Low-dimensional representations of finite orthogonal groups

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Abstract

We determine the smallest irreducible Brauer characters for finite quasi-simple orthogonal type groups in non-defining characteristic. Under some restrictions on the characteristic we also prove a gap result showing that the next larger irreducible Brauer characters have a degree roughly the square of those of the smallest non-trivial characters.

1. *Introduction*

This paper is devoted to studying low-dimensional irreducible representations of finite orthogonal groups in non-defining characteristic. Our aim is a gap result showing that there are a few well-understood representations of very small degree, and all other irreducible representations have degree which is roughly the square of the smallest ones. Knowing the low-dimensional irreducible representations of quasi-simple groups has turned out to be of considerable importance in many applications, most notably in the determination of maximal subgroups of almost simple groups. More specifically we prove:

THEOREM 1. *Let* $G = \mathrm{Spin}_{2n}^{\epsilon}(q)$ *with* $\epsilon \in \{\pm\}$, *q odd and* $n \geq 6$. *Assume that* $\ell \geq 5$ *is a prime not dividing* $q(q+1)$. *Let* φ *be an* ℓ-*modular irreducible Brauer character of* G *of degree less than* $q^{4n-10} - q^{n+4}$. *Then* $\varphi(1)$ *is one of*

$$1, \quad q\frac{(q^n - \epsilon 1)(q^{n-2} + \epsilon 1)}{q^2 - 1} - \kappa_{\ell, q^{n-1} + \epsilon 1}, \quad q^2 \frac{(q^{2n-2} - 1)}{q^2 - 1} - \kappa_{\ell, q^n - \epsilon 1},$$
$$\frac{1}{2}\frac{(q^n - \epsilon 1)(q^{n-1} \pm \epsilon 1)}{q \mp 1}, \qquad \frac{(q^n - \epsilon 1)(q^{n-1} \pm \epsilon 1)}{q \mp 1},$$

where for an integer m, $\kappa_{\ell, m} := 1$ *if* $\ell | m$ *and* $\kappa_{\ell, m} := 0$ *otherwise*.

THEOREM 2. *Let* $G = \mathrm{Spin}_{2n+1}(q)$ *with q odd and* $n \geq 5$. *Assume that* $\ell \geq 5$ *is a prime such that the order of q modulo* ℓ *is either odd, or bigger than* $n/2$. *Let* φ *be an* ℓ-*modular irreducible Brauer character of* G *of degree less than* $(q^{4n-8} - q^{2n})/2$. *Then* $\varphi(1)$ *is one of*

$$1, \quad \frac{q^{2n} - 1}{q^2 - 1}, \quad \frac{1}{2}q\frac{(q^n - 1)(q^{n-1} - 1)}{q + 1}, \quad \frac{1}{2}q\frac{(q^n + 1)(q^{n-1} + 1)}{q + 1},$$
$$\frac{1}{2}q\frac{(q^n + 1)(q^{n-1} - 1)}{q - 1} - \kappa_{\ell, q^n - 1}, \quad \frac{1}{2}q\frac{(q^n - 1)(q^{n-1} + 1)}{q - 1} - \kappa_{\ell, q^n + 1}, \quad q\frac{q^{2n} - 1}{q^2 - 1}, \quad \frac{q^{2n} - 1}{q \pm 1}.$$

Observe that the character degrees listed in the theorems are of the order of magnitude about q^{2n-2}, q^{2n-1} respectively, which is only slightly larger than the square root of the given bound.

Gap results of the form described above have already been proved for all other series of finite classical

quasi-simple groups of Lie type. The situation for orthogonal groups is considerably harder since the smallest dimensional representations have comparatively much larger degree than for the other series. For odd-dimensional orthogonal groups over fields of even characteristic Guralnick–Tiep [8] obtained gap results similar to ours without any restriction on the non-defining characteristic ℓ for which the representations are considered. Their approach crucially relies on the exceptional isomorphism to symplectic groups.

Our results do not cover all characteristics ℓ as our proofs rely on uni-triangularity of a suitable part of the ℓ-modular decomposition matrix of the groups considered which in turn is proved using properties of generalised Gelfand–Graev characters. Since this has not been established in full generality (although it is expected to hold), the present state of knowledge makes it necessary to impose certain restrictions on the prime numbers ℓ considered, as well as, more seriously, on the underlying characteristic having to be odd.

The paper is structured as follows. In Section 2 we determine the small dimensional complex irreducible characters of spin groups using Deligne–Lusztig theory. In Section 3 we investigate the restriction of small dimensional Brauer characters to an end node parabolic subgroup. Finally, with this information we determine the precise dimensions of the smallest Brauer characters for all three series of spin groups in Section 4 and derive the gap results in Theorems 1 and 2 including the precise values of the κ_i, see Theorem 4·4 and Corollary 4·10.

Kay and I started work on this paper around 2011. Sadly, he passed away very unexpectedly shortly before the completion of the manuscript. I would like to dedicate this paper to his memory.

2. Small degree complex irreducible characters

In this section we recall the classification of the smallest degrees of complex irreducible characters of the finite spin groups and fix some notation. Let G be a finite reductive group, that is, the group of fixed point under a Steinberg map of a connected reductive algebraic group over an algebraically closed field of positive characteristic. Recall the partition

$$\mathrm{Irr}(G) = \coprod_r \mathcal{E}(G, r)$$

of the ordinary irreducible characters of G into Lusztig series parametrised by conjugacy classes of semisimple elements $r \in G^*$, the dual group of G. Here, $\mathcal{E}(G, 1)$ are the so-called unipotent characters of G. Then, for any r Lusztig's Jordan decomposition (see e.g. [5, Thm. 2.6.22]) gives a bijection

$$J_r : \mathcal{E}(G, r) \longrightarrow \mathcal{E}(C_{G^*}(r), 1)$$

with the unipotent characters of the centraliser $C_{G^*}(r)$, under which the character degrees transform by the formula

$$\chi(1) = |G^* : C_{G^*}(r)|_{p'} \, J_r(\chi)(1).$$

2·1. The odd-dimensional spin groups $\mathrm{Spin}_{2n+1}(q)$

Let q be a power of a prime p and $G = \mathrm{Spin}_{2n+1}(q)$ with $n \geq 2$. We start by enumerating unipotent characters of small degree. Here, we allow for slightly larger degrees than in the general case, since this will be needed later on and moreover we believe that this information may be of independent interest.

Let us recall that a *symbol* is a pair $S = (X, Y)$ of strictly increasing sequences $X = (x_1 < \ldots < x_r)$, $Y = (y_1 < \ldots < y_s)$ of non-negative integers. The *rank* of S is then defined to be

$$\sum_{i=1}^{r} x_i + \sum_{j=1}^{s} y_j - \left\lfloor \left(\frac{r + s - 1}{2} \right)^2 \right\rfloor.$$

The symbol $S' = (\{0\} \cup (X+1), \{0\} \cup (Y+1))$ is said to be *equivalent* to S, and so is the symbol (Y, X). The rank is constant on equivalence classes. The *defect* of S is $d(S) = ||X| - |Y||$, which clearly is also invariant under equivalence.

The unipotent characters of the groups $\mathrm{Spin}_{2n+1}(q)$ are parametrised by equivalence classes symbols of rank n and odd defect (see e.g. [**5**, Thm. 4.5.1]). The following result is due to Nguyen [**12**, Prop. 3.1] for $n \geq 6$:

PROPOSITION 2·1. *Let* $G = \mathrm{Spin}_{2n+1}(q)$ *or* $\mathrm{Sp}_{2n}(q)$. *Let* χ *be a unipotent character of* G *of degree*

$$\chi(1) \leq \begin{cases} q^{6n-16} - q^{4n-5} & \text{when } n \geq 6, \\ q^{15} - q^{12} & \text{when } n = 5, \\ q^{11} - q^9 - q^8 & \text{when } n = 4, \\ q^7 - q^5 & \text{when } n = 3. \end{cases}$$

Then χ *is as given in Table 1 where we also record the degree of* $\chi(1)$ *as a polynomial in* q.

Table 1. *Small unipotent characters in types* B_n *and* C_n

S	$\chi_S(1)$	$\deg_q(\chi_S(1))$	conditions
$\binom{n}{-}$	1	0	
$\binom{0,1,n}{-}$	$\frac{1}{2}q\frac{(q^n-1)(q^{n-1}-1)}{q+1}$	$2n-1$	
$\binom{0,1}{n}$	$\frac{1}{2}q\frac{(q^n+1)(q^{n-1}+1)}{q+1}$	$2n-1$	
$\binom{1,n}{0}$	$\frac{1}{2}q\frac{(q^n+1)(q^{n-1}-1)}{q-1}$	$2n-1$	
$\binom{0,n}{1}$	$\frac{1}{2}q\frac{(q^n-1)(q^{n-1}+1)}{q-1}$	$2n-1$	
$\binom{0,2,n-1}{-}$	$\frac{1}{2}q^2\frac{(q^{2n}-1)(q^{n-1}-1)(q^{n-3}-1)}{q^4-1}$	$4n-6$	$n>3$
$\binom{0,2}{n-1}$	$\frac{1}{2}q^2\frac{(q^{2n}-1)(q^{n-1}+1)(q^{n-3}+1)}{q^4-1}$	$4n-6$	$n>3$
$\binom{2,n-1}{0}$	$\frac{1}{2}q^2\frac{(q^{2n}-1)(q^{n-1}+1)(q^{n-3}-1)}{(q^2-1)^2}$	$4n-6$	$n>3$
$\binom{0,n-1}{2}$	$\frac{1}{2}q^2\frac{(q^{2n}-1)(q^{n-1}-1)(q^{n-3}+1)}{(q^2-1)^2}$	$4n-6$	$n>3$
$\binom{1,n-1}{1}$	$q^3\frac{(q^n+1)(q^n-1)(q^{2n-4}-1)}{(q^2-1)^2}$	$4n-5$	$n>5$
$\binom{0,1,2,n}{1}$	$\frac{1}{2}q^4\frac{(q^n-1)(q^{2n-2}-1)(q^{n-2}-1)}{q^4-1}$	$4n-4$	$n>5$ or $(n,q)=(3,2)$
$\binom{0,1,2}{1,n}$	$\frac{1}{2}q^4\frac{(q^n+1)(q^{2n-2}-1)(q^{n-2}+1)}{q^4-1}$	$4n-4$	$n>5$
$\binom{1,2,n}{0,1}$	$\frac{1}{2}q^4\frac{(q^n+1)(q^{2n-2}-1)(q^{n-2}-1)}{(q^2-1)^2}$	$4n-4$	$n>5$
$\binom{0,1,n}{1,2}$	$\frac{1}{2}q^4\frac{(q^n-1)(q^{2n-2}-1)(q^{n-2}+1)}{(q^2-1)^2}$	$4n-4$	$n>5$
$\binom{0,2}{2}$	$q^2\frac{q^6-1}{q^2-1}$	6	$n=3$

Proof. The degree polynomials of unipotent characters for $n \leq 5$ can be computed using `Chevie` [**4**]. For $q < 20$, a direct evaluation of these polynomials gives the claim. For $q > 20$, the claim then follows by easy estimates using the explicit formulas. For $n \geq 6$ the assertion is shown in [**12**, Prop. 3.1]. $\square$

We now enumerate the complex irreducible characters of $\mathrm{Spin}_{2n+1}(q)$ of small degree. The irreducible character degrees of families of groups of fixed Lie type over the field $\mathbb{F}_q$ can be written as polynomials in q. It turns out that the smallest such degree polynomials for groups of type B_n have degree in q around $2n$, while the next larger ones have degree in q around $4n$. We list all irreducible characters whose degrees lie in the first range. Note that the complex irreducible character of smallest non-trivial degree for orthogonal groups was determined in [**13**]. For $n \geq 5$, the following has been shown in [**12**, Th. 1.2].

THEOREM 2·2. *Let* $G = \mathrm{Spin}_{2n+1}(q)$ *with* $n \geq 3$. *If* $\chi \in \mathrm{Irr}(G)$ *is such that*

$$\chi(1) < \begin{cases} q^{4n-8} & \text{when } n \geq 5, \\ (q^{2n} - q^{2n-1})/2 & \text{when } n \in \{3, 4\}, \end{cases}$$

then χ *is as given in Table 2, where* $1_G, \rho_1, \ldots, \rho_4$ *are the first five unipotent characters listed in Table 1.*

Table 2. *Smallest complex characters of* $\mathrm{Spin}_{2n+1}(q)$, $n \geq 3$

χ	$\chi(1)$	# (q odd)	# (q even)	$\deg_q(\chi(1))$
1_G	1	1	1	0
$\rho_{s,1}$	$(q^{2n} - 1)/(q^2 - 1)$	1	0	$2n - 2$
ρ_1	$\frac{1}{2}q(q^n - 1)(q^{n-1} - 1)/(q + 1)$	1	1	$2n - 1$
ρ_2	$\frac{1}{2}q(q^n + 1)(q^{n-1} + 1)/(q + 1)$	1	1	$2n - 1$
ρ_3	$\frac{1}{2}q(q^n + 1)(q^{n-1} - 1)/(q - 1)$	1	1	$2n - 1$
ρ_4	$\frac{1}{2}q(q^n - 1)(q^{n-1} + 1)/(q - 1)$	1	1	$2n - 1$
ρ_t^-	$(q^{2n} - 1)/(q + 1)$	$(q - 1)/2$	$q/2$	$2n - 1$
$\rho_{s,q}$	$q(q^{2n} - 1)/(q^2 - 1)$	1	0	$2n - 1$
ρ_t^+	$(q^{2n} - 1)/(q - 1)$	$(q - 3)/2$	$(q - 2)/2$	$2n - 1$

Proof. For $3 \leq n \leq 4$, the complete list of ordinary irreducible characters of G and their degrees can be found on the website [**10**]. For $q < 30$, the claim can then be checked by computer, while for $q > 30$, an easy estimate, using the known degrees in q of the degree polynomials, shows that the given list is complete.

For $n \geq 5$ the result is in [**12**, Thm. 1.2] and [**8**, Cor. 6.2]. For later use let us recall the origin of the various non-unipotent characters listed in Table 2 in Lusztig's parametrisation of characters in terms of semisimple classes in the dual group $G^* = \mathrm{PCSp}_{2n}(q)$.

Let $s \in G^*$ be an isolated involution with centraliser $\mathrm{Sp}_2(q) \circ \mathrm{Sp}_{2n-2}(q)$. The corresponding Lusztig series $\mathcal{E}(G, s)$ is in bijective correspondence under Jordan decomposition with the unipotent characters of $\mathrm{Sp}_2(q) \circ \mathrm{Sp}_{2n-2}(q)$. Thus, we obtain the semisimple character $\rho_{s,1}$ and the character $\rho_{s,q}$ corresponding to the Steinberg character in the $\mathrm{Sp}_2(q)$-factor (both given in Table 2), while all other characters in that series have degree at least $\rho_{s,1}(1)q(q^{n-1} - 1)(q^{n-2} - 1)/(q + 1)/2$ (see Table 1), which is larger than our bound.

The other characters arise from elements in G^* with centraliser $\mathrm{Sp}_{2n-2}(q) \times \mathrm{GL}_1(q)$ or $\mathrm{Sp}_{2n-2}(q) \times \mathrm{GU}_1(q)$. There are $q - 3$ central elements t in $\mathrm{GL}_1(q)$ of order larger than 2, which are fused to their inverses in G^*. The corresponding Lusztig series contain the semisimple characters ρ_t^+ from Table 2. Moreover, the $q - 1$ elements in $\mathrm{GU}_1(q)$ of order larger than 2 give rise to the $(q - 1)/2$ semisimple characters ρ_t^-. All other characters in these Lusztig series have too large degree. $\square$

2·2. *The even-dimensional spin groups* $\mathrm{Spin}_{2n}^{\pm}(q)$

For the even-dimensional spin groups $\mathrm{Spin}_{2n}^+(q)$ of plus-type, the unipotent characters are parametrised by symbols of rank n and defect $d \equiv 0 \pmod 4$, while for those of minus-type, the parametrisation is by symbols of defect $d \equiv 2 \pmod 4$.

PROPOSITION 2·3. *Let* χ *be a unipotent character of* $\mathrm{Spin}_{2n}^+(q)$ *of degree*

$$\chi(1) \leq \begin{cases} (q^{6n-16} - q^{6n-17})/2 & \text{when } n \geq 8, \\ q^{4n-5} - q^{4n-7} & \text{when } 4 \leq n \leq 7. \end{cases}$$

Then χ *is as given in Table 3.*

Table 3. *Small unipotent characters in type D_n*

S	$\chi_S(1)$	$\deg_q(\chi_S(1))$	conditions
$\binom{n}{0}$	1	0	
$\binom{n-1}{1}$	$q\dfrac{(q^n-1)(q^{n-2}+1)}{q^2-1}$	$2n-3$	
$\binom{1,n}{0,1}$	$q^2\dfrac{q^{2n-2}-1}{q^2-1}$	$2n-2$	
$\binom{n-2}{2}$	$q^2\dfrac{(q^n-1)(q^{2n-2}-1)(q^{n-4}+1)}{(q^2-1)(q^4-1)}$	$4n-10$	$n>4$
$\binom{0,1,2,n-1}{-}$	$\frac12 q^3\dfrac{(q^n-1)(q^{n-1}-1)(q^{n-2}-1)(q^{n-3}-1)}{(q+1)^2(q^2+1)}$	$4n-7$	
$\binom{0,n-1}{1,2}$	$\frac12 q^3\dfrac{(q^n-1)(q^{n-1}-1)(q^{n-2}+1)(q^{n-3}+1)}{(q^2-1)^2}$	$4n-7$	
$\binom{1,n-1}{0,2}$	$\frac12 q^3\dfrac{(q^n-1)(q^{n-1}+1)(q^{n-2}-1)(q^{n-3}+1)}{(q-1)^2(q^2+1)}$	$4n-7$	
$\binom{2,n-1}{0,1}$	$\frac12 q^3\dfrac{(q^n-1)(q^{n-1}+1)(q^{n-2}+1)(q^{n-3}-1)}{(q^2-1)^2}$	$4n-7$	
$\binom{1,2,n}{0,1,2}$	$q^6\dfrac{(q^{2n-2}-1)(q^{2n-4}-1)}{(q^2-1)(q^4-1)}$	$4n-6$	
$\binom{n-3}{3}$	$q^3\dfrac{(q^n-1)(q^{2n-2}-1)(q^{2n-4}-1)(q^{n-6}+1)}{(q^2-1)(q^4-1)(q^6-1)}$	$6n-21$	$n>6$
$\binom{2}{2}$	$q^2\dfrac{q^6-1}{q^2-1}$ (2×)	6	$n=4$
$\binom{1,2}{1,2}$	$q^6\dfrac{q^6-1}{q^2-1}$ (2×)	10	$n=4$
$\binom{0,3}{1,3}$	$q^4\dfrac{(q^8-1)(q^5-1)}{(q-1)(q^2-1)}$	14	$n=5,\ q>2$
$\binom{0,1,2,3,4}{1}$	$\frac12 q^7(q^5-1)(q^3-1)(q-1)^2$	17	$n=5,\ q=2$
$\binom{3}{3}$	$q^3\dfrac{(q^4+1)(q^{10}-1)}{q^2-1}$ (2×)	15	$n=6$
$\binom{0,1,3,4}{-}$	$\frac12 q^4\dfrac{(q^{10}-1)(q^3-1)^2(q-1)}{q+1}$	20	$n=6,\ q=2,3$
$\binom{0,1,3,5}{-}$	$\frac12 q^4\dfrac{(q^{12}-1)(q^7-1)(q^5-1)(q-1)}{q^3+1}$	26	$n=7$
$\binom{4}{4}$	$q^4\dfrac{(q^{14}-1)(q^{10}-1)(q^6+1)}{(q^2-1)(q^4-1)}$ (2×)	28	$n=8$
$\binom{5}{4}$	$q^4\dfrac{(q^9-1)(q^8+1)(q^{14}-1)(q^6+1)}{(q-1)(q^4-1)}$	36	$n=9,\ q>2$

Proof. The (more involved) case $n \leq 8$ can be handled computationally as indicated in the proof of Proposition 2·1. For $n \geq 9$ this is shown by a slight variation of the arguments used in the proof of [**12**, Prop. 3.4] which gives the list of unipotent characters of degree at most q^{4n-10}. $\square$

Similarly we obtain:

PROPOSITION 2·4. *Let χ be a unipotent character of* $\mathrm{Spin}_{2n}^-(q)$ *of degree*

$$\chi(1) \leq \begin{cases} (q^{6n-16}-q^{6n-17})/2 & when\ n \geq 8, \\ q^{4n-5}-q^{4n-7} & when\ 4 \leq n \leq 7. \end{cases}$$

Then χ is as given in Table 4.

Again, see [**12**, Prop. 3.3] for the list of unipotent characters of degree at most q^{4n-10}.
The following two results have already been shown in [**12**, Th. 1.3 and 1.4] when $n \geq 5$.

THEOREM 2·5. *Let $G = \mathrm{Spin}_{2n}^+(q)$. If $\chi \in \mathrm{Irr}(G)$ is such that*

$$\chi(1) < \begin{cases} q^{4n-10} & when\ n \geq 6,\ or\ n = 5\ and\ q\ is\ odd, \\ q^{10}-q^8 & when\ n = 5\ and\ q\ is\ even, \\ (q^8-2q^6)/4 & when\ n = 4\ and\ q\ is\ odd, \\ q^8-q^7+q^5 & when\ n = 4\ and\ q\ is\ even, \end{cases}$$

then χ is as given in Table 5, *or $(n,q) = (4,2)$ and $\chi(1) = 28$.*

Table 4. *Small unipotent characters in type* ^2D_n

S	$\chi_S(1)$	$\deg_q(\chi_S(1))$	conditions
$\binom{0,n}{-}$	1	0	
$\binom{1,n-1}{-}$	$q\frac{(q^n+1)(q^{n-2}-1)}{q^2-1}$	$2n-3$	
$\binom{0,1,n}{1}$	$q^2\frac{(q^{2n-2}-1)}{q^2-1}$	$2n-2$	
$\binom{2,n-2}{-}$	$q^2\frac{(q^n+1)(q^{2n-2}-1)(q^{n-4}-1)}{(q^4-1)(q^2-1)}$	$4n-10$	$n>4$
$\binom{1,2,n-1}{0}$	$\frac{1}{2}q^3\frac{(q^n+1)(q^{n-1}+1)(q^{n-2}-1)(q^{n-3}-1)}{(q^2-1)^2}$	$4n-7$	
$\binom{0,2,n-1}{1}$	$\frac{1}{2}q^3\frac{(q^n+1)(q^{n-1}-1)(q^{n-2}+1)(q^{n-3}-1)}{(q-1)^2(q^2+1)}$	$4n-7$	
$\binom{0,1,n-1}{2}$	$\frac{1}{2}q^3\frac{(q^n+1)(q^{n-1}-1)(q^{n-2}-1)(q^{n-3}+1)}{(q^2-1)^2}$	$4n-7$	
$\binom{0,1,2}{n-1}$	$\frac{1}{2}q^3\frac{(q^n+1)(q^{n-1}+1)(q^{n-2}+1)(q^{n-3}+1)}{(q+1)^2(q^2+1)}$	$4n-7$	
$\binom{0,1,2,n}{1,2}$	$q^6\frac{(q^{2n-2}-1)(q^{2n-4}-1)}{(q^2-1)(q^4-1)}$	$4n-6$	
$\binom{3,n-3}{-}$	$q^3\frac{(q^n+1)(q^{2n-2}-1)(q^{2n-4}-1)(q^{n-6}-1)}{(q^2-1)(q^4-1)(q^6-1)}$	$6n-21$	$n>6$
$\binom{4,5}{-}$	$q^4\frac{(q^9+1)(q^8+1)(q^{14}-1)(q^6+1)(q-1)}{(q^2-1)(q^4-1)}$	36	$n=9$
$\binom{0,1,3}{3}$	$q^4\frac{(q^8-1)(q^5+1)}{(q+1)(q^2-1)}$	14	$n=5$

Proof. For $n=4$, the complete list of ordinary irreducible characters of G and their degrees can be found on the website [**10**]. For $q<50$, the claim can then be checked by computer, while for $q>50$, an easy estimate shows that the given list is complete.

For $n \geq 5$ we refer to [**12**, §6,7]. Here, $1_g, \rho_1, \rho_2$ denote the first three unipotent characters listed in Table 3. The characters $\rho_{s,a}^\pm, \rho_{s,b}^\pm$ are the semisimple characters in the Lusztig series of involutions with disconnected centraliser of type $\mathrm{PCO}_{2n-2}^\pm(q)$, and the characters $\rho_t^\pm$ are the semisimple characters in the Lusztig series of semisimple elements with (connected) centraliser of type $\mathrm{PCSO}_{2n-2}^\pm(q)$. $\square$

Table 5. *Smallest complex characters of* $\mathrm{Spin}_{2n}^\epsilon(q)$, $n \geq 4$

χ	$\chi(1)$	# (q odd)	# (q even)	$\deg_q(\chi(1))$
1_G	1	1	1	0
ρ_1	$q(q^n-\epsilon 1)(q^{n-2}+\epsilon 1)/(q^2-1)$	1	1	$2n-3$
$\rho_{s,a}^-, \rho_{s,b}^-$	$\frac{1}{2}(q^n-\epsilon 1)(q^{n-1}-\epsilon 1)/(q+1)$	2	0	$2n-2$
$\rho_{s,a}^+, \rho_{s,b}^+$	$\frac{1}{2}(q^n-\epsilon 1)(q^{n-1}+\epsilon 1)/(q-1)$	2	0	$2n-2$
ρ_t^-	$(q^n-\epsilon 1)(q^{n-1}-\epsilon 1)/(q+1)$	$(q-1)/2$	$q/2$	$2n-2$
ρ_2	$q^2(q^{2n-2}-1)/(q^2-1)$	1	1	$2n-2$
ρ_t^+	$(q^n-\epsilon 1)(q^{n-1}+\epsilon 1)/(q-1)$	$(q-3)/2$	$(q-2)/2$	$2n-2$

THEOREM 2·6. *Let* $G = \mathrm{Spin}_{2n}^-(q)$. *If* $\chi \in \mathrm{Irr}(G)$ *is such that*

$$\chi(1) < \begin{cases} q^{4n-10} & \text{when } n \geq 6, \\ q^{10}-q^9 & \text{when } n = 5, \\ (q^8-2q^6)/2 & \text{when } n = 4 \text{ and } q \text{ is odd}, \\ q^8-q^6 & \text{when } n = 4 \text{ and } q \text{ is even}, \end{cases}$$

then χ *is as given in Table* 5.

Proof. Again, the case $n=4$ can be settled using the data in [**10**], while for $n \geq 8$, we refer to [**12**, §6,7] (see Table 4 for the unipotent characters). As before, $1_G, \rho_1, \rho_2$ denote the first three unipotent

characters listed in Table 4. The characters $\rho_{s,a}^\pm, \rho_{s,b}^\pm$ lie in the Lusztig series of involutions with disconnected centraliser of type $\mathrm{PCO}_{2n-2}^\pm(q)$, and the $\rho_t^\pm$ are the semisimple characters in the Lusztig series of semisimple elements with centraliser of type $\mathrm{PCSO}_{2n-2}^\pm(q)$. $\quad\square$

3. *Locating Brauer characters of low degree*

Here we study the restriction of small dimensional irreducible ℓ-Brauer characters of spin groups to an end node parabolic subgroup. This requires no assumptions on ℓ or on q. Throughout this section let $G = \mathrm{Spin}_m^{(\pm)}(q)$ with $m \geq 5$ and let $P = QL$ be a fixed maximal parabolic subgroup of G stabilising a singular 1-space of the natural module of $\mathrm{SO}_m^{(\pm)}(q)$, with unipotent radical Q and Levi factor L. Observe that $Q \cong \mathbb{F}_q^{m-2}$ is the natural module for $L' := [L, L]$ of type $\mathrm{Spin}_{m-2}^{(\pm)}(q)$. For $\chi \in Q^* := \mathrm{Irr}(Q) = \mathrm{Hom}\,(Q, \mathbb{C}^\times)$ we denote by L_χ its inertia group in L and set $L'_\chi := (L')_\chi$, the stabiliser of χ in L'.

Let k be an algebraically closed field of characteristic ℓ not dividing q. Let W be a kG-module. Then the restriction of W to Q is semisimple and we have a direct sum decomposition $W|_Q = \bigoplus_\chi W_\chi$ into the Q-weight spaces

$$W_\chi := \{w \in W \mid x.w = \chi(x)w \text{ for all } x \in Q\} \qquad \text{for } \chi \in Q^*,$$

that is, the Q-isotypic components. A kG-module W is called Q-*small* if for all $\chi \in Q^*$ the module W_χ has only linear L_χ-composition factors. This notion is closely related to but somewhat stronger than that of being Q-*linear small* in [**11**, Def. 1.3], where only the constituents in the socle are required to be linear. We will nevertheless see that results for the latter do also apply to the former.

The following is well-known:

LEMMA 3·1. *Let $\phi \in \mathrm{Hom}\,(\mathbb{F}_q, \mathbb{C}^\times)$ be a non-trivial linear character. Then $\sum_{a \in \mathbb{F}_q} \phi(a) = 0$ and thus $\sum_{a \in \mathbb{F}_q^\times} \phi(a) = -1$.*

Proof. Clearly the second assertion follows from the first. To see the first, note that the values of ϕ are the p-th roots of unity, where q is a power of p. Also ϕ is constant on the cosets of $\ker(\phi)$, thus

$$\sum_{a \in \mathbb{F}_q} \phi(a) = \sum_{k \in \mathbb{F}_p} \sum_{a \in \ker(\phi)+k} \phi(a) = \sum_{k \in \mathbb{F}_p} |\ker(\phi)|\phi(k) = |\ker(\phi)| \sum_{k \in \mathbb{F}_p} \phi(k) = 0.$$

$\square$

3·1. *The odd-dimensional spin groups*

Let $G = \mathrm{Spin}_{2n+1}(q)$. We first recall the $L' \cong \mathrm{Spin}_{2n-1}(q)$-orbit structure on $Q \cong \mathbb{F}_q^{2n-1}$ and its dual. The L'-module Q admits an L'-invariant non-degenerate quadratic form F. Then two non-zero elements x_1, x_2 of Q lie in the same L'-orbit if and only if $F(x_1) = F(x_2)$. Thus apart from the trivial orbit there is one orbit of singular vectors of length $q^{2n-2} - 1$, $(q-1)/2$ orbits of length $q^{2n-2} + q^{n-1}$ of plus-type, and $(q-1)/2$ orbits of length $q^{2n-2} - q^{n-1}$ of minus-type.

Let W be a Q-small kG-module. Any P'-composition factor V of W has a Q-weight space decomposition $V = \bigoplus_\chi V_\chi$, and only χ of one type $\eta \in \{0, \pm\}$ can occur as V is irreducible. Furthermore, V_χ is then an irreducible, hence 1-dimensional L'_χ-module, with Brauer character μ say. We write $V =: V^{(\eta,\mu)}$, and denote its Brauer character by $\varphi^{(\eta,\mu)}$.

LEMMA 3·2. *Let $G = \mathrm{Spin}_{2n+1}(q)$ with $n \geq 2$ and q odd, and $x \in Q$ be a long root element of G. Then for all $\chi \in Q^*$ and all linear $\mu \in \mathrm{IBr}(L'_\chi)$ we have*

$$\varphi^{(\eta,\mu)}(x) = \begin{cases} -1 & \text{if } \eta = 0, \\ \eta q^{n-1} & \text{if } \eta \in \{\pm\}. \end{cases}$$

Proof. We represent elements of Q by row vectors and elements of its dual $\mathrm{Hom}\,(Q, \mathbb{F}_q)$ by column vectors. Note that as Q is a self dual L'-module, the L'-orbit structure on Q and $\mathrm{Hom}\,(Q, \mathbb{F}_q)$ is identical. We call the elements of $\mathrm{Hom}\,(Q, \mathbb{F}_q)$ functionals. So for example a singular functional is an element of $\mathrm{Hom}\,(Q, \mathbb{F}_q)$ on which the L'-invariant quadratic form F vanishes.

We choose a basis $\{e_1, \ldots, e_{n-1}, g, f_{n-1}, \ldots, f_1\}$ of Q and its dual basis in $\mathrm{Hom}\,(Q, \mathbb{F}_q)$ in such a way that the Gram matrix of the L'-invariant symmetric bilinear form with respect to this basis is the matrix all of whose non-zero entries are 1 and appear on the anti-diagonal.

Without loss we may assume that $x = [1, 0, \ldots, 0]$ as all singular vectors in Q are L'-conjugate and G-conjugate to a long root element. Let $\bar{t} = [a, \bar{b}, c]^{\mathrm{tr}} \in \mathrm{Hom}\,(Q, \mathbb{F}_q)$ with $a, c \in \mathbb{F}_q$ and $\bar{b} \in \mathbb{F}_q^{2n-3}$. Note that $\bar{t}(x) = a$.

Let $\phi \in \mathrm{Hom}\,(\mathbb{F}_q, \mathbb{C}^\times)$ be a non-trivial character. Then for each $\chi \in Q^*$ there exists a unique $\bar{t}_\chi \in \mathrm{Hom}\,(Q, \mathbb{F}_q)$ such that $\chi(x) = \phi(\bar{t}_\chi(x))$.

We can now calculate the character values on $V^{(0,\mu)}$. First observe that a functional $\bar{t} = [a, \bar{b}, c]^{\mathrm{tr}}$ is singular if and only if one of the following is true:

(A) $F(\bar{b}) = ac = 0$, or
(B) $F(\bar{b}) = -ac/2 \neq 0$.

The number of $\bar{t}$ of type (B) is equal to the number of nonsingular vectors in $\mathbb{F}_q^{2n-3}$ which is $q^{2n-3} - q^{2n-4}$ times the number of non-trivial choices for a which is $q - 1$. By Lemma 3.1 these contribute $-(q^{2n-3} - q^{2n-4})$ to the trace of x on $V^{(0,\mu)}$.

The elements $\bar{t}$ of type (A) come in two flavours depending on whether or not $a = 0$. If $a = 0$, then if $c \neq 0$ there are q^{2n-4} choices for $\bar{b}$, while if $c = 0$ there are $q^{2n-4} - 1$ choices for $\bar{b}$. In total these $\bar{t}$ contribute

$$(q - 1)q^{2n-4} + q^{2n-4} - 1 = q^{2n-3} - 1$$

to the trace of x. Finally if $a \neq 0$, then $c = 0$ while there are q^{2n-4} choices for $\bar{b}$ which yields a contribution of $-q^{2n-4}$ to the trace. Thus the trace of x on $V^{(0,\mu)}$ is

$$\varphi^{(0,\mu)}(x) = -(q^{2n-3} - q^{2n-4}) + (q^{2n-3} - 1) - q^{2n-4} = -1$$

as claimed.

Next we calculate the character value of x on $V^{(+,\mu)}$. Observe that the form F evaluates to a fixed square, say 1, on the functional $\bar{t} = [a, \bar{b}, c]^{\mathrm{tr}}$ if and only if $F([a, \bar{0}, c]^{\mathrm{tr}}) + F(\pm \bar{b}) = 1$, that is, if and only if one of the following is true:

(A) $F(\bar{b}) = 1$ and $ac = 0$, or
(B) $1 - F(\bar{b}) = ac/2 \neq 0$.

The contribution to the trace of x by functionals of type (A) occurs in one of two ways: If $a \neq 0$, then $c = 0$ and then there are $q^{2n-4} + q^{n-2}$ choices for $\bar{b}$ which yields

$$-(q^{2n-4} + q^{n-2}).$$

If $a = 0$, then there are q choices for c and $q^{2n-4} + q^{n-2}$ choices for $\bar{b}$ which yields

$$q(q^{2n-4} + q^{n-2}).$$

To compute the contribution by functionals of type (B) we observe that $a \neq 0$ and that for every choice of a there are $q^{2n-3} - (q^{2n-4} + q^{n-2})$ choices for $\bar{b}$ after which c is determined uniquely. Thus functionals of type (B) contribute

$$-q^{2n-3} + q^{2n-4} + q^{n-2}$$

to the trace of x. Summing up the contributions yields

$$\varphi^{(+,\mu)}(x) = -(q^{2n-4} + q^{n-2}) + (q^{2n-3} + q^{n-1}) - q^{2n-3} + q^{2n-4} + q^{n-2} = q^{n-1}.$$

Finally we calculate the character value of x on $V^{(-,\mu)}$. Observe that F evaluates to a fixed non-square α on $\bar{t} = [a, \bar{b}, c]^{\mathrm{tr}}$, if and only if $F([a, \bar{0}, c]^{\mathrm{tr}}) + F(\bar{b}) = \alpha$, that is, if and only if one of the following is true:

(A) $F(\bar{b}) = \alpha$ and $ac = 0$, or
(B) $\alpha - F(\bar{b}) = ac/2 \neq 0$.

The contribution by functionals of type (A) occurs in one of two ways: If $a \neq 0$, then $c = 0$ and then there are $q^{2n-4} - q^{n-2}$ choices for $\bar{b}$ which yields

$$-(q^{2n-4} - q^{n-2}).$$

If $a = 0$, then there are q choices for c and $q^{2n-4} - q^{n-2}$ choices for $\bar{b}$ which yields

$$q(q^{2n-4} - q^{n-2}).$$

To compute the contribution by functionals of type (B) we observe that $a \neq 0$ and that for every choice of a there are $q^{2n-3} - (q^{2n-4} - q^{n-2})$ choices for $\bar{b}$ after which c is determined uniquely. Thus functionals of type (B) contribute

$$-q^{2n-3} + q^{2n-4} - q^{n-2}$$

to the trace of x. Summing up the contributions yields

$$\varphi^{(-,\mu)}(x) = -(q^{2n-4} - q^{n-2}) + (q^{2n-3} - q^{n-1}) - q^{2n-3} + q^{2n-4} - q^{n-2} = -q^{n-1}$$

as claimed. $\square$

Remark 3·3. While a similar result holds for the case of even q, we do not consider this here as character bounds for $\mathrm{Spin}_{2n+1}(q)$ with q even have already been obtained in [**8**].

We next compute the trace on a long root element in the Levi factor.

LEMMA 3·4. *Let $G = \mathrm{Spin}_{2n+1}(q)$ with $n \geq 2$ and q odd. If $y \in L$ is a long root element then for all $\chi \in Q^*$ and all linear $\mu \in \mathrm{IBr}(L'_\chi)$ we have*

$$\varphi^{(\eta,\mu)}(y) = \begin{cases} q^{2n-4} - 1 & \text{if } \eta = 0, \\ q^{2n-4} + \eta q^{n-1} & \text{if } \eta \in \{\pm\}. \end{cases}$$

Proof. Let $\chi \in Q^*$ be of type η. By definition $V^{(\eta,\mu)} = \mu \uparrow_{P'_\chi}^{P'}$ where μ is a linear character of P'_χ. The element y is unipotent and hence conjugate to an element of L'_χ thus $\varphi^{(\eta,\mu)}(y) = \varphi^{(\eta,1)}(y)$ for all μ. Hence it suffices to compute $\varphi^{(\eta,1)}(y)$.

Now $\varphi^{(\eta,1)}|_{L'}$ is the permutation character of L' on the cosets of L'_χ. Thus $\varphi^{(\eta,1)}(y)$ can be computed by counting the fixed points of y on the cosets of L'_χ in L'. This amounts to counting vectors v in $C_Q(y)$ with $F(v) = 0$, $F(v) = 1$, and $F(v)$ a fixed non-square respectively. To make the count we observe that $C_Q(y)$ is the orthogonal direct sum of a totally singular 2-space with a non-degenerate space of dimension $2n - 5$.

Thus the number of singular non-zero vectors in $C_Q(y)$ is equal to $q^2 q^{2n-6} - 1 = q^{2n-4} - 1$, the number of vectors in $C_Q(y)$ with $F(y) = 1$ is equal to $q^2(q^{2n-6} + q^{n-3}) = q^{2n-4} + q^{n-1}$, while the number of vectors with $F(y)$ a fixed non-square is equal to $q^2(q^{2n-6} - q^{n-3}) = q^{2n-4} - q^{n-1}$. The claim follows. $\square$

PROPOSITION 3·5. *Let $G = \mathrm{Spin}_{2n+1}(q)$ with $n \geq 2$ and q odd. Then $C_W(Q) \neq \{0\}$ for any faithful Q-small kG-module W.*

Proof. Let $x \in Q$ and $y \in L$ be long root elements of G, such that x and y are G-conjugate. As x, y are ℓ'-elements we can work with Brauer characters.

Now $W|_Q$ decomposes as $C_W(Q) \oplus W^0 \oplus W^+ \oplus W^-$ where we write W^η for the sum of all W_χ of type η. Denote the sum of the multiplicities of the characters $\varphi^{(\eta,\mu)}$ in the character of W^η by a_η, denote the character of the P'-module $C_W(Q)$ by φ^c and the character of W by φ. So

$$\varphi(x) = \varphi^c(1) - a_0 + a_+ q^{n-1} - a_- q^{n-1}$$

by Lemma 3·2 and

$$\varphi(y) = \varphi^c(y) + (a_0 + a_+ + a_-)q^{2n-4} - a_0 + a_+ q^{n-1} - a_- q^{n-1}$$

by Lemma 3·4. As x and y are G-conjugate, $\varphi(x) = \varphi(y)$. Thus we find that

$$\varphi^c(1) - \varphi^c(y) = (a_0 + a_+ + a_-)q^{2n-4} \geq q^{2n-4} > 0$$

as $a_0 + a_+ + a_- > 0$ (since W is faithful) and so $C_W(Q) \neq \{0\}$. $\square$

PROPOSITION 3·6. *Let* $G = \mathrm{Spin}_{2n+1}(q)$ *with* $n \geq 2$ *and* q *odd and let* W *be an irreducible Q-small kG-module. Then W occurs as an ℓ-modular composition factor of the Harish-Chandra induction from L to G of one of the modules in Table* 2.

Proof. As W is Q-small Proposition 3·5 shows that $C_W(Q) \neq 0$. An application of [**7**, Lemma 4.2(ii) and (iii)] then gives that the L-constituents of $C_W(Q)$ are amongst those of $[W, Q]$. Recall that the P'-composition factors of $[W, Q]$ are all of the form $V^{(\eta,\mu)}$. Thus the L'-composition factors of the latter are precisely those occurring in $V^{(\eta,\mu)}$. By assumption for all $\chi \in Q^*$ the L'_χ-submodules of $\mathrm{Soc}(W_\chi)$ are linear, that is, any L'-composition factor ψ occurring in $V^{(\eta,\mu)}$ is a constituent of an induced module $\mu \uparrow^{L'}_{L'_\chi}$, where μ is a linear character of L'_χ. In particular, $\psi(1) \leq |L' : L'_\chi| = |L : L_\chi| \leq q^{2n-2} - 1$. Then by Theorem 2·2, ψ is one of the modules in Table 2. $\square$

3·2. *The even-dimensional spin groups*

We now turn to the even-dimensional spin groups $G = \mathrm{Spin}^\epsilon_{2n}(q)$, $\epsilon \in \{\pm\}$, with $n \geq 3$. Here, the $L' \cong \mathrm{Spin}^\epsilon_{2n-2}(q)$-orbit structure on $Q \cong \mathbb{F}_q^{2n-2}$ is as follows. Apart from the trivial orbit there is one orbit of singular vectors of length $q^{2n-3} + \epsilon(q^{n-1} - q^{n-2}) - 1$ and $q - 1$ orbits of length $q^{2n-3} - \epsilon q^{n-2}$. When q is odd, then half of the $q-1$ orbits of length $q^{2n-3} - \epsilon q^{n-2}$ are of plus type whereas the others are of minus type (i.e., lie in distinct L-orbits). When q is even all orbits of length $q^{2n-3} - \epsilon q^{n-2}$ are in the same L-orbit. As Q is a self dual L-module, the L-orbit structures on Q and $\mathrm{Hom}(Q, \mathbb{F}_q)$ are identical. As in Section 3·1 we let $V^{(\eta,\mu)}$ be the corresponding simple P'-modules, with character $\varphi^{(\eta,\mu)}$.

We first prove the analogue of Proposition 3·5.

LEMMA 3·7. *Let* $G = \mathrm{Spin}^\epsilon_{2n}(q)$ *with* $n \geq 3$, *and* $x \in Q$ *be a long root element of* G. *Then for all* $\chi \in Q^*$ *and all linear* $\mu \in \mathrm{IBr}(L'_\chi)$ *we have*

$$\varphi^{(\eta,\mu)}(x) = \begin{cases} \epsilon(q^{n-1} - q^{n-2}) - 1 & \text{if } \eta = 0, \\ -\epsilon q^{n-2} & \text{if } \eta \neq 0. \end{cases}$$

Proof. We argue as in Lemma 3·2 and keep the same notation. Choose $\{e_1, \ldots, e_{n-1}, f_{n-1}, \ldots, f_1\}$ as basis of Q and its dual basis in such a way that the Gram matrix of the L'-invariant symmetric bilinear form with respect to this basis is the matrix all of whose non-zero entries are 1 and appear on the anti-diagonal if $\epsilon = +$, while for $\epsilon = -$, the Gram matrix is of this form except that the middle 2×2 square is not anti-diagonal.

Recall that we represent elements of Q by row vectors and elements of $\mathrm{Hom}(Q, \mathbb{F}_q)$ by column vectors.

Without loss we may assume that $x = [1, 0, \ldots, 0]$ as all singular vectors in Q are L'-conjugate. Let $\bar{t} = [a, \bar{b}, c]^{\mathrm{tr}}$, where $a, c \in \mathbb{F}_q$, and $\bar{b}$ is an element of $\mathbb{F}_q^{2n-4}$.

We start with the character values on $V^{(0,\mu)}$. Observe that a functional $\bar{t} = [a, \bar{b}, c]^{\mathrm{tr}}$ is singular if and only if one of the following is true:

(A) $F(\bar{b}) = ac = 0$, or
(B) $F(\bar{b}) = -ac \neq 0$.

The number of $\bar{t}$ of type (B) is equal to the number of nonsingular vectors in $\mathbb{F}_q^{2n-4}$ which is $q^{2n-4} - q^{2n-5} - \epsilon(q^{n-2} - q^{n-3})$ times the number of non-trivial choices for a which is $q - 1$. By Lemma 3·1 these contribute

$$-(q^{2n-4} - q^{2n-5} - \epsilon(q^{n-2} - q^{n-3}))$$

to the trace of x on $V^{(0,\mu)}$.

The elements $\bar{t}$ of type (A) come in two flavours depending on whether or not $a = 0$. If $a = 0$, then if $c \neq 0$ there are $q^{2n-5} + \epsilon(q^{n-2} - q^{n-3})$ choices for $\bar{b}$, while if $c = 0$ there are $q^{2n-5} + \epsilon(q^{n-2} - q^{n-3}) - 1$ choices for $\bar{b}$. In total these $\bar{t}$ contribute

$$(q - 1)(q^{2n-5} + \epsilon(q^{n-2} - q^{n-3})) + (q^{2n-5} + \epsilon(q^{n-2} - q^{n-3}) - 1 = q^{2n-4} + \epsilon(q^{n-1} - q^{n-2}) - 1$$

to the trace of x. Finally if $a \neq 0$, then $c = 0$ while there are $q^{2n-5} + \epsilon(q^{n-2} - q^{n-3})$ choices for $\bar{b}$ which yields a contribution of $-(q^{2n-5} + \epsilon(q^{n-2} - q^{n-3}))$. Thus the trace of x on $V^{(0,\mu)}$ is

$$-q^{2n-4} - q^{2n-5} + \epsilon(q^{n-2} - q^{n-3}) + q^{2n-4} + \epsilon(q^{n-1} - q^{n-2} - 1) - (q^{2n-5} + \epsilon(q^{n-2} - q^{n-3}))$$
$$= \epsilon(q^{n-1} - q^{n-2}) - 1$$

as claimed.

Next we calculate the character value of x on $V^{(\eta,\mu)}$ for $\eta \neq 0$. Note that all L'-orbits of nonsingular vectors in Q are of length $q^{2n-3} - \epsilon q^{n-2}$. We observe that the form F evaluates to $\alpha \neq 0$ on the functional $\bar{t} = [a, \bar{b}, c]^{\mathrm{tr}}$ if and only if $F([a, \bar{0}, c]^{\mathrm{tr}}) + F(\bar{b}) = \alpha$, that is, if and only if one of the following holds:

(A) $F(\bar{b}) = \alpha$ and $ac = 0$, or
(B) $\alpha - F(\bar{b}) = ac \neq 0$.

The contribution to the trace of x by functionals of type (A) occurs in one of two ways: If $a \neq 0$, then $c = 0$ and then there are $q^{2n-5} - \epsilon q^{n-3}$ choices for $\bar{b}$ which contributes

$$-(q^{2n-5} - \epsilon q^{n-3}).$$

If $a = 0$, then there are q choices for c and $q^{2n-5} - \epsilon q^{n-3}$ choices for $\bar{b}$ which contributes

$$q(q^{2n-5} - \epsilon q^{n-3}).$$

To compute the contribution by functionals of type (B) we observe that $a \neq 0$ and that for every choice of a there are $q^{2n-4} - q^{2n-5} + q^{n-3}$ choices for $\bar{b}$ after which c is determined uniquely. Thus functionals of type (B) contribute

$$-q^{2n-4} + q^{2n-5} - q^{n-3}$$

to the trace of x. Summing up the contributions yields that

$$\varphi^{(\eta,\mu)}(x) = -(q^{2n-5} - \epsilon q^{n-3}) + (q^{2n-4} - \epsilon q^{n-2}) - q^{2n-4} + q^{2n-5} - q^{n-3} = -\epsilon q^{n-2}$$

as claimed. $\square$

LEMMA 3·8. *Let $G = \mathrm{Spin}_{2n}^{\epsilon}(q)$ with $n \geq 3$. If $y \in L$ is a long root element, then for all $\chi \in Q^*$ and all linear $\mu \in \mathrm{IBr}(L'_\chi)$ we have*

$$\varphi^{(\eta,\mu)}(y) = \begin{cases} q^{2n-5} + \epsilon(q^{n-1} - q^{n-2}) - 1 & \text{if } \eta = 0, \\ q^{2n-5} - \epsilon q^{n-2} & \text{if } \eta \neq 0. \end{cases}$$

Proof. Let χ be an element of type η from Q^*. By definition $V^{(\eta,\mu)} = \mu \uparrow_{P'_\chi}^{P'}$ where μ is a linear character of P'_χ. As in the proof of Lemma 3·4 it suffices to compute $\varphi^{(\eta,1)}(y)$.

Now $\varphi_{L'}^{(\eta,1)}$ is the permutation character of L' on the cosets of L'_χ. Thus $\varphi^{(\eta,1)}(y)$ can be computed by counting vectors v in $C_Q(y)$ with $F(v) = 0$, and with $F(v) = \alpha \neq 0$. To make the count we observe that $C_Q(y)$ is the orthogonal direct sum of a totally singular 2-space with a non-degenerate space of dimension $2n - 6$. Thus the number of singular non-zero vectors in $C_Q(y)$ is equal to

$$q^2(q^{2n-7} + \epsilon(q^{n-3} - q^{n-4})) - 1 = q^{2n-5} + \epsilon(q^{n-1} - q^{n-2}) - 1$$

whereas the number of vectors $v \in C_Q(y)$ with $F(v) = \alpha \neq 0$ equals $q^2(q^{2n-7} - \epsilon q^{n-4}) = q^{2n-5} - \epsilon q^{n-2}$. The claim follows. $\square$

PROPOSITION 3·9. *Let $G = \mathrm{Spin}_{2n}^{\epsilon}(q)$ with $n \geq 3$. Then $C_W(Q) \neq \{0\}$ for any faithful Q-small kG-module W.*

Proof. We argue as in the proof of Proposition 3·5. Let $x \in Q$ and $y \in L$ be long root elements of G. As a Q-module, W decomposes as $C_W(Q) \oplus W^0 \oplus W^{\neq 0}$. Denote the sum of the multiplicities of the characters $\varphi^{(0,\mu)}$ in the character of W^0 by a_0, and the sum of the multiplicities of the $\varphi^{(\eta,\mu)}$, $\eta \neq 0$ in $W^{\neq 0}$ by a_1. Further, denote the character of the P'-module $C_W(Q)$ by χ^c and the character of W by φ. So

$$\varphi(x) = \chi^c(1) + a_0(\epsilon(q^{n-1} - q^{n-2}) - 1) - \epsilon a_1 q^{n-2}$$

by Lemma 3·7 and

$$\varphi(y) = \chi^c(y) + (a_0 + a_1)q^{2n-5} + a_0(\epsilon(q^{n-1} - q^{n-2}) - 1) - \epsilon a_1 q^{n-2}$$

by Lemma 3·8. As x, y are G-conjugate we have $\varphi(x) = \varphi(y)$. Noting that $a_0 + a_1 > 0$ as W is faithful we see that

$$\chi^c(1) - \chi^c(y) = (a_0 + a_1)q^{2n-5} \geq q^{2n-5} > 0$$

whence $C_W(Q) \neq \{0\}$. $\square$

PROPOSITION 3·10. *Let $G = \mathrm{Spin}_{2n}^{\epsilon}(q)$ with $\epsilon \in \{\pm\}$ and $n \geq 3$ and let W be an irreducible Q-small kG-module. Then W occurs as an ℓ-modular composition factor of the Harish-Chandra induction from L to G of one of the modules in Table 5.*

Proof. As W is Q-small Proposition 3·9 shows that $C_W(Q) \neq 0$. As in the proof of Proposition 3·6 this implies that any constituent of $W|_{L'}$ has dimension not larger than $(q^{n-1} - 1)(q^{n-2} + 1)$ and then we may conclude using Theorem 2·5. $\square$

4. *The main result*

We are finally in a position to obtain the sought after gap result. For this, we keep the notation from the previous sections. In particular $P = QL$ is an end-node maximal parabolic subgroup of $G = \mathrm{Spin}_m^{(\pm)}(q)$ with unipotent radical Q and Levi factor L. Furthermore, we keep the notation ρ_i, ρ_s, $\rho_t^{\pm}, \ldots$ for small dimensional irreducible characters as in Section 2 with the specific semisimple elements $s, t \in G^*$

introduced there. For an ordinary character χ we denote by χ^0 its ℓ-modular Brauer character, that is, its restriction to ℓ-regular classes. Throughout, for an integer m, we set

$$\kappa_{\ell,m} := \begin{cases} 1 & \text{if } \ell | m, \\ 0 & \text{otherwise.} \end{cases}$$

4·1. *The odd-dimensional spin groups*

Let $n \geq 2$, $G = \mathrm{Spin}_{2n+1}(q)$ and ℓ a prime not dividing q. We first collect some results on the decomposition numbers of low dimensional ordinary representations in non-defining characteristic.

LEMMA 4·1. *Let $G = \mathrm{Spin}_{2n+1}(q)$, $n \geq 2$, and $\epsilon \in \{\pm\}$. Then ρ_t^ϵ remains irreducible modulo ℓ when $\ell \nmid (q - \epsilon 1)$ or when $o(t) \neq \ell^f$ or $2\ell^f$, and otherwise*

$$(\rho_t^\epsilon)^0 = \begin{cases} (\rho_1 + \rho_2 + \epsilon 1_G)^0 & \text{if } o(t) = \ell^f > 1, \\ (\rho_{s,q} + \epsilon\rho_{s,1})^0 & \text{if } o(t) = 2\ell^f > 2,\ \ell \neq 2. \end{cases}$$

Proof. According to the description in the proof of Theorem 2·2, the character ρ_t^- is semisimple in the Lusztig series indexed by an element $t \in G^*$ of order dividing $q+1$. But by the observation in [**9**, Prop. 1], the semisimple characters in a Lusztig series $\mathcal{E}(G, t)$ remain irreducible modulo all primes ℓ for which the ℓ'-part of t has the same centraliser as t. By our description of the parameters t, this is the case unless this ℓ'-part has order at most 2.

When $o(t)$ is a power of ℓ, then by Broué–Michel [**2**, Thm. 9.12] ρ_t^- lies in a unipotent bloc. By the explicit formula for semisimple characters in terms of Deligne–Lusztig characters [**5**, Thm. 2.6.11], we find that $(\rho_t^-)^0 = (\rho_1 + \rho_2 - 1_G)^0$. Finally, if the order of t is twice a power of ℓ, then its 2-part is conjugate to s and hence ρ_t^- lies in the same ℓ-block as the Lusztig series of $\rho_{s,1}$. Again the claim follows from the explicit formula [**5**, Thm. 2.6.11] for the semisimple character ρ_t^- in terms of Deligne–Lusztig characters.

The argument for the characters ρ_t^+ is entirely similar. □

Thus, either $\rho_t^\pm$ remains irreducible modulo ℓ, or its ℓ-modular constituents are known if we know them for the remaining characters in Table 2. We therefore henceforth only consider the latter.

LEMMA 4·2. *Let $G = \mathrm{Spin}_{2n+1}(q)$ with q odd and $n \geq 3$, and ℓ a prime not dividing $q(q+1)$. Assume that the ℓ-modular decomposition matrix of the union of blocks $\mathcal{E}_\ell(G, s)$ is uni-triangular. Then the entries in its first ten rows are approximated from above by Table 6, where $k := n - 3$. In particular, both $\rho_{s,1}$ and $\rho_{s,q}$ remain irreducible modulo ℓ.*

ρ	a_ρ										
$1 \boxtimes 1$	0	1									
$\mathrm{St} \boxtimes 1$	1	.	1								
$1 \boxtimes \rho_1$	1	.	.	1							
$1 \boxtimes \rho_2$	1	.	.	.	1						
$1 \boxtimes \rho_3$	1	$k+1$	.	.	.	1					
$1 \boxtimes \rho_4$	1	$k+1$	.	.	.	.	1				
$\mathrm{St} \boxtimes \rho_1$	2	.	.	.	.	.	.	1			
$\mathrm{St} \boxtimes \rho_2$	2	.	.	.	.	.	.	.	1		
$\mathrm{St} \boxtimes \rho_3$	2	.	$k+1$	.	.	.	.	.	.	1	
$\mathrm{St} \boxtimes \rho_4$	2	.	$k+1$	.	.	.	.	.	.	.	1

Table 6. *Approximate decomposition matrices for $\mathcal{E}(\mathrm{Spin}_{2n+1}(q), s)$, $n \geq 3$*

Here, q^{a_ρ} is the precise power of q dividing $\rho(1)$.

Proof. By a result of Geck, see [**2**, Thm. 14.4], the Lusztig series $\mathcal{E}(G, s)$ of the semisimple involution $s \in G^*$ forms a basic set for the union of ℓ-blocks $\mathcal{E}_\ell(G, s)$. By Lusztig's Jordan decomposition (see [**5**, Thm. 2.6.22]) $\mathcal{E}(G, s)$ is in bijection with $\mathcal{E}(C, 1)$, where $C = C_{G^*}(s) \cong \mathrm{Sp}_2(q) \times \mathrm{Sp}_{2n-2}(q)$, hence with $\mathcal{E}(\mathrm{Sp}_2(q), 1) \times \mathcal{E}(\mathrm{Sp}_{2n-2}(q), 1)$. Accordingly, we may and will denote the elements of $\mathcal{E}(G, s)$ by exterior tensor products of unipotent characters, so that $\rho_{s,1} = 1 \boxtimes 1$ and $\rho_{s,q} = \mathrm{St} \boxtimes 1$.

The character $\rho_{s,1} \in \mathcal{E}(G, s)$ is semisimple, so remains irreducible modulo all odd primes (see [**9**, Prop. 1]). We next claim that $\rho_{s,1}^0$ does not occur as a composition factor of $\rho_{s,q}^0$. Indeed, by the known decomposition numbers for $\mathrm{Spin}_5(q) \cong \mathrm{Sp}_4(q)$ (see [**14**]), $\rho_{s,q}$ remains irreducible for $n = 2$ unless $\ell | (q+1)$ (which is excluded by assumption). Now assume the assertion has already been shown for $\mathrm{Spin}_{2n-1}(q)$. Thus the upper left-hand corner of the ℓ-modular decomposition matrix for $\mathcal{E}_\ell(\mathrm{Spin}_{2n-1}(q), s)$ has the form:

$$\begin{array}{c|cc} \rho_{s,1} & 1 & \\ \rho_{s,q} & . & 1 \end{array}$$

Harish-Chandra inducing the projective characters corresponding to the two columns of this matrix yields projective characters of G of the same form, and thus, by uni-triangularity, $\rho_{s,q}^0$ is irreducible. The upper bounds on the remaining entries given in Table 6 are now obtained inductively exactly as in the proof of [**3**, Thm. 6.3]: upper bounds on the entries for the corresponding columns for the Levi subgroup of an end node parabolic subgroup are known by induction. The Harish-Chandra induction of those columns can be calculated by the comparison theorem [**5**, Thm. 3.2.7] in terms of usual induction in the Weyl group, and this gives the stated entries. □

PROPOSITION 4·3. *Let $G = \mathrm{Spin}_{2n+1}(q)$ with q odd and $n \geq 4$, and ℓ a prime not dividing $q(q+1)$ such that the ℓ-modular decomposition matrix of $\mathcal{E}_\ell(G, s)$ is uni-triangular. Assume that $(n, q) \neq (4, 3), (5, 3)$. Then any ℓ-modular irreducible Brauer character φ of G of degree less than $q^{4n-8} - q^{2n}$ is a constituent of the ℓ-modular reduction of one of the complex characters listed in Table 2.*

Proof. By assumption we have that $\varphi(1)$ is smaller than the constant

$$b_2 = \Big(\frac{q^{2n-4} - 1}{q^2 - 1} - \frac{q^{n-2} - 1}{q - 1} \Big)(q^{2n-2} - 1)$$

in [**11**, Table 4]. Then by [**11**, Prop. 5.3] the module W is Q-linear small, unless we are in one of the exceptions listed in [**11**, Rem. 5.4]. Now note that the proof in [**11**, Prop. 5.3] actually shows that W must in fact even be Q-small, since all of its L_χ-composition factors must be linear, not only those in the socle. The only exceptions in [**11**, Rem. 5.4] relevant here are $\mathrm{Spin}_9(3), \mathrm{Spin}_{11}(3)$, which were excluded. Then Proposition 3·6 applies to show that φ occurs in the ℓ-modular reduction of a constituent χ of the Harish-Chandra induction of some character ψ of L as in Table 2. In particular χ is either unipotent, or in $\mathcal{E}(G, s)$ or $\mathcal{E}(G, t)$ with s, t as introduced before.

If χ is unipotent, then by [**3**, Cor. 6.5] we obtain that φ is in fact a constituent of one of the complex characters in Table 2. Now assume that $\chi \in \mathcal{E}(G, t)$. By the main result of [**1**], the union of ℓ-blocks in $\mathcal{E}_\ell(G, t)$ is Morita equivalent to the union of unipotent ℓ-blocks in $\mathcal{E}_\ell(C, 1)$, where C is dual to $C_{G^*}(t) \cong \mathrm{Sp}_{2n-2}(q) \cdot (q - \epsilon 1)$. By [**7**, Thm. 2.1] the smallest non-trivial degree of any unipotent ℓ-modular Brauer character of C is at least $c := \frac{1}{2}(q^{n-1} - 1)(q^{n-1} - q)/(q + 1)$ (observe that the Weil modules are not unipotent as $\ell \neq 2$). Hence any $\varphi \neq (\rho_t^\epsilon)^0$ in $\mathcal{E}_\ell(G, t)$ has degree at least

$$|G^* : C_{G^*}(t)|_{q'} \cdot c = \frac{1}{2} \frac{(q^{2n} - 1)(q^{n-1} - 1)(q^{n-1} - q)}{(q + 1)^2}$$

which is larger than our bound.

Finally, assume that $\chi \in \mathcal{E}(G, s)$. The Harish-Chandra induction of $\rho_{s,1}$ and $\rho_{s,q}$ from L to G only

contains the characters denoted $\rho_{s,1}$, $\rho_{s,q}$, $1 \boxtimes \rho_i$ and $\mathrm{St} \boxtimes \rho_i$, for $i = 2, 3, 4$, from Table 6. Since we assume that the ℓ-modular decomposition matrix of $\mathcal{E}_\ell(G, s)$ is uni-triangular, lower bounds for the degrees of the corresponding Brauer characters can be derived from Lemma 4·2. These show that φ must be equal to one of $\rho_{s,1}^0$, $\rho_{s,q}^0$. $\square$

Let $d_\ell(q)$ denote the order of q modulo ℓ. We then obtain Theorem 2 in the following form:

THEOREM 4·4. *Let $G = \mathrm{Spin}_{2n+1}(q)$ with q odd and $n \geq 4$, and $\ell \geq 5$ a prime not dividing q such that $d_\ell(q)$ is either odd, or $d_\ell(q) > n/2$. Let φ be an ℓ-modular irreducible Brauer character of G of degree less than $\frac{1}{2}(q^{4n-8} - q^{2n})$. Then $\varphi(1)$ is one of*

$$1, \quad \frac{q^{2n}-1}{q^2-1}, \quad \frac{1}{2}q\frac{(q^n-1)(q^{n-1}-1)}{q+1}, \quad \frac{1}{2}q\frac{(q^n+1)(q^{n-1}+1)}{q+1},$$

$$\frac{1}{2}q\frac{(q^n+1)(q^{n-1}-1)}{q-1} - \kappa_{\ell,q^n-1}, \quad \frac{1}{2}q\frac{(q^n-1)(q^{n-1}+1)}{q-1} - \kappa_{\ell,q^n+1},$$

$$q\frac{q^{2n}-1}{q^2-1}, \quad \frac{q^{2n}-1}{q\pm1}.$$

Proof. We claim that the assumption of Proposition 4·3 is satisfied in our situation. First, it is well-known that decomposition matrices for blocks with cyclic defect groups are uni-triangular, so we are done in that case. Now let $G \hookrightarrow \tilde{G}$ be a regular embedding, that is, $\tilde{G}$ is a group coming from an algebraic group with connected centre and with the same derived subgroup as G (see [**5**, Def. 1.7.1 and Exmp. 1.7.4]). By the result of Gruber–Hiss [**6**, Thm. 8.2(c)] the decomposition matrix of any classical group with connected centre is uni-triangular whenever $\ell > 2$ is a linear prime. (Recall that a prime ℓ is *linear* for $\tilde{G}$ if the order $d_\ell(q)$ of q modulo ℓ is odd.) Then, the proof of Proposition 4·3 shows that all ℓ-modular Brauer characters of $\tilde{G}$ of degree less than $q^{4n-8} - q^{2n}$ are as claimed. Now $|\tilde{G}/C_{\tilde{G}}(G)| = |Z(G)| = 2$ (see [**5**, Exmp. 1.7.4]), so any irreducible (Brauer) character of $\tilde{G}$ restricted to G has at most two irreducible constituents. Thus our claim holds for G as well.

It remains to discuss the groups $\mathrm{Spin}_9(3)$ and $\mathrm{Spin}_{11}(3)$ excluded in the statement of Proposition 4·3. Their Sylow ℓ-subgroups are cyclic for all primes $\ell > 5$, and then all small-dimensional Brauer characters can easily be determined from the known ordinary character degrees. For the prime $\ell = 5$ we have $d_\ell(q) = d_5(3) = 4$ is even, so it is excluded in our conclusion. $\square$

Remark 4·5. For even q we have $\mathrm{Spin}_{2n+1}(q) \cong \mathrm{Sp}_{2n}(q)$, and for these groups it was shown by Guralnick–Tiep [**8**, Thm. 1.1] that the conclusion of Theorem 4·4 continues to hold for $n \geq 5$, while for $n = 4$ the lower bound has to be replaced by $q^2(q^4 - 1)(q^3 - 1)$ when $q > 2$ and by 203 when $q = 2$.

4·2. *The even-dimensional spin groups*

Here again, s, t denote certain semisimple elements as specified in Section 2.

THEOREM 4·6. *Let q be odd. Let either $G = \mathrm{Spin}_{2n}^+(q)$ with $n \geq 5$, and $\ell \geq 5$ a prime not dividing $q(q + 1)$, or let $G = \mathrm{Spin}_{2n}^-(q)$ with $n \geq 6$, and $\ell \geq 5$ a prime not dividing q. Then any ℓ-modular irreducible Brauer character φ of G of degree less than $q^{4n-10} - q^{n+4}$ is a constituent of the ℓ-modular reduction of one of the complex characters listed in Table 5.*

Proof. By comparing we see that $\varphi(1)$ is smaller than the constant

$$b_2 = \begin{cases} \left(\frac{q(q^{2n-6}-1)}{q^2-1} - \frac{q^{n-3}-1}{q-1} - 7\right)(q^{2n-3} + q^{n-1} - q^{n-2} - 1) & \text{if } G = \mathrm{Spin}_{2n}^+(q), \\ \frac{q^4-1}{q+1}(q^7 - q^4 + q^3 - 1) & \text{if } G = \mathrm{Spin}_{10}^-(q), \\ \left(\frac{q(q^{2n-6}-1)}{q^2-1} - q^{n-3} - n + 4\right)(q^{2n-3} - q^{n-1} + q^{n-2} - 1) & \text{if } G = \mathrm{Spin}_{2n}^-(q), n \geq 6 \end{cases}$$

in [**11**, Table 4]. Again by [**11**, Prop. 5.3] the module W is Q-small, unless we are in one of the exceptions listed in [**11**, Rem. 5.4] (noting that the proof of [**11**, Prop. 5.3] actually yields this stronger conclusion). Then Proposition 3·10 applies to show that φ is a constituent of the ℓ-modular reduction of the Harish-Chandra induction of some character ψ of L as in Table 5. The only exception from [**11**, Rem. 5.4] relevant here is $\mathrm{Spin}_{10}^+(3)$, Here, the smallest degree of a non-trivial character of $L'_\chi = \mathrm{Spin}_6^+(3) = \mathrm{SL}_4(3)$ is 26 and $|L : L_\chi| = (3^4 - 1)(3^3 + 1) = 2240$, so if W has no trivial L'_χ-constituent then $\dim W \geq 58240$, while the bound in the statement is $3^{20-10} - 3^{5+4} = 39366$, so the conclusion holds here as well.

We consider the various possibilities for ψ. If ψ is unipotent, so one of 1_L, ρ_1 or ρ_2, then its Harish-Chandra induction only contains characters occurring in Table 6 or 7 of [**3**]. Our claim in this case follows from [**3**, Cor. 5.8].

Next assume that ψ is one of ρ_t^- or ρ_t^+. Then its Harish-Chandra induction lies in the Lusztig series $\mathcal{E}(G, t)$. By the main result of [**1**] the ℓ-blocks in $\mathcal{E}_\ell(G, t)$ are Morita equivalent to the unipotent ℓ-blocks of a group dual to $C_{G^*}(t) \cong \mathrm{CSO}_{2n-2}^\epsilon(q)$. In particular, the decomposition matrices are the same. For the latter we may apply [**3**, Prop. 5.7] to see that all Brauer characters in that series apart from $\rho_t^\pm$ have degree at least

$$\frac{(q^n - \epsilon 1)(q^{n-1} - \epsilon 1)}{(q+1)} \Big(q \frac{(q^{n-1} - \epsilon 1)(q^{n-3} + \epsilon 1)}{q^2 - 1} - 1 \Big)$$

which is larger than our bound. A similar argument applies to the constituents of the Harish-Chandra induction of $\rho_{s,a}^\pm$ and $\rho_{s,b}^\pm$. In this case, [**1**] yields a Morita equivalence between the blocks in $\mathcal{E}_\ell(G, s)$ and the unipotent blocks of the disconnected group $\mathrm{CO}_{2n-2}^\epsilon(q)$, with connected component of index 2. Another application of [**3**, Prop. 5.7] shows our assertion in this last case. $\square$

In order to make the previous result more explicit, we determine the ℓ-modular reductions of some of the low-dimensional $\mathrm{Spin}_{2n}^\pm(q)$-modules in Table 5. The first result extends [**3**, Thm. 5.5]:

PROPOSITION 4·7. *Let $G = \mathrm{Spin}_{2n}^-(q)$ with q odd and $n \geq 6$, and $\ell \geq 5$ a prime dividing $q + 1$. Then the entries in the first eight rows of the decomposition matrix of the union of the unipotent ℓ-blocks of G are approximated from above by Table 7, where $k := n - 6$.*

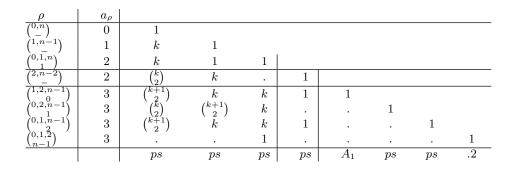*Here, the bottom row of the table indicates the Harish-Chandra series of the Brauer character in the respective column.*

ρ	a_ρ								
$\binom{0,n}{-}$	0	1							
$\binom{1,n-1}{-}$	1	k	1						
$\binom{0,1,n}{1}$	2	k	1	1					
$\binom{2,n-2}{-}$	2	$\binom{k}{2}$	k	.	1				
$\binom{1,2,n-1}{0}$	3	$\binom{k+1}{2}$	k	k	1	1			
$\binom{0,2,n-1}{1}$	3	$\binom{k}{2}$	$\binom{k+1}{2}$	k	.	.	1		
$\binom{0,1,n-1}{2}$	3	$\binom{k+1}{2}$	k	k	1	.	.	1	
$\binom{0,1,2}{n-1}$	3	.	.	1	.	.	.	.	1
		ps	ps	ps	ps	A_1	ps	ps	$.2$

Table 7. *Approximate decomposition matrices for $\mathrm{Spin}_{2n}^-(q)$, $n \geq 6$*

Proof. This is proved by induction along the very same lines as [**3**, Thm. 5.5]. We start with the case $n = 6$. Here, the six principal series PIMs are obtained from the decomposition matrix of the Hecke algebra $\mathcal{H}(B_5; q^2; q)$. The projective character in the A_1-series comes by Harish-Chandra induction from a PIM of a Levi subgroup of type A_4, while the projective character in the ".2"-series is obtained from a Levi

subgroup of type $^2D_4 \times A_1$. This shows the claim for $n = 6$ (with $k = 0$). Then Harish-Chandra induction of these eight projective characters yields projective characters of G with the stated decompositions for all $n \geq 7$.

No other Harish-Chandra series can contribute to characters of a-value at most 3 by [**3**, Prop. 5.3]. $\square$

Remark 4·8. For $n = 5$ there is at least one unipotent PIM of $^2D_5(q)$ in the Harish-Chandra series of type A_1^2 of a-value 2, and we do not see how to rule out that there might be several of them.

LEMMA 4·9. *Let* $G = \mathrm{Spin}_{2n}^\epsilon(q)$, $\epsilon \in \{\pm\}$ *and* $n \geq 3$. *Then*

$$(\rho_t^\epsilon)^0 = \begin{cases} (\epsilon\rho_1 + \rho_2 + 1_G)^0 & \text{if } o(t) = \ell^f > 1, \\ (\rho_{s,a}^\epsilon + \rho_{s,b}^\epsilon)^0 & \text{if } o(t) = 2\ell^f > 2, \ \ell \neq 2, \end{cases}$$

and ρ_t^ϵ *remains irreducible modulo* ℓ *otherwise. Furthermore,* $\rho_{s,a}^\pm, \rho_{s,b}^\pm$ *remain irreducible modulo all primes* $\ell \neq 2$.

Proof. According to the description in the proof of Theorem 2·5, the characters $\rho_{s,a}^\pm$ and $\rho_{s,b}^\pm$ are semisimple in Lusztig series indexed by elements of order 2, so we may argue as in the proof of Lemma 4·2 using [**9**, Prop. 1]. The proof for $\rho_t^\pm$ is completely analogous to the one of Lemma 4·1. $\square$

COROLLARY 4·10. *Keep the assumptions on* n, q *and* ℓ *from Theorem 4·6. If* φ *is an* ℓ-*modular irreducible Brauer character of* $\mathrm{Spin}_{2n}^\epsilon(q)$ *of degree* $\varphi(1) < q^{4n-10} - q^{n+4}$ *then* $\varphi(1)$ *is one of*

$$1, \quad q\frac{(q^n - \epsilon 1)(q^{n-2} + \epsilon 1)}{q^2 - 1} - \kappa_{\ell, q^{n-1} + \epsilon 1}, \quad q^2\frac{(q^{2n-2} - 1)}{q^2 - 1} - \kappa_{\ell, q^n - \epsilon 1},$$

$$\frac{1}{2}\frac{(q^n - \epsilon 1)(q^{n-1} \pm \epsilon 1)}{q \mp 1}, \qquad \frac{(q^n - \epsilon 1)(q^{n-1} \pm \epsilon 1)}{q \mp 1}.$$

Proof. This follows directly from Theorem 4·6 with the partial decomposition matrix for the unipotent characters in [**3**, Prop. 5.7] and the statement of Lemma 4·9. $\square$

This implies Theorem 1.

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