1 = \ell^a$ is a power of ℓ.*

This happens if $\ell = 2^{2^f} + 1 \geq 5$ is a Fermat prime, or if $\ell = 2$ and $q = 2^a - 1$ is a Mersenne prime, or if $(q, \ell^a) = (8, 9)$.

The further examples occurring in projective special linear groups are generalizations of this case:

Proposition 4.4. *Let $G = \mathrm{L}_n(q)$ with $n \geq 3$. Then $c_\ell(G) = 1$ for some prime divisor ℓ of $|G|$, $\ell \nmid q$, if and only if $(q^n - 1)/(q - 1) = \ell^a$ is a power of the prime ℓ. In the latter case, n is necessarily a prime.*

Proof. First note that $\ell = 2$ does not give an example. Indeed, for $\ell = 2$ by Hiss [5, Thm. B], $\mathrm{ch}(\tilde{\Phi}_1^G)$ contains the Steinberg character of degree $q^{n(n-1)/2}$, while the Sylow 2-subgroup has order bounded above by $(q+1)^{n-1}(n!)_2$, too small for $(n, q) \neq (3, 3)$. The latter case can be discarded by using the precise order of the Sylow 2-subgroup.

Hence from now on we may assume that $\ell \neq 2$. We work with the group $\tilde{G} := \mathrm{PGL}_n(q)$ instead of G, as we may by a prior observation. By Lemma 4.2 there exists a maximal torus T of $\tilde{G}$ whose order is a power of ℓ. Now $|T|(q-1) = \prod_{i=1}^r (q^{m_i} - 1)$ for some partition $(m_1, \ldots, m_r)$ of n. If this is a power of ℓ, then either $r = 1$ or $\ell | (q - 1)$. In the former case $|T| = (q^n - 1)/(q - 1)$ is a power of ℓ, and ℓ is a Zsigmondy prime for $q^n - 1$. Thus the end node parabolic subgroups are ℓ'-Hall subgroups and so $c_\ell(G) = 1$ in this case.

Hence we may assume that $\ell | (q-1)$, and then $|T| = (q-1)^{n-1}$ is a power of ℓ. The normalizer $N_{\tilde{G}}(T)$ is an extension of T by the symmetric group $\mathfrak{S}_n$. Corollary 3.3 together with Lemma 2.4(a) force that $\ell \geq n$ if $c_\ell(G) = 1$, or $(\ell, n) = (3, 4)$ (since $\ell \neq 2$ by our first reduction). If $\ell > n$ then the order formula for $\tilde{G}$ shows that T

contains a Sylow ℓ-subgroup of $\tilde{G}$, and T is contained in the subgroup $\mathrm{GL}_{n-1}(q)$. By Lemma 2.4(a) and induction, with induction base the case $n = 2$ in Proposition 4.3, this forces $(q^{n-1} - 1)/(q - 1) = \ell^a$ (note that this is also true for the solvable cases $n = 3$, $q \in \{2, 3\}$). So both $q - 1$ and $(q^{n-1} - 1)$ are powers of ℓ, which is only possible if $n = 2$ or $\ell = 2$, both of which are excluded.

So we are left with $\ell = n$, or $(\ell, n) = (3, 4)$. In the first case, $\ell = n$ divides $q - 1$, and ℓ divides $|\mathfrak{S}_n|$ exactly once, so a Sylow ℓ-subgroup of G has order $(q-1)^{n-1}$. But by [14, Thm. 1.1] the smallest non-trivial character degree of G is $(q^n - q)/(q - 1)$, which is bigger.

If $(\ell, n) = (3, 4)$ and $q - 1$ is a power of 3, then necessarily $q = 4$, so $G = \mathrm{L}_4(4)$, which is not an example. □

The case of unitary groups is rather similar, except that we find no examples since there are no end node parabolics:

Proposition 4.5. *Let $G = \mathrm{U}_n(q)$ with $n \geq 3$, $(n, q) \neq (3, 2)$. Then $c_\ell(G) > 1$ for all prime divisor ℓ of $|G|$, $\ell \not| q$.*

Proof. In the cases $\ell = 2$, or $n = 3$ and $\ell|(q^3+1)$, $\tilde{\Phi}_1^G$ contains the Steinberg module of dimension $q^{n(n-1)/2}$ (see [5, Thm. B]), which is larger than the order of the Sylow ℓ-subgroup of G, except when $(n, q) = (3, 3)$. The latter case can be ruled out by [9]. Thus we may assume $\ell > 2$, and $n > 3$ when $\ell|(q^3 + 1)$.

Let T be a maximal torus of $\tilde{G} := \mathrm{PGU}_n(q)$ whose normalizer contains a Sylow ℓ-subgroup of $\tilde{G}$. Since

$$|T|(q + 1) = \prod_{i=1}^{r}(q^{m_i} - (-1)^{m_i})$$

for some partition $(m_1, \ldots, m_r)$ of n, we conclude by Lemma 4.2 that either $\ell|(q+1)$, or else ℓ is a Zsigmondy prime for $q^n - (-1)^n$. In any case, when $n = 3$ we find that $\ell|(q^3 + 1)$, which was excluded above, so we have $n \geq 4$. In the case that ℓ is a Zsigmondy prime for $q^n - (-1)^n$, the Brauer trees have been determined by Fong and Srinivasan [4]. The trivial character is connected to a character of degree

$$q^3 \frac{(q^{n-1} - (-1)^{n-1})(q^{n-2} - (-1)^{n-2})}{(q^2 - 1)(q + 1)}$$

which is larger than the order of a Sylow ℓ-subgroup.

Hence we have $\ell|(q + 1)$, and $|T| = (q + 1)^{n-1}$ is a power of ℓ. The normalizer of T is an extension by the symmetric group $\mathfrak{S}_n$, whence $\ell \geq n$ or $(\ell, n) = (3, 4)$ if $c_\ell(G) = 1$ by Corollary 3.3. If $\ell > n$ then T itself contains a Sylow ℓ-subgroup, as does the subgroup $\mathrm{GU}_{n-1}(q)$. Since $n \geq 4$, the latter is not an example, except for $q = 2$ when $\mathrm{GU}_3(2)$ is solvable. But $\mathrm{U}_4(2)$ is in [9] and gives no example. If $\ell = n$, a Sylow ℓ-subgroup of G has order $|L| = (q + 1)^{n-1}$. On the other hand, for $(n, q) \neq (4, 2), (4, 3)$, the only non-trivial character degrees $\chi(1)$ of G below this bound are

$$\frac{q^n - (-1)^n}{q + 1}, \ \frac{q^n + (-1)^n q}{q + 1}$$

(see [14, Table V]), which both satisfy $|L|/2 < \chi(1) < |L| - 1$. Thus, $|L| - 1$ is not a nonnegative integral linear combination of the two degrees and consequently $c_\ell(G) > 1$. The cases $(n, q) \in \{(4, 2), (4, 3)\}$ can be excluded using [9].

Finally, if $(\ell, n) = (3, 4)$, note that q and $q + 1$ both being prime powers forces $q \in \{2, 8\}$ and $\ell = 3$. The first case has already been considered. For $q = 8$, so $G = \mathrm{U}_4(8)$, the Sylow 3-subgroup has order $3^7 = 2187$, and the smallest character degrees are 455, 456, 3705. It is easily seen that $2187 - 1$ is not a nonnegative integral linear combination of 455 and 456, which completes the proof. □

Proposition 4.6. *Let $G = \mathrm{S}_{2n}(q)$ with $n \geq 2$, $(n, q) \neq (2, 2)$. Then $c_\ell(G) > 1$ for all prime divisor ℓ of $|G|$, $\ell \nmid q$.*

Proof. First assume that $\ell | (q+1)$ (this covers in particular the case $\ell = 2$). Then by [5, Thm. B] $\mathrm{ch}(\tilde{\Phi}_1^G)$ contains the Steinberg character of G, so $\mathrm{ch}(\tilde{\Phi}_1^G)(1) \geq q^{n^2} + 1$. On the other hand, the order of a Sylow ℓ-subgroup of G is bounded above by

$$(q + 1)^n \, |2^n.\mathfrak{S}_n|_\ell \leq (q + 1)^n \, 2^{2n}.$$

This is too small unless $(n, q) \in \{(2, 2), (2, 3), (2, 4), (3, 2)\}$. In the last two cases, the actual order of a Sylow ℓ-subgroup is still smaller than $q^{n^2} + 1$, while the first two groups are contained in [9].

So from now on $\ell \nmid (q+1)$. The image H in G of the standard subgroup $\mathrm{Sp}_{2n-2}(q) \times \mathrm{Sp}_2(q)$ of $\mathrm{Sp}_{2n}(q)$ has index $(q^{2n}-1)/(q^2-1)$ times a power of q. Hence, by Lemma 2.4 we have $c_\ell(G) > 1$ unless $c_\ell(H) = 1$ or $\ell | (q^{2n} - 1)/(q^2 - 1)$. Now by induction, with induction base the case $n = 1$ in Proposition 4.3, $c_\ell(H) = 1$ only if $n = 2$ and either $\ell | (q + 1)$, which is excluded here, or $q \leq 3$, which is in [9].

So we may assume that $\ell | (q^{2n} - 1)/(q^2 - 1)$. Let $d := \min\{m \mid \ell | (q^{2m} - 1)\}$, and H the image in G of the field extension subgroup $\mathrm{Sp}_{2m}(q^d)$ of $\mathrm{Sp}_{2n}(q)$, $m := \lfloor n/d \rfloor$. Its index is prime to ℓ, and it is a proper subgroup for $d \neq 1$. In that case by induction we have $c_\ell(H) > 1$ unless $d = n$.

If $d = 1$ then $\ell | \gcd(q^2 - 1, (q^{2n} - 1)/(q^2 - 1))$, so $\ell | n$. Moreover, by the first part of the proof, we may assume that in fact $\ell | (q - 1)$. The Sylow ℓ-subgroups are then contained in the normalizers of tori of order $(q - 1)^n$. The normalizer quotient is the Weyl group of type B_n, which itself is a wreath product $2 \wr \mathfrak{S}_n$. If $\ell < n$, this is not an example by Corollary 3.3. If $\ell = n$, the Sylow ℓ-subgroup has order $n(q - 1)_\ell^n$. By [14, Th. 5.2] the smallest degrees of non-trivial irreducible characters of G are $(q^n \pm 1)/2$, and all other degrees are at least $\frac{1}{2}(q^{2n} - 1)/(q+1)$. (Note that $n = \ell \geq 3$ by assumption.) But by loc. cit. the characters of degree $(q^n \pm 1)/2$ lie in Lusztig series indexed by elements of order 2 in the dual group, hence not in the principal ℓ-block when $\ell \neq 2$ by [1]. Lemma 2.3 now shows that $c_\ell(G) > 1$.

If $d = n$ then $\ell | (q^{2n} - 1)$, but ℓ does not divide $q^{2m} - 1$ for $m < n$. Hence $\ell | (q^n + 1)$, or $\ell | (q^n - 1)$ and n is odd. In both cases, ℓ is a Zsigmondy prime for $q^n \pm 1$, and the Sylow ℓ-subgroups of G are cyclic. By [4], the trivial character is connected to a character of degree

$$\frac{q}{2} \frac{(q^n - 1)(q^{n-1} + 1)}{q - 1} \qquad \text{or} \qquad \frac{q}{2} \frac{(q^n + 1)(q^{n-1} - 1)}{q - 1}$$

on the ℓ-Brauer tree. Both are bigger than the maximal order $n(q^n+1)$ of a Sylow ℓ-subgroup. □

Proposition 4.7. *Let $G = \mathrm{O}_n^{(\pm)}(q)$ with $n \geq 7$. Then $c_\ell(G) > 1$ for all prime divisor ℓ of $|G|$, $\ell \nmid q$.*

Proof. We argue by induction on n, the induction base being given by the cases of $\mathrm{O}_6^+(q) = \mathrm{L}_4(q)$ and $\mathrm{O}_6^-(q) = \mathrm{U}_4(q)$ which were treated in Propositions 4.4 and 4.5.

By [5], the prime $\ell = 2$ can be discarded, and also the prime divisors of $q+1$ when $G \neq \mathrm{O}_n^-(q)$. Now first assume that $n = 2m+1$ is odd. Then $\mathrm{SO}_{2m+1}(q)$ contains $\mathrm{SO}_{2m}^{\pm}(q)$, with index $q^n \pm 1$ times a power of q. Thus, by induction and Lemma 2.4 examples can only occur for $\ell | \gcd(q^n-1, q^n+1) = 2$, which was already excluded.

So we may assume that $n = 2m$ is even. The group $G = \mathrm{SO}_{2m}^+(q)$ contains $\mathrm{SO}_{2m-1}(q)$ with index $q^m - 1$, so by induction $\ell | (q^m - 1)$. By Lemma 2.8 this implies that $m = d\ell^i$ where d is the order of q modulo ℓ. In particular, a Sylow ℓ-subgroup of G is contained in the normalizer N of a maximal torus T of order $(q^d-1)^{m/d}$, where N/T has a quotient $\mathfrak{S}_k$, $k = \ell^i$. If $i > 1$, this is no example by Corollary 3.3. If $i = 1$, a Sylow ℓ-subgroup of G has order at most $(q^d-1)^{m/d}(q^d-1)$, while the smallest non-trivial character degree of G is at least $(q^m-1)(q^{m-1}-1)/(q^2-1)$ [14, Table II], which is too big. If $i = 0$, so $d = m$, the Sylow ℓ-subgroup has order $(q^m - 1)$, which is again too small.

Finally, in the case $G = \mathrm{SO}_{2m}^-(q)$, the subgroup $\mathrm{SO}_{2m-1}(q)$ of index q^m+1 shows that $\ell | (q^m+1)$. As above this forces $m = d\ell^i$, and a Sylow ℓ-subgroup of G is contained in the normalizer N of a maximal torus T of order $(q^d+1)^{m/d}$, where N/T has a quotient $\mathfrak{S}_k$, $k = \ell^i$. We may now argue as in the previous case, using that the smallest non-trivial character degree of G equals $(q^m+1)(q^{m-1}-q)/(q^2-1)$ by [14, Table II], to show that no example can arise. □

The previous propositions constitute the proof of the following:

Theorem 4.8. *Let G be a finite simple classical group of Lie type and ℓ a prime different from the defining characteristic of G. Then $c_\ell(G) = 1$ if and only if $G = \mathrm{L}_n(q)$ and $(q^n-1)/(q-1)$ is a power of ℓ.*

5. Groups of Lie type in defining characteristic

Let $\mathbb{F} := \bar{\mathbb{F}}_p$ denote the algebraic closure of the finite field $\mathbb{F}_p$. Let $\mathbf{G}$ be a simple simply-connected affine algebraic group defined over $\mathbb{F}$, and let $F\colon \mathbf{G} \to \mathbf{G}$ be a Frobenius map with finite group of fixed points $G := \mathbf{G}^F$, a finite group of Lie type.

Let $\mathbf{B}$ be an F-stable Borel subgroup of $\mathbf{G}$ and $\mathbf{T} \leq \mathbf{B}$ an F-stable maximal torus contained in $\mathbf{B}$. Let $\mathbf{U}_\alpha \leq \mathbf{B}$ be the root group corresponding to the highest long root with respect to $(\mathbf{B}, \mathbf{T})$. Then $\mathbf{U}_\alpha$ is also F-stable provided G is not of type ${}^2B_2$, ${}^2G_2$ or ${}^2F_4$, and we let $q := |(U_\alpha)^F|$. Note that in this case q is always an integral power of p. In the previously mentioned exceptional cases the subgroup $\mathbf{U}_\alpha \times \mathbf{U}_\beta$ is F-stable, where β denotes the highest short root. Here we put $q^2 := |(\mathbf{U}_\alpha \times \mathbf{U}_\beta)^F|$.

The Lie rank of G is defined to be the dimension of the maximal torus $\mathbf{T}$.

5.1. **Parabolic descent.** Let $\mathbf{P} \leq \mathbf{G}$ be an F-stable parabolic subgroup of $\mathbf{G}$ and $\mathbf{L} \leq \mathbf{P}$ an F-stable Levi subgroup of $\mathbf{P}$. Then $\mathbf{H}\colon = [\mathbf{L}, \mathbf{L}]$ is an F-stable semisimple subgroup of $\mathbf{G}$. For short we call $\mathbf{P}$ of type X, if $(\mathbf{H}, F)$ has Lie type X.

Let $\pi\colon \tilde{\mathbf{H}} \to \mathbf{H}$ denote the simply-connected cover of $\mathbf{H}$. Then there exists a Frobenius map $F\colon \tilde{\mathbf{H}} \to \tilde{\mathbf{H}}$ such that π commutes with the action of F. One has the following "parabolic descent lemma".

Lemma 5.1. *Let $G = \mathbf{G}^F$ be a finite group of Lie type defined in characteristic p satisfying $c_p(G) = 1$, and let $\mathbf{P} \leq \mathbf{G}$ be an F-stable parabolic subgroup of type X. Let $\tilde{\mathbf{H}}$ be the simply-connected cover of $\mathbf{H}\colon = [\mathbf{L}, \mathbf{L}]$ and $F\colon \tilde{\mathbf{H}} \to \tilde{\mathbf{H}}$ the Frobenius map of $\tilde{\mathbf{H}}$ as constructed above such that $(\tilde{\mathbf{H}}, F)$ is of type X. Then $c_p(\tilde{\mathbf{H}}^F) = 1$.*

Proof. Since $\mathbf{P}^F$ has p'-index in G, the hypothesis implies that $c_p(\mathbf{P}^F) = 1$ (cf. Lemma 2.4(a)). Furthermore, as $\mathbf{P}^F = \mathbf{L}^F.R_u(\mathbf{P})^F$, Lemma 2.4(b) shows that $c_p(\mathbf{L}^F) = 1$. Since $\mathbf{H}^F$ has p'-index in $\mathbf{L}^F$, the previously mentioned argument shows that $c_p(\mathbf{H}^F) = 1$. The surjective map π induces a homomorphism of finite groups $\pi^F\colon \tilde{\mathbf{H}}^F \to \mathbf{H}^F$ whose kernel is a p'-group and whose image has p' index in $\mathbf{H}^F$. Thus Lemma 2.4 yields the claim. □

We first consider some small rank cases.

5.2. G **of type** A_1**.** The following result was proved by J.E. Humphreys in [6].

Theorem 5.2. *Let $G\colon = \mathrm{SL}_2(q)$, $q = p^f$. Then $c_p(G) = 2^f - 1$. In particular, $c_p(G) = 1$, if and only if $f = 1$.*

5.3. G **of type** A_2 **or** ${}^2A_2$**.** The following theorem is a combination of results of L. Chastkofsky [2, Thm.3, Cor.] and L. Chastkofsky and W. Feit [3].

Theorem 5.3.

(a) *Let $G\colon = \mathrm{SL}_3(2^f)$ or $G\colon = \mathrm{SU}_3(2^f)$. Then $c_2(G) = 6^f - 5^f$. In particular, $c_2(G) = 1$, if and only if $f = 1$.*
(b) *Let $G\colon = \mathrm{SL}_3(q)$, $q = p^f$, $p \neq 2$. Then $c_p(G) = 12^f - 6^f + 1$. In particular, $c_p(G) \neq 1$.*
(c) *Let $G\colon = \mathrm{SU}_3(q)$, $q = p^f$, $p \neq 2$. Then $c_p(G) = 12^f - 6^f - 1$. In particular, $c_p(G) \neq 1$.*

5.4. G **of type** B_2 **or** ${}^2B_2$**.** In [7], J.E. Humphreys has described the lift of the character of Φ_1 for the groups $G\colon = \mathrm{Sp}_4(p)$, $p > 7$, in terms of irreducible characters, and given the dimension of Φ_1 for $p \geq 5$. The groups $\mathrm{Sp}_4(2)$ and $\mathrm{Sp}_4(3)$ can be analyzed with [9]. All the necessary information is collected in the following theorem.

Theorem 5.4. *We have $c_2(\mathrm{Sp}_4(2)) = 5$, $c_3(\mathrm{Sp}_4(3)) = 2$ and $c_p(\mathrm{Sp}_4(p)) = 9$ for $p \geq 5$.*

The dimension of Φ_1 for the Suzuki groups has also been analyzed by L. Chastkofsky and W. Feit in [3]. They showed the following:

Theorem 5.5. *Let $G\colon = \mathrm{Suz}(2^f)$, $f = 2m + 1 \geq 1$. Then $c_2(G) = 2^{2f} - T_f 2^f - 1$, where*

$$T_n\colon = \left(\tfrac{1+\sqrt{5}}{2}\right)^n + \left(\tfrac{1-\sqrt{5}}{2}\right)^n \tag{5.1}$$

is the n^{th}-Lucas number. In particular, $c_2(G) = 1$ if and only if $f = 1$.

5.5. G **of type** G_2 **(or** $^2G_2$**).** The dimension of Φ_1 for $G = G_2(p)$ was analyzed by D. Mertens in [12]. He computed $c_p(G_2(p))$ for $p \leq 7$ explicitly, and showed that for $p > 7$ the decomposition of the projective $\mathbb{F}[G_2(p)]$-modules $Q(\mu)$ is generic (see [8, 18.6, Table 11]).

Theorem 5.6. *We have $c_2(G_2(2)) = 7$, $c_3(G_2(3)) = 21$, $c_5(G_2(5)) = 13$ and $c_p(G_2(p)) = 91$ for $p \geq 7$.*

For the groups of type $^2G_2$, note that $^2G_2(3) = \mathrm{Aut}(\mathrm{L}_2(8))$, so $c_3(^2G_2(3)) = 1$ by Proposition 2.2.

Proposition 5.7. *Let $G = {}^2G_2(q^2)$, $q^2 = 3^{2f+1} > 3$. Then $c_3(G) > 1$.*

Proof. For $q^2 = 27$, the claim follows with [9], so now assume that $q^2 > 27$. We show that $c_3(G) > 1$ by studying the decomposition

$$\mathrm{ch}(\tilde{\Phi}_1) = \sum_{\chi \in \mathrm{Irr}(G)} n_\chi \, \chi$$

of the 1-PIM $\tilde{\Phi}_1$ of G, with non-negative integers n_χ. Let's assume that $c_3(G) = 1$. Then we have

(a) $n_{1_G} = 1$,
(b) $n_\chi = 0$ if $\chi(1) > |G|_3 - 1$,
(c) $n_\chi = n_\psi$ if χ, ψ are algebraically conjugate over $\mathbb{Q}_3$, since the 1-PIM is rational over $\mathbb{Q}_3$,
(d) $n_\chi = 0$ if $1 \neq \chi$ is not cuspidal.

In order to see (d) assume that $n_\chi > 0$ for some non-cuspidal character χ. Then the Harish-Chandra restriction of χ is contained in the restriction of $\mathrm{ch}(\tilde{\Phi}_1)$, which is just $\mathrm{ch}(\tilde{\Phi}_1^T)$. But this is just the trivial character.

The degrees and most values of the ordinary irreducible characters of $G = {}^2G_2(q^2)$, $q^2 = 3^{2f+1}$ were determined by Ward [15]. Here, $|G|_3 = q^6$. The tables in loc. cit. show that the character degrees of cuspidal characters of G smaller than q^6 are

$$\frac{r}{2}(q^2-1)(q^2-3r+1), \; \frac{r}{2}(q^2-1)(q^2+3r+1),$$
$$r(q^4-1), \; (q^4-1)(q^2-3r+1), \; (q^2-1)(q^4-q^2+1),$$

with $r := q/\sqrt{3}$. We proceed to show that $n_\chi = 0$ for the characters χ of the last degree: The only non-trivial degree smaller than $q^6 - 1 - (q^2-1)(q^4-q^2+1) = 2(q^4-q^2)$ is $q^4 - q^2 + 1$ (if $q^2 > 27$), but this is not a divisor of $2(q^4-q^2)$.

Now the two characters of degree $r(q^2-1)(q^2-3r+1)/2$ have are irrational over $\mathbb{Q}_3$, with values in $\mathbb{Q}_3(\sqrt{-3})$, so either both or none occur in $\mathrm{ch}(\tilde{\Phi}_1)$, and the same applies to the characters of degrees $r(q^2-1)(q^2+3r+1)/2$ and $r(q^4-1)$. Thus we are left with degrees

$$r(q^2-1)(q^2-3r+1), \; r(q^2-1)(q^2+3r+1),$$
$$2r(q^4-1), \; (q^4-1)(q^2-3r+1).$$

From [15] we deduce the following values on 3-singular elements:

χ	$\chi(1)$	$3A$	$3BC$	$9A$
$\chi_2+\chi_3$	$r(q^2-1)(q^2-3r+1)$	q^2-r	$-r$	$2r$
$\chi_4+\chi_5$	$r(q^2-1)(q^2+3r+1)$	$-q^2-r$	$-r$	$2r$
$\chi_6+\chi_7$	$2r(q^4-1)$	$-2r$	$-2r$	$-2r$
χ_T	$(q^4-1)(q^2-3r+1)$	$-q^2+3r-1$	-1	-1
$\mathrm{ch}(\tilde{\Phi}_1)-1_G$	q^6-1	-1	-1	-1

Thus, the values of these characters on an element of order 9 are divisible by 3, except for the last ones. Since projective characters vanish on all p-singular elements, it follows that exactly one character of degree $(q^4-1)(q^2-3r+1)$ appears in $\tilde{\Phi}_1$, with multiplicity 1. The only way the remainder $3r(q^2-1)(q^2-r+1)$ may be written as a non-negative integral linear combination of the first three degrees is $2(\chi_2+\chi_3)(1)+(\chi_6+\chi_7)(1)$. But the linear combination

$$1_G+\chi_T+2(\chi_2+\chi_3)+\chi_6+\chi_7$$

does not vanish on elements of order 9. This contradiction shows that $c_3(G)>1$. □

5.6. **Finite groups of Lie type satisfying $c_p(G)=1$ in natural characteristic.** Collecting the information from the previous subsections we deduce the following theorem.

Theorem 5.8. *Let $G=\mathbf{G}^F$ be a simply-connected finite group of Lie type defined in characteristic p satisfying $c_p(G)=1$. Then one of the following holds:*

(i) $G=\mathrm{SL}_2(p)$,
(ii) $G=\mathrm{SL}_3(2)$,
(iii) $G=\mathrm{SU}_3(2)$,
(iv) $G=\mathrm{Suz}(2)$,
(v) $G={}^2G_2(3)$.

If G satisfies one of (i)-(v)*, then $c_p(G)=1$.*

Proof. From Theorems 5.2, 5.3 and 5.5 one concludes that the groups G in (i), (ii), (iii) or (iv) of the theorem satisfy $c_p(G)=1$. Also, $c_3({}^2G_2(3))=1$ by [9]. Hence it suffices to show that a group of Lie type G defined in characteristic p and satisfying $c_p(G)=1$ must be one of the examples (i), (ii), (iii), (iv) or (v).

Suppose first that $G=\mathbf{G}^F$ is an untwisted group of Lie type satisfying $c_p(G)=1$. If G is of type A_1, Humphreys' theorem (cf. Thm. 5.2) implies the claim. Furthermore, if G is of Lie rank greater than 1, the parabolic descent lemma (cf. Lemma 5.1) implies that q must be equal to p. Thus, if the Lie rank of G equals 2, one concludes from Theorem 5.3(a) and (b), Theorem 5.4 and 5.6 that G has to be of type A_2 and $q=2$. If the Lie rank of G equals 3, the parabolic descent lemma implies that G must be of Lie type A_3 and $q=2$. In particular, $G\simeq\mathrm{SL}_4(2)\simeq\mathfrak{A}_8$. But $c_2(\mathfrak{A}_8)=7$ (see Theorem 3.2), a contradiction. Hence, by the parabolic descent lemma, there are no examples of untwisted groups of Lie type of Lie rank greater than 2.

Suppose that $G=\mathbf{G}^F$ is of type ${}^2A_n$. If $n\geq 3$, then $\mathbf{G}$ has an F-stable parabolic subgroup of type (A_1,F^2) and the parabolic descent lemma and Humphreys' theorem (cf. Thm. 5.2) shows that there are no examples in this case. If $n = 2$, Theorem 5.3(a) and (c) apply showing that the only possible example is (iii).

If G is of type ${}^2D_n$, $n \geq 4$, $\mathbf{G}$ has an F-stable parabolic subgroup of type ${}^2A_3$. Hence the parabolic descent lemma shows that there are no examples in this case. If G is of type ${}^3D_4$, $\mathbf{G}$ has an F-stable parabolic subgroup of type (A_1, F^3), and if G is of type ${}^2E_6$, $\mathbf{G}$ has an F-stable parabolic subgroup of type ${}^2D_4$. Thus by the parabolic descent lemma there are no examples in this case as well.

If G is of type ${}^2B_2$, Theorem 5.5 shows that ${}^2B_2(2)$ is the only possible example. But $c_2({}^2F_4(2)) = 13$ by [9]. This fact — together with the parabolic descent lemma — shows also that there are no examples of type ${}^2F_4$. If G is of type ${}^2G_2$, Proposition 5.7 shows that ${}^2G_2(3)$ is the only possible example. □

6. Sporadic groups

It remains to treat the sporadic simple groups. Here we have:

Theorem 6.1. *Let G be a sporadic simple group. Then $c_\ell(G) > 1$ unless one of:*

(a) $G = M_{11}$, $\ell = 11$, *or*

(b) $G = M_{23}$, $\ell = 23$.

Proof. The groups in cases (a) and (b) are examples, because there exists an ℓ'-Hall subgroup. Using Lemma 2.3 and the tables in the modular Atlas, one is left with the groups

$$\{Fi_{22}, HN, Ly, Th, Fi_{23}, Co_1, J_4, F_3^+, B, M\}$$

for some small primes. The decomposition numbers for HN at $\ell = 5$ can be found at the home page of the modular Atlas

`http://www.math.rwth–aachen.de/~MOC/.`

In the subsequent table, we present triples (G, ℓ, H) where H is a subgroup of G of ℓ'-index, and such that H is an extension of a soluble group by a simple group S for which $c_\ell(S) > 1$ by previously proved results. It follows from Lemma 2.4(a) that the corresponding pairs (G, ℓ) do not lead to examples.

Table 1. Some ℓ'-index subgroups

G	ℓ	H	G	ℓ	H
Fi_{22}	2	$2^{10}.M_{22}$	J_4	2	$2^{11}.M_{24}$
Fi_{22}	3	$\mathrm{O}_7(3)$	F_3^+	2	$2^{11}.M_{24}$
HN	3	$3^{1+4}.4.\mathfrak{A}_5$	F_3^+	3	$[3^{13}].2^2.\mathrm{L}_3(3)$
Ly	5	$5^3.\mathrm{L}_3(5)$	B	2	$2^{1+22}.Co_2$
Th	2	$2^{1+8}.\mathfrak{A}_9$	B	3	Fi_{23}
Fi_{23}	2	$2.Fi_{22}$	B	5	$5^3.\mathrm{L}_3(5)$
Fi_{23}	3	$[3^{10}].2.\mathrm{L}_3(3)$	M	2	$2^{1+24}.Co_1$
Co_1	2	$2^{11}.M_{24}$	M	3	$[3^{17}].2.\mathfrak{S}_4.M_{11}$
Co_1	3	$3^6.2.M_{12}$	M	5	$[5^9].2.\mathrm{L}_3(5)$

Using this, we are left with the following three cases:

$$\{(HN, 2), (Th, 3), (Co_1, 5)\}.$$

For $\ell = 5$ the only non-trivial character degrees of Co_1 below $5^4 = |Co_1|_5$ are 276 and 299. Clearly, $5^4 - 1$ is not a non-negative integral linear combination of these two. The Harada-Norton group HN contains the alternating group $\mathfrak{A}_{12}$. The 1-PIM of $\mathfrak{A}_{12}$ has dimension 204288, which is larger than $|HN|_2 = 2^{14}$. Thus this is no example by Lemma 2.1.

Let now $G := Th$, with $\ell = 3$. Clearly the character of the 1-PIM Φ_1^H of the maximal subgroup $H := {}^3D_4(2)\colon 3$ contains all three linear characters of H. But the two non-trivial linear characters of H are only contained in restrictions of characters of G of degree at least 4 881 384. In particular

$$\mathrm{ch}(\Phi_1^G)(1) > 4\,881\,384 > 3^{10} = |G|_\ell.$$

This completes the proof. □

7. Finite groups with an ℓ'-Hall subgroup

In this section we assume that G is a finite group and that ℓ is a prime number. In order to prove Theorem C we make use of the following lemma.

Lemma 7.1. *Let G be a finite group satisfying $c_\ell(G) = 1$, and let N be a normal subgroup of G with the following property:*

(i) *G/N has an ℓ'-Hall subgroup, and there exists a unique G/N-conjugacy class of ℓ'-Hall subgroups.*

(ii) *N has an ℓ'-Hall subgroup, and there exists a unique N-conjugacy class of ℓ'-Hall subgroups.*

Then G has an ℓ'-Hall subgroup, and there exists a unique G-conjugacy class of ℓ'-Hall subgroups.

Proof. Let $H_N \leq N$ be an ℓ'-Hall subgroup in N. Since there is a unique N-conjugacy class of such groups, the Frattini argument implies that $G = N_G(H_N).N$. Let $\pi\colon N_G(H_N) \to G/N$ denote the canonical projection, and let $\bar{H} \leq G/N$ be an ℓ'-Hall subgroup. Put $X\colon = \{\, g \in N_G(H_N) \mid \pi(h) \in \bar{H} \,\}$. Then X has normal subgroups $H_N \leq N_N(H_N)$, and $N_N(H_N)/H_N$ is an ℓ-group, and $X/N_N(H_N) \simeq \bar{H}$ is an ℓ'-group. In particular, X is ℓ-soluble and thus contains an ℓ'-Hall subgroup H. By construction, this subgroup is also an ℓ'-Hall subgroup of G.

Assume that H_1 and H_2 are ℓ'-Hall subgroup of G. Then $N \cap H_1$ is an ℓ'-Hall subgroup of N, and, by hypothesis (ii), we may assume that $H_N\colon = H_1 \cap N = H_2 \cap N$. In particular, $H_1, H_2 \leq N_G(H_N)$. From hypothesis (i) one concludes that H_1 and H_2 are conjugate in $N_G(H_N)$. This yields the claim. □

From Theorem A and Lemma 7.1 the proof of Theorem C can be deduced as follows:

Proof of Theorem C. It suffices to show that if G is a finite group satisfying $c_\ell(G) = 1$, $\ell \in \{2, 3, 5\}$, then G contains an ℓ'-Hall subgroup and that there exists a unique G-conjugacy class of such subgroups. Suppose that the assertion is false, and that

G is a counterexample of minimal order. Then by Lemma 7.1, G must be simple and non-abelian. Theorem A implies that either $G = \mathfrak{A}_5$, $\ell = 5$, or $G = \mathrm{L}_n(q)$, $(q^n - 1)/(q - 1) = \ell^f$. As $\mathfrak{A}_5$ contains a unique conjugacy class of subgroups of index 5, one can eliminate the first case. In the latter case Zsigmondy's theorem (cf. Lemma 2.9) implies that either (a) $n = 2$, or (b) $q^n = 64$, $n \geq 3$ or (c) n is prime, $n \geq 3$ and $\ell \equiv 1 \mod n$. Since we assumed that $\ell \in \{2, 3, 5\}$, one can discard the cases (b) and (c). Thus assume that case (a) holds. Then $q + 1 = \ell^f$ and (q, ℓ^f) are consecutive prime powers. If $\ell = 2$ one has $q = p$ and $G = \mathrm{L}_2(p)$ for a Mersenne prime number p. In particular, the normalizer of a Sylow p-subgroup is an $2'$-Hall subgroup of G, and there is a unique G-conjugacy class of such groups. If $\ell = 3$, one has $\ell^f = 3^2$ and $q = 8$, and if $\ell = 5$ one has $\ell^f = 5$ and $q = 4$. Both cases can be eliminated by the previously mentioned argument. Hence such a counter example cannot exist and this yields the claim. □

Remark 7.2. The classification of subgroups of $\mathrm{L}_2(q)$ shows that $\mathrm{L}_2(\ell)$ does not contain an ℓ'-Hall subgroup for $\ell \geq 13$, and $\mathrm{PGL}_2(\ell)$ does not contain an ℓ'-Hall subgroup for $\ell = 7$ or $\ell = 11$. Hence Theorem C(a) does not hold for $\ell \geq 7$.

Both end node parabolics are $7'$-Hall subgroups of $\mathrm{L}_3(2)$. Hence Theorem C(b) certainly fails for $\ell = 7$.

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