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The Largest Irreducible Representations of Simple Groups

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Abstract

Answering a question of I. M. Isaacs, we show that the largest degree of irreducible complex representations of any finite non-abelian simple group can be bounded in terms of the smaller degrees. We also study the asymptotic behavior of this largest degree for finite groups of Lie type. Moreover, we show that for groups of Lie type, the Steinberg character has largest degree among all unipotent characters.

1. *Introduction*

For a finite group G, let $b(G)$ denote the largest degree of any irreducible complex representation of G. Certainly, $b(G)^2 \leq |G|$, and this trivial bound is best possible in the following sense. One can write $|G| = b(G)(b(G) + e)$ for some non-negative integer e. Then $e = 0$ if and only if $|G| = 1$. Y. Berkovich showed that $e = 1$ precisely when $|G| = 2$ or G is a 2-transitive Frobenius group, cf. [**2**, Thm. 7]. In particular, there is no upper bound on $|G|$ when $e = 1$. On the other hand, it turns out that $|G|$ *can* be bounded in terms of e if $e > 1$. Indeed, N. Snyder showed in [**25**] that then $|G| \leq ((2e)!)^2$.

One can ask whether the largest degree $b(G)$ can be bounded in terms of the remaining degrees of G. More precisely, can one bound

$$\varepsilon(G) := \frac{\sum_{\chi\in\mathrm{Irr}(G),\ \chi(1)<b(G)} \chi(1)^2}{b(G)^2}$$

away from 0 for all non-abelian finite groups G? The aforementioned result of Berkovich immediately implies a negative answer to this question for general groups. M. Isaacs raised the question whether there exists a universal constant $\varepsilon > 0$ such that $\varepsilon(S) \geq \varepsilon$ for all *simple* groups S. Assuming an affirmative answer to this question, he has improved Snyder's bound to the polynomial bound $|G| \leq Be^6$ (for some universal constant B and for all finite groups G with $e > 1$), cf. [**12**].

In this paper we answer Isaacs' question in the affirmative:

Theorem 1.1. *There exists a universal constant $\varepsilon > 0$ such that $\varepsilon(S) \geq \varepsilon$ for all finite non-abelian simple groups S.*

One can give an explicit value for ε in Theorem 1.1, say $\varepsilon \geq 2/(120,000!)$, (moreover, $\varepsilon(S) > 1/209$ for all but finitely many finite non-abelian simple groups S), cf. Corollary 4.9. This is certainly very far from best possible, and it comes from the proof of Theorem 2.1. Note that, for alternating groups $S = \mathsf{A}_n$ only asymptotic formulae are known for $b(S)$, see [**28**] and [**15**]. It would be interesting to improve on this bound for ε. We do not know of any non-abelian

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simple group S with $\varepsilon(S) < 1$ (in fact, $\varepsilon(S) > 1$ for the majority of simple classical groups, and for all simple exceptional groups of Lie type as well as sporadic simple groups, see Theorem 4.7 and Proposition 4.3). As pointed out by Isaacs in [**12**], if $\varepsilon(S) \geq 1$ for all non-abelian simple groups S, then his polynomial bound Be^6 can be improved to $|G| \leq e^6 + e^4$.

Lusztig's classification of irreducible characters of finite groups of Lie type provides us with a way to find $b(S)$ for any given finite simple Lie-type group S in principle. In particular, $b(S)$ is known if S is defined over a field of large enough cardinality (in comparison to the rank), cf. Corollary 4.6. But because of the delicate combinatorics involved in the classification, the right asymptotic for $b(S)$ for simple classical groups over small fields $\mathbb{F}_q$ has not been determined. In fact, even the question whether the Steinberg character of S has largest degree among the unipotent characters of S has not been answered in the literature. Our second main result answers this question in the affirmative:

Theorem 1.2. *Let $\mathcal{G}$ be a simple algebraic group in characteristic p, $F : \mathcal{G} \to \mathcal{G}$ a Steinberg endomorphism, and $G = \mathcal{G}^F$ be the corresponding finite group of Lie type. Then the degree of the Steinberg character of G is strictly larger than the degree of any other unipotent character.*

The next result yields a lower and upper bound for $b(S)$ in the case of finite groups of Lie type:

Theorem 1.3. *For any $1 > \varepsilon > 0$, there are some (explicit) constants $A, B > 0$ depending on ε such that, for any simple algebraic group $\mathcal{G}$ in characteristic p of rank n and any Steinberg endomorphism $F : \mathcal{G} \to \mathcal{G}$, the largest degree $b(G)$ of the corresponding finite group $G := \mathcal{G}^F$ over $\mathbb{F}_q$ satisfies the following inequalities:*

$$A(\log_q n)^{(1-\varepsilon)/\gamma} < \frac{b(G)}{|G|_p} < B(1 + \log_q n)^{2.54/\gamma}$$

if G is classical, and

$$1 \leq \frac{b(G)}{|G|_p} < B$$

if G is an exceptional group of Lie type. Here, $\gamma = 1$ if G is untwisted of type A, and $\gamma = 2$ otherwise.

In particular, if we fix q and let the rank n grow, then the ratio $b(G)/|G|_p$ also grows unbounded – a fact we find rather surprising (note that $|G|_p$ is the degree of the Steinberg character of G). One can view Theorem 1.3 as a Lie-type analogue of the results of [**28**] and [**15**]. Even more explicit lower and upper bounds for $b(G)$ are proved in §5 for finite classical groups G, cf. Theorems 5.1, 5.2, and 5.3.

Certainly, any upper bound for $b(G)$ also holds for the largest degree $b_\ell(G)$ of the ℓ-modular irreducible representations of G. Here is a lower bound for $b_\ell(G)$:

Theorem 1.4. *There exists an (explicit) constant $C > 0$ such that, for any simple algebraic group $\mathcal{G}$ in characteristic p, any Steinberg endomorphism $F : \mathcal{G} \to \mathcal{G}$, and any prime ℓ, the largest degree $b_\ell(G)$ of ℓ-modular irreducible representations of $G := \mathcal{G}^F$ satisfies the inequality $b_\ell(G)/|G|_p \geq C$.*

The paper is organized as follows. In Section 2 we prove Theorem 1.1 for S an alternating group. In Section 3 we prove various comparison results between unipotent character degrees, which may be of independent interest. We then prove Theorem 1.1 for groups of Lie type in Section 4. In the final Section 5 we complete the proof of Theorem 1.3 and give even better upper and lower bounds for classical groups of Lie type, as well as the proof of Theorem 1.4.

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2. *Symmetric and Alternating Groups*

In this section we analyze the largest degree of irreducible characters of the alternating groups. The main idea is that if one starts with a character $\chi \in \mathrm{Irr}(\mathsf{S}_n)$, by considering the irreducible constituents of $\mathrm{Ind}_{\mathsf{S}_{n-1}}^{\mathsf{S}_n} \mathrm{Res}_{\mathsf{S}_{n-1}}^{\mathsf{S}_n}(\chi)$ then one can show that χ itself cannot be too large. Some further technicalities arise because the symmetric group is easier to analyze, but we need the result for alternating groups.

We recall some basic combinatorics connected with symmetric groups. By a *Young diagram*, we mean a finite subset Δ of $\mathbb{Z}^{>0} \times \mathbb{Z}^{>0}$ such that for all $(x, y) \in \mathbb{Z}^{>0} \times \mathbb{Z}^{>0}$, $(x+1, y) \in \Delta$ or $(x, y+1) \in \Delta$ implies $(x, y) \in \Delta$. Elements of Δ are called *nodes*. We denote by $Y(n)$ the set of Young diagrams of cardinality n. For any fixed Δ, we let l and k denote the largest x-coordinate and y-coordinate in Δ respectively and define a_j for $1 \le j \le k$ and b_i for $1 \le i \le l$ by

$$a_j := \max\{i \mid (i, j) \in \Delta\}$$

and likewise

$$b_i := \max\{j \mid (i, j) \in \Delta\}.$$

Thus, for each Δ, we have a pair of mutually transpose partitions

$$n = a_1 + \cdots + a_k = b_1 + \cdots + b_l.$$

For each $(i, j) \in \Delta$, we define the *hook* $H_{i,j} := H_{i,j}(\Delta)$ to be the set of $(i', j') \in \Delta$ such that $i' \ge i$, $j' \ge j$, and equality holds in at least one of these two inequalities. We define the *hook length*

$$h(i, j) := h_{i,j}(\Delta) := |H_{i,j}(\Delta)| = 1 + a_j - i + b_i - j,$$

and set

$$P := P(\Delta) := \prod_{(i,j)\in\Delta} h_{i,j}.$$

Define $\mathcal{A}(\Delta)$ (resp. $\mathcal{B}(\Delta)$) to be the set of nodes that can be added (resp. removed) from Δ to produce another Young diagram:

$$\mathcal{A}(\Delta) := \{(i, j) \in \mathbb{Z}^{>0} \times \mathbb{Z}^{>0} \mid \Delta \cup \{(i, j)\} \in Y(n+1)\}$$

and

$$\mathcal{B}(\Delta) := \{(i, j) \in \Delta \mid \Delta \setminus \{(i, j)\} \in Y(n-1)\}.$$

Thus $\mathcal{A}(\Delta)$ consists of the pair $(1, k+1)$ and pairs $(a_j + 1, j)$ where $j = 1$ or $a_j < a_{j-1}$. In particular, the values i for $(i, j) \in \mathcal{A}(\Delta)$ are pairwise distinct, so

$$n \ge \sum_{(i,j)\in\mathcal{A}(\Delta)} (i-1) \ge \frac{|\mathcal{A}(\Delta)|^2 - |\mathcal{A}(\Delta)|}{2},$$

and $|\mathcal{A}(\Delta)| < \sqrt{2n}+1$. Similarly, $\mathcal{B}(\Delta)$ consists of the pairs (a_j, j) where either $j = k$ or $a_j > a_{j+1}$. Hence

$$n \geq \sum_{(i,j)\in\mathcal{B}(\Delta)} i \geq \frac{|\mathcal{B}(\Delta)|^2 + |\mathcal{B}(\Delta)|}{2},$$

and $|\mathcal{B}(\Delta)| < \sqrt{2n}$. For $(i,j) \in \mathcal{A}(\Delta)$, the symmetric difference between $\mathcal{A}(\Delta)$ and $\mathcal{A}(\Delta \cup \{(i,j)\})$ consists of at most three elements: (i,j) itself and possibly $(i+1,j)$ and/or $(i,j+1)$. Likewise, the symmetric difference between $\mathcal{B}(\Delta)$ and $\mathcal{B}(\Delta \setminus \{(i,j)\})$ consists of at most three elements: (i,j) and possibly $(i-1,j)$ and/or $(i,j-1)$.

There are bijective correspondences between elements of $Y(n)$, partitions $n = \sum_j a_j$, dual partitions $n = \sum_i b_i$, and complex irreducible characters of S_n. By the hook length formula, the degree of the character associated to Δ is $n!/P(\Delta)$.

The branching rule for $\mathsf{S}_{n-1} < \mathsf{S}_n$ asserts that the restriction to S_{n-1} of the irreducible representation $\rho(\Delta)$ of S_n associated to $\Delta \in Y(n)$ is the direct sum of $\rho(\Delta \setminus (i,j))$ over all $(i,j) \in \mathcal{B}(\Delta)$. By Frobenius reciprocity, it follows that the induction from S_n to S_{n+1} of $\rho(\Delta)$ is the direct sum of $\rho(\Delta \cup \{(i,j)\})$ over all $(i,j) \in \mathcal{A}(\Delta)$. We can now prove the main theorem of this section.

THEOREM 2.1. *Let $S \subset \mathbb{R}$ be a finite set. Then there exists N and $\delta > 0$ depending only on S such that for all $n > N$ and every irreducible character ϕ of S_n, there exists an irreducible character ψ of S_n such that*

$$\frac{\psi(1)}{\phi(1)} \in [\delta, \infty) \setminus S.$$

In particular, if $S = S_0 := \{1, 2, 1/2\}$, then one can choose $N = 120,000$ and $\delta = 9/65$.

Proof. Equivalently, we prove that there exists N and $\delta > 0$ such that, for all $n > N$ and for all $\Delta \in Y(n)$, there exists $\Gamma \in Y(n)$ such that

$$\frac{\dim \rho(\Gamma)}{\dim \rho(\Delta)} = \frac{P(\Delta)}{P(\Gamma)} \in [\delta, \infty) \setminus S.$$

Consider the decomposition of

$$\mathrm{Ind}_{\mathsf{S}_{n-1}}^{\mathsf{S}_n} \mathrm{Res}_{\mathsf{S}_{n-1}}^{\mathsf{S}_n} \rho(\Delta) \tag{2.1}$$

into irreducible summands of the form $\rho(\Gamma)$. These summands are indexed by the set $N(\Delta)$ of quadruples (i_1, j_1, i_2, j_2), where $(i_1, j_1) \in \mathcal{B}(\Delta)$ and $(i_2, j_2) \in \mathcal{A}(\Delta \setminus \{(i_1, j_1)\})$. Clearly, $|N(\Delta)| < \sqrt{2n}(\sqrt{2(n-1)}+1) < (2.01)n$ (if $n > 20,000$), while the degree of the representation (2.1) equals $n \dim \rho(\Delta)$. Choosing $\delta \leq 9/65$, we see that the sum of the dimensions of all the constituents $\rho(\Gamma)$ with $\Gamma \in N(\Delta)$ and

$$\frac{\dim \rho(\Gamma)}{\dim \rho(\Delta)} < \delta$$

is less than $(0.28)n(\dim \rho(\Delta))$. Now, choosing

$$\epsilon = \frac{0.72}{\max S}$$

we see that, either there exists an element of $N(\Delta)$ with corresponding diagram $\Gamma \in Y(n)$ such that

$$\frac{\dim \rho(\Gamma)}{\dim \rho(\Delta)} \in [\delta, \infty) \setminus S$$

or there exist at least ϵn elements of $N(\Delta)$ with corresponding diagrams Γ such that

$$\frac{\dim \rho(\Gamma)}{\dim \rho(\Delta)} \in S. \tag{2.2}$$

We need only treat the latter case. Note that $\epsilon = 0.36$ if $S = S_0$.

Consider octuples $(i_1, j_1, \ldots, i_4, j_4)$ such that (i_1, j_1, i_2, j_2) and (i_3, j_3, i_4, j_4) are in $N(\Delta)$, every Γ corresponding to either of them satisfies (2.2), the coordinates i_1, i_2, i_3, i_4 are pairwise distinct, and the same is true for the coordinates j_1, j_2, j_3, j_4. The number of such octuples must be at least

$$(\epsilon n - 1)(\epsilon n - 7(\sqrt{2n} + 1)).$$

(Indeed, there are at least $\epsilon n - 1$ choices for $(i_1, j_1, i_2, j_2) \in N(\Delta)$ with $i_1 \neq i_2$ and $j_1 \neq j_2$ yielding Γ satisfying (2.2). Consider any $(i_3, j_3, i_4, j_4) \in N(\Delta)$ yielding Γ satisfying (2.2). One can check that $i_3 = i_1$ iff $j_3 = j_1$, $i_3 = i_2$ implies $j_3 = j_2 - 1$, and $j_3 = j_2$ implies $i_3 = i_2 - 1$. Similarly, $i_4 = i_3$ iff $j_4 = j_3$, $i_4 = i_2$ iff $j_4 = j_2$, $i_4 = i_1$ implies $j_4 = j_1 + 1$, and $j_4 = j_1$ implies $i_4 = i_1 + 1$.) Observe that

$$(\epsilon n - 1)(\epsilon n - 7(\sqrt{2n} + 1)) > (9/10)\epsilon^2 n^2$$

if n is sufficiently large ($n \geq 10^5$ would suffice for $S = S_0$).

Let us fix one such octuple. We set

$$\Delta_{12} := (\Delta \setminus \{(i_1, j_1)\}) \cup \{(i_2, j_2)\}$$

and

$$\Delta_{34} := (\Delta \setminus \{(i_3, j_3)\}) \cup \{(i_4, j_4)\}.$$

By the distinctness of the i and j coordinates, we have

$$(i_3, j_3) \in \mathcal{B}(\Delta_{12})$$

and

$$(i_4, j_4) \in \mathcal{A}(\Delta_{12} \setminus \{(i_3, j_3)\}).$$

Let

$$\Delta_{1234} := ((\Delta_{12} \setminus \{(i_3, j_3)\}) \cup \{(i_4, j_4)\}.$$

Given $(i, j), (i', j') \in \mathcal{A}(\Delta)$, we can compare $h_{(i',j')}(\Delta)$ to $h_{(i',j')}(\Delta \cup \{(i, j)\})$. If $i \neq i'$ and $j \neq j'$, the hook lengths are equal, but if $i = i'$ or $j = j'$, then

$$h_{(i',j')}(\Delta \cup \{(i, j)\}) = h_{(i',j')}(\Delta) + 1.$$

From this formula, we deduce that

$$\frac{P(\Delta)P(\Delta_{1234})}{P(\Delta_{12})P(\Delta_{34})} = \frac{a(a+2)}{(a+1)^2} \cdot \frac{b(b-2)}{(b-1)^2},$$

where

$$a = h_{(\min(i_2,i_4),\min(j_2,j_4))}(\Delta), \;\; b = h_{(\min(i_1,i_3),\min(j_1,j_3))}(\Delta).$$

Letting $S^2 = \{s_1 s_2 \mid s_1, s_2 \in S\}$, we conclude that

$$\frac{P(\Delta_{1234})}{P(\Delta)} \in \left(\frac{a(a+2)}{(a+1)^2} \cdot \frac{b(b-2)}{(b-1)^2}\right) S^2.$$

As long as δ is chosen less than $(9/16)(\min S)^2$, this value is automatically greater than δ. For instance, if $S = S_0$, then we can choose $\delta = 9/65$. It remains to show that we can choose the octuple $(i_1, \ldots, j_4)$ such that the value is not in S.

There are finitely many values of t such that $tS^2 \cap S$ is non-empty, and we need only consider values of a and b for which

$$\frac{a(a+2)}{(a+1)^2} \cdot \frac{b(b-2)}{(b-1)^2}$$

lies in this finite set. For instance, if $S = S_0$, then there are exactly seven such values for t: $t = 2^i$ with $-3 \leq i \leq 3$. We claim that for each value t, the set of octuples which achieves this value is $o(n^2)$. The claim implies the theorem. To prove the claim we note that there are at most $(\sqrt{2n}+1)^3 = O(n^{3/2})$ possibilities for $(i_1, j_1, i_2, j_2, i_3, j_3)$. Given one such value, b is determined, so if t is fixed, so is a. For a given value of a and given i_2 and j_2, there are at most two possibilities for $(i_4, j_4) \in \mathcal{A}(\Delta)$ with $h_{(\min(i_2,i_4),\min(j_2,j_4))}(\Delta)$ achieving this fixed value. The claim follows. More precisely, if $S = S_0$, then the number of octuples in question is at most $14(\sqrt{2n}+1)^3$, which is less than $(9/10)\epsilon^2 n^2$ when $n > 120,000$. □

COROLLARY 2.2. *There is some constant $\varepsilon > 0$ such that $\varepsilon(\mathsf{A}_n) \geq \varepsilon$ for all $n \geq 5$. In fact, one can choose $\varepsilon = 2/(120,000!)$. Furthermore, $\varepsilon(\mathsf{A}_n) > 1/209$ if $n > 120,000$.*

Proof. Choose $S = S_0 = \{2, 1, 1/2\}$ and apply Theorem 2.1. Let $\chi \in \mathrm{Irr}(\mathsf{A}_n)$ be of degree $b := b(\mathsf{A}_n)$ and let $\phi \in \mathrm{Irr}(\mathsf{S}_n)$ be lying above χ; in particular $\phi(1) = rb$ with $r = 1$ or 2. By Theorem 2.1, if $n > N := 120,000$, then there is some $\psi \in \mathrm{Irr}(\mathsf{S}_n)$ such that $\psi(1)/\phi(1) \geq \delta =: 9/65$ and $\psi(1)/\phi(1) \notin S$. Now let $\rho \in \mathrm{Irr}(\mathsf{A}_n)$ be lying under ψ; in particular, $\rho(1) = \psi(1)/s$ with $s = 1$ or 2. Then $\rho(1)/\chi(1) = (\psi(1)/\phi(1)) \cdot (r/s)$, and so $\rho(1)/\chi(1) \geq \delta/2$ and $\rho(1)/\chi(1) \neq 1$. It follows that $\varepsilon(\mathsf{A}_n) \geq \delta^2/4 > 1/209$ for $n \geq N$.

On the other hand, if $5 \leq n \leq N$, then we can use the trivial bound that $b(S)^2 < |S|$ and so $\varepsilon(S) > 1/|S|$, giving $\varepsilon(\mathsf{A}_n) \geq 2/(N!)$. □

3. *Comparing Unipotent Character Degrees of Simple Groups of Lie Type*

Each finite simple group S of Lie type has an irreducible character St of degree $\mathrm{St}(1) = |S|_p$, where p is the underlying characteristic, the so-called *Steinberg character* (and there is a unique such character except when $S = {}^2F_4(2)'$). We refer the reader to [**3**, Chap. 6] and [**5**] for this, as well as for basic facts on Deligne–Lusztig theory. The main aim of this section is to prove Theorem 1.2 which compares the degree of St with that of the other unipotent characters.

By the results of Lusztig, unipotent characters of isogenous groups have the same degrees, so it is immaterial here whether we speak of groups of adjoint or of simply connected type; moreover, all unipotent characters have the center in their kernel, so they can all be considered as characters of the corresponding simple group.

It is easily checked from the formulas in [**3**, §13] and the data in [**17**] that Theorem 1.2 does in fact hold for exceptional groups of Lie type. The six series of classical groups are handled in Corollaries 3.3 and 3.8 and Proposition 3.9 after some combinatorial preparations. On the way we derive some further interesting relations between unipotent character degrees.

3.1. *Type* GL_n

For $q > 1$ and $\underline{c} = (c_1 < \ldots < c_s)$ a strictly increasing sequence we set

$$[\underline{c}] := \prod_{i=1}^{s} (q^{c_i} - 1)$$

and $\underline{c} + m := (c_1 + m < \ldots < c_s + m)$ for an integer m.

LEMMA 3.1. *Let $q \geq 2$, $s \geq 1$.*

(i) *Then $\frac{q^a-1}{q^{a-1}-1} \leq \frac{q^b-1}{q^{b-1}-1}$ if and only if $a \geq b$.*

(ii) *Let $\underline{c} = (c_1 < \ldots < c_s)$ be a strictly increasing sequence of integers, with $c_1 \geq 2$. Then:*

$$q^s < \frac{[\underline{c}]}{[\underline{c}-1]} < q^{s+1}.$$

Proof. The first part is obvious, and then the second follows by a $2s$-fold application of (i) since

$$q^s < \frac{q^{c_s}-1}{q^{c_s-s}-1} = \prod_{i=1}^{s} \frac{q^{c_s-s+i}-1}{q^{c_s-s+i-1}-1} \leq \frac{[\underline{c}]}{[\underline{c}-1]} \leq \prod_{i=1}^{s} \frac{q^{i+1}-1}{q^i-1} = \frac{q^{s+1}-1}{q-1} < q^{s+1}.$$

□

We denote by χ_λ the unipotent character of $\mathrm{GL}_n(q)$ parametrized by the partition λ of n. Its degree is given by the quantized hook formula

$$\chi_\lambda(1) = q^{a(\lambda)} \frac{(q-1)\cdots(q^n-1)}{\prod_h (q^{l(h)}-1)},$$

where h runs over the hooks of $\lambda = (a_1 \geq \ldots \geq a_r)$, and $a(\lambda) = \sum_{i=1}^r (i-1)a_i$ (see for example [**22**, (21)] or [**18**]).

PROPOSITION 3.2. *Let $\lambda = (a_1 \geq \ldots \geq a_{r-1} > 0) \vdash n-1$ be a partition of $n-1$ and μ, ν the partitions of n obtained by adding a node at $(r,1)$, (i,j) respectively, where $i < r$ and $a_i = j-1$. Then for all $q \geq 2$ the corresponding unipotent character degrees of $\mathrm{GL}_n(q)$ satisfy*

$$q^{-j-1}\chi_\mu(1) < \chi_\nu(1) < q^{2-j}\chi_\mu(1) \leq \chi_\mu(1).$$

Proof. According to the hook formula, we have to consider the hooks in μ, ν of different lengths. These lie in the 1st column, the ith rows and in the jth column. Let $\underline{h} = (1 < h_2 < \ldots < h_r)$ denote the hook lengths in the 1st column, $\underline{k} = (k_1 < \ldots < k_{j-1})$ the hook lengths in the ith row and $\underline{l} = (l_1 < \ldots < l_{i-1})$ the hook lengths in the jth column of μ. Write $\underline{h}' = (h_2 < \ldots < h_r)$ and $\underline{k}' = (0 < k_1 < \ldots < k_{j-1})$. Then a threefold application of Lemma 3.1(ii) shows that

$$\begin{aligned} \chi_\nu(1) &= q^{a(\nu)-a(\mu)} \tfrac{[\underline{h}]}{[\underline{h}'-1]} \tfrac{[\underline{k}]}{[\underline{k}'+1]} \tfrac{[\underline{l}]}{[\underline{l}+1]} \chi_\mu(1) \\ &< q^{-r+i} q^r q^{1-j} q^{1-i} \chi_\mu(1) = q^{2-j}\chi_\mu(1) \leq \chi_\mu(1), \end{aligned}$$

since $j \geq 2$. The other inequality is then also immediate. □

Note that ν is the partition obtained from μ by moving one node from the last row (which contains a single node) to some row higher up. Since clearly any partition of n can be reached by a finite number of such operations from $(1)^n$, we conclude:

COROLLARY 3.3. *Any unipotent character of* $\mathrm{GL}_n(q)$ *other than the Steinberg character* St *has smaller degree than* St.

A better result can be obtained when $q \geq 3$, since then the upper bound in Lemma 3.1(ii) can be improved to $q^{s+1/2}$. In that case, 'moving up' any node in a partition leads to a smaller unipotent degree:

PROPOSITION 3.4. *Let $q \geq 3$, and $\nu \neq \mu$ be two partitions of n with $\nu \rhd \mu$ in the dominance order. Then the corresponding unipotent character degrees of $\mathrm{GL}_n(q)$ satisfy $\chi_\nu(1) < \chi_\mu(1)$.*

Proof. In our situation, ν can be reached from μ by a sequence of steps of moving up a node in a partition. Consider one such step, where the node at position (r, s) is moved to position (i, j), with $j > s$. A similar estimate as in the proof of Proposition 3.2, but with the improved upper bound from Lemma 3.1, leads to the result. □

EXAMPLE 3.5. *The previous result fails for $q = 2$; the smallest counterexample occurs for $n = 6$, $\mu = (2)^3 \lhd \nu = (3)(2)(1)$, where $\chi_\mu(1) = 5952 < \chi_\nu(1) = 6480$.*

3.2. *Type* GU_n

The analogue of Proposition 3.2 is no longer true for the unipotent characters of unitary groups, in general. Still, we can obtain a characterization of the Steinberg character by comparing with character degrees in $\mathrm{GL}_n(q)$.

PROPOSITION 3.6. *Any partition λ of n has $r = \lceil n/2 \rceil$ distinct hooks $h_1, \ldots, h_r$ of odd lengths $l(h_i) \leq 2i - 1$, $1 \leq i \leq r$.*

Proof. We proceed by induction on n. The result is clear for 2-cores, i.e., triangular partitions. Now let $\lambda = (a_1 \leq \ldots \leq a_r)$ be a partition of n which is not a 2-core, with corresponding β-set $B = \{a_1, a_2 + 1, \ldots, a_r + r - 1\}$. The hook lengths of λ are just the differences $j - i$ with $j \in B$, $i \notin B$, $i < j$ (see [**22**, Lemma 2]). Since λ is not a 2-core, there exists $j \in B$ with $j - 2 \notin B$. Let $B' = \{j - 2\} \cup B \setminus \{j\}$, the β-set of a partition μ of $n - 2$. We now compare hook lengths in B' and in B: hooks in B' from $k > j$, $k \in B'$, to j become hooks from k to $j - 2$ in B, and hooks from $j - 2$ to $k \notin B'$, $k < j - 2$, become hooks from j to k in B. In both cases, the length has increased by 2. But we have one further new hook in B: either from j to $j - 1$ (if $j - 1 \notin B'$), or from $j - 1$ to $j - 2$ (if $j - 1 \in B'$), of length 1. So indeed, in both cases we have produced hooks of the required odd lengths in λ. □

PROPOSITION 3.7. *Let λ be a partition of n. Then the degree of the unipotent character of $\mathrm{GL}_n(q)$ indexed by λ is at least as big as the corresponding one of $\mathrm{GU}_n(q)$.*

Proof. It is well-known that the degree of the unipotent character of $\mathrm{GU}_n(q)$ indexed by λ is obtained from the one for $\mathrm{GL}_n(q)$ by formally replacing q by$-q$ in the hook formula above and adjusting the sign. Now let $h_1, \ldots, h_r$ denote the sequence of hooks of odd length from the previous result. Observe that the numerators in the hook formula for $\mathrm{GL}_n(q)$ and $\mathrm{GU}_n(q)$ differ by the factor $\prod_{i=1}^{r}(q^{2i-1} + 1)/(q^{2i-1} - 1)$. Since $(q^a + 1)/(q^b + 1) < (q^a - 1)/(q^b - 1)$ when $b < a$, the claim now follows from the hook formula. □

It seems that the only case with equality, apart from the trivial cases 1 and St, occurs for the partition $(2)^2$ of 4.

Since the degree of the Steinberg character of $\mathrm{GL}_n(q)$ and $\mathrm{GU}_n(q)$ is the same, the following is immediate from Corollary 3.3 and Proposition 3.7:

COROLLARY 3.8. *Any unipotent character of* $\mathrm{GU}_n(q)$ *other than the Steinberg character* St *has smaller degree than* St.

3.3. *Other classical types*

The unipotent characters of the remaining classical groups $G = G(q)$ (i.e. symplectic and orthogonal groups) are labelled by *symbols*, whose definition and basic combinatorics we now recall (we refer to [**18**] and [**22**, Prop. 5] for the hook formula given here). A *symbol* $S = (X, Y)$ is a pair of strictly increasing sequences $X = (x_1 < \ldots < x_r)$, $Y = (y_1 < \ldots < y_s)$ of non-negative integers. The *rank* of S is then

$$\mathrm{rk}(S) = \sum_{i=1}^{r} x_i + \sum_{j=1}^{s} y_j - \left\lfloor \left(\frac{r+s-1}{2}\right)^2 \right\rfloor .$$

The symbol $S' = (\{0\} \cup (X+1), \{0\} \cup (Y+1))$ is said to be *equivalent* to S, and so is the symbol (Y, X). The rank is constant on equivalence classes. The *defect* of S is $d(S) = ||X| - |Y||$, which clearly is also invariant under equivalence.

Lusztig has shown that the unipotent characters of classical groups of rank n are naturally parametrized by equivalence classes of symbols of rank n, with those of odd defect parametrizing characters in types B_n and C_n, those of defect $\equiv 0 \pmod 4$ characters in type D_n, and those of defect $\equiv 2 \pmod 4$ characters in type ${}^2D_n$. (Here, each so-called degenerate symbol, where $X = Y$, parametrizes two unipotent characters in type D_n.)

The degrees of unipotent characters are most conveniently given by an analogue of the hook formula for $\mathrm{GL}_n(q)$, as follows. A *hook of* S is a pair $(b, c) \in \mathbb{N}_0^2$ with $b < c$ and either $b \notin X$, $c \in X$, or $b \notin Y$, $c \in Y$. Thus, a hook of S is nothing else but a hook (as considered in Section 2 and for type A above) of the permutation with associated β-set either X or Y. A *cohook of* S is a pair $(b, c) \in \mathbb{N}_0^2$ with $b < c$ and either $b \notin Y$, $c \in X$, or $b \notin X$, $c \in Y$. We also set

$$a(S) := \sum_{\{b,c\} \subseteq S} \min\{b, c\} - \sum_{i \geq 1} \binom{r+s-2i}{2},$$

where the first sum runs over all 2-element subsets of the multiset $X \cup Y$ of entries of S, and $b(S) = \lfloor |S-1|/2 \rfloor - |X \cap Y|$ if $X \neq Y$, respectively $b(S) = 0$ else. The degree of the unipotent character χ_S of a finite classical group $G = G(q)$ parametrized by S is then given as

$$\chi_S(1) = q^{a(S)} \frac{|G|_{q'}}{2^{b(S)} \prod_{(b,c) \text{ hook}} (q^{c-b} - 1) \prod_{(b,c) \text{ cohook}} (q^{c-b} + 1)},$$

where the products run over hooks, respectively cohooks of S (see [**18**, Bem. 3.12 and 6.8]). It can be checked that this is constant on equivalence classes. It is also clear from this that the unipotent characters in types B_n and C_n have the same degrees.

For $q > 1$ and $\underline{c} = (c_1 < \ldots < c_s)$ a strictly increasing sequence we set

$$[\underline{c}]^- := \prod_{i=1}^{s} (q^{c_i} + 1).$$

PROPOSITION 3.9. *Let S be a symbol of rank n, parametrizing a unipotent character of $G = G(q)$ of rank n. Then $\chi_S(1) \leq \mathrm{St}(1)$, where* St *denotes the Steinberg character of G, with equality only if $\chi_S = \mathrm{St}$.*

Proof. We argue by induction on n, the case $n \leq 2$ being trivial. So let now $S = (X, Y)$ be a symbol of rank at least 3. Replacing S by an equivalent symbol, we may assume that $0 \notin X$.

We first discuss the case where $a := \max(X \cup Y) \in X$. Writing $X = (h_1 < \ldots < h_r < a)$ (so $h_1 > 0$), $Y = (k_1 < \ldots < k_s)$, we consider the symbol $S' = (X', Y)$ where $S' = (0 < h_1 < \ldots < h_r)$. Thus

$$\mathrm{rk}(S') = \mathrm{rk}(S) - a, \qquad a(S') = a(S) - \sum h_i - \sum k_j.$$

We also write

$$\{\tilde{h}_1, \ldots, \tilde{h}_t\} := \{0, \ldots, a-1\} \setminus X, \quad \{\tilde{k}_1, \ldots, \tilde{k}_t\} := \{0, \ldots, a-1\} \setminus Y.$$

Then, with G' denoting a group of rank $\mathrm{rk}(S')$ and of the same type as G, the hook formula gives

$$\chi_S(1) = q^{\sum h_i + \sum k_j} \frac{|G|_{q'}}{|G'|_{q'}} \cdot \frac{1}{[\underline{h}] \cdot [a - \underline{\tilde{h}}] \cdot [\underline{k}]^- \cdot [a - \underline{\tilde{k}}]^-} \chi_{S'}(1).$$

Application of Lemma 4.1 and the inductive hypothesis for $\chi_{S'}(1)$ then gives

$$\chi_S(1) \le d^2 q^{a - \sum(a - \tilde{h}_i) - \sum(a - \tilde{k}_j)} \ \mathrm{St}(1),$$

with $d = \frac{32}{9}$ when $q = 2$, respectively $d = \frac{9}{5}$ when $q > 2$. Now note that $\tilde{h}_1 = 0$, and $\tilde{h}_i, \tilde{k}_j < a$, so the exponent at q is negative unless $X = (1, \ldots, a)$, $Y = (0, \ldots, a-1)$ or $Y = (0, \ldots, a)$, when $\chi_S = \mathrm{St}$. Moreover, the required inequality holds as soon as $\sum_{i \ge 2}(a - \tilde{h}_i) + \sum_j (a - \tilde{k}_j)$ is at least 4 (when $q = 2$), respectively at least 2 when $q > 2$. Also note that when $\underline{\tilde{h}} = (0)$ consists of just one entry, then one of the factors d in the estimate goes away, whence we only need $\sum_j (a - \tilde{k}_j) \ge 2$. There remain exactly eight possibilities for S which can be dealt with case by case.

In the case when $a = \max(X \cup Y) \in Y \setminus X$, write $X = (h_1 < \ldots < h_r)$ (so still $h_1 > 0$), $Y = (k_1 < \ldots < k_s < a)$, and consider the symbol $S' = (X \cup \{0\}, Y \setminus \{a\})$. Then exactly the same estimates as before apply to show that $\chi_S(1) < \mathrm{St}(1)$. $\square$

This completes the proof of Theorem 1.2.

4. *Theorem 1.1 for Simple Groups of Lie Type*

Note that to prove Theorem 1.1 we can ignore any finite number of non-abelian simple groups, in particular the 26 sporadic groups (of course one can find out the exact value of $\varepsilon(S)$ for each of them; in particular, one can check using [**4**] that $\varepsilon(S) > 1$ for all the sporadic groups). Thus, in view of Corollary 2.2, it remains to prove Theorem 1.1 for simple groups of Lie type.

We begin with some estimates:

LEMMA 4.1. *Let $q \ge 2$. Then the following inequalities hold.*

(i) $\prod_{i=1}^{\infty}(1 - 1/q^i) > 1 - 1/q - 1/q^2 + 1/q^5 \ge \exp(-\alpha/q)$, *where* $\alpha = 2\ln(32/9) \approx 2.537$,
(ii) $\prod_{i=2}^{\infty}(1 - 1/q^i) > 9/16$,
(iii) $\prod_{i=k}^{\infty}(1 + 1/q^i)$ *is smaller than* 2.4 *if* $k = 1$, 1.6 *if* $k = 2$, 1.28 *if* $k = 3$, *and* 16/15 *if* $k = 5$, *and*
(iv) $1 < \prod_{i=1}^{n}(1 - (-1/q)^i) \le 3/2$.

Proof. (i) As mentioned in [**8**] (see the paragraph after Lemma 3.4 of [**8**]), a convenient way to prove these estimates is to use Euler's pentagonal number theorem [**1**, p. 11]:

$$\begin{aligned} \prod_{i=1}^{\infty}(1 - \tfrac{1}{q^i}) &= 1 + \sum_{n=1}^{\infty}(-1)^n (q^{-n(3n-1)/2} + q^{-n(3n+1)/2}) \\ &= 1 - q^{-1} - q^{-2} + q^{-5} + q^{-7} - q^{-12} - q^{-15} + \cdots \end{aligned} \tag{4.1}$$

Since $q^{-m} \geq \sum_{i=m+1}^{\infty} q^{-i}$, finite partial sums of this series yield arbitrarily accurate upper and lower bounds for $\prod_{i=1}^{\infty}(1-q^{-i})$. In particular, truncating the series (4.1) at the term q^{-5} yields the first inequality. Next, consider the function

$$f(x) := 1 - x - x^2 + x^5 - \exp(-\alpha x)$$

for the chosen α. The choice of α ensures that $f(1/2) = 0 = f(0)$, and $f''(x) < 0$ for all $x \in [0, 1/2]$. It follows that $f(x) \geq 0$ on $[0, 1/2]$, yielding the second inequality.

(ii) Clearly,

$$\prod_{i=2}^{\infty}(1-\frac{1}{q^i}) \geq \prod_{i=2}^{\infty}(1-\frac{1}{2^i}) = 2\prod_{i=1}^{\infty}(1-\frac{1}{2^i}) > 2(1-\frac{1}{2}-\frac{1}{4}+\frac{1}{32}) = \frac{9}{16}.$$

(iii) Applying (4.1) with $q = 4$ and truncating the series at the term q^{-7} we get $\prod_{i=1}^{\infty}(1-4^{-i}) < 0.6876$. Applying (4.1) with $q = 2$ and truncating the series at the term q^{-15} we get $\prod_{i=1}^{\infty}(1-2^{-i}) > 0.2887$. Now

$$\prod_{i=1}^{\infty}(1+\frac{1}{q^i}) \leq \prod_{i=1}^{\infty}(1+\frac{1}{2^i}) = \frac{\prod_{i=1}^{\infty}(1-\frac{1}{4^i})}{\prod_{i=1}^{\infty}(1-\frac{1}{2^i})} < \frac{0.6876}{0.2887} < 2.382.$$

The other bounds can be obtained by using this bound and noting that

$$\prod_{i=k}^{\infty}(1+\frac{1}{q^i}) \leq \prod_{i=k}^{\infty}(1+\frac{1}{2^i}) = \frac{\prod_{i=1}^{\infty}(1+\frac{1}{2^i})}{\prod_{i=1}^{k-1}(1+\frac{1}{2^i})}.$$

(iv) follows from the estimates

$$(1-\frac{1}{q^{2k}})(1+\frac{1}{q^{2k+1}}) < 1 < (1+\frac{1}{q^{2k-1}})(1-\frac{1}{q^{2k}})$$

for any $k \geq 1$. □

4.1. *Theorem 1.1 for exceptional groups of Lie type*

The following result of G. M. Seitz [**24**, Thm. 2.1] will be basic to our discussion not only of exceptional but also of classical groups of arbitrary rank:

THEOREM 4.2. (Seitz) *Let $\mathcal{G}$ be a simple, simply connected algebraic group over the algebraic closure of a finite field of characteristic p, $F : \mathcal{G} \to \mathcal{G}$ a Steinberg endomorphism of $\mathcal{G}$, and let $L := \mathcal{G}^F$. Then $b(L) \leq |L|_{p'}/|T_0|$, where T_0 is a maximal torus of L of minimal order. For q sufficiently large, this is in fact an equality (namely, whenever* $\mathrm{Irr}(T_0)$ *contains at least one character in general position).*

In the above result as well as in Proposition 4.5 and Corollary 4.6, q is the common absolute value of the eigenvalues of F acting on the character group of an F-stable maximal torus of $\mathcal{G}$. If $L = \mathcal{G}^F$ is untwisted and $\mathrm{rank}(\mathcal{G}) = r$, then $|T_0| = (q-1)^r$. For the convenience of the reader, we list $|T_0|$ for twisted groups L in Table 1.

PROPOSITION 4.3. *Let S be a simple exceptional group of Lie type. Then $\varepsilon(S) > 1$.*

Proof. According to the result of Seitz quoted in Theorem 4.2 the maximal degree $b(S)$ is bounded above by $|L|_{p'}/|T_0|$, where L is the group of fixed points of a simple, simply connected algebraic group under a Steinberg endomorphism with $S = L/Z(L)$, and T_0 is a maximal torus of L of minimal order. For each of the ten series of exceptional groups of Lie type we give in

TABLE 1. *Maximal tori of smallest order in twisted groups*

L	$\lvert T_0\rvert$	L	$\lvert T_0\rvert$
${}^2B_2(q^2)$	$q^2-\sqrt{2}q+1$	${}^2A_n(q),\ n=2m-1$	$(q^2-1)^m/(q+1)$
${}^2G_2(q^2)$	$q^2-\sqrt{3}q+1$	${}^2A_n(q),\ n=2m$	$(q^2-1)^{m-1}(q^2-q+1)$
${}^3D_4(q)$	$(q^2-q+1)^2$	${}^2D_n(q),\ n\geq 4$	$(q-1)^{n-1}(q+1)$
${}^2F_4(q^2)$	$(q^2-\sqrt{2}q+1)^2$	${}^2E_6(q),\ q\geq 7$	$(q^2-q+1)^3$

Table 2 another maximal torus T of L and a lower bound n_T for the number of irreducible Deligne–Lusztig characters $R_{T,\theta}$ parametrized by characters θ in general position of this torus, such that $n_T(R_{T,\theta}(1))^2 > (|L|_{p'}/|T_0|)^2$ at least for $q>3$ (resp. $q^2>2$ for the Suzuki and Ree groups), and such that $R_{T,\theta}(1) < b(S)$. The finitely many small cases can be checked directly. (The list of character degrees of L is given on Frank Lübeck's website [**17**].) □

TABLE 2. *Tori in exceptional groups*

L	$\lvert T\rvert$	n_T	L	$\lvert T\rvert$	n_T
${}^2B_2(q^2)$	Φ_8''	$(\Phi_8''-1)/4$	$F_4(q)$	Φ_{12}	$(\Phi_{12}-1)/12$
${}^2G_2(q^2)$	q^2-1	$(q^2-3)/2$	$E_6(q)$	Φ_9	$(\Phi_9-3)/9$
$G_2(q)$	Φ_6	$(\Phi_6-3)/6$	${}^2E_6(q)$	Φ_{18}	$(\Phi_{18}-3)/9$
${}^3D_4(q)$	Φ_{12}	$(\Phi_{12}-1)/12$	$E_7(q)$	$\Phi_{14}\Phi_2$	$(\Phi_{14}-7)/14$
${}^2F_4(q^2)$	Φ_{24}''	$(\Phi_{24}''-1)/12$	$E_8(q)$	Φ_{15}	$(\Phi_{15}-1)/30$

Here, Φ_m is the m^{th} cyclotomic polynomial in q, $\Phi_8''=q^2-\sqrt{2}q+1$, $\Phi_{24}''=q^4-\sqrt{2}q^3+q^2-\sqrt{2}q+1$.

The same proof as above establishes Theorem 1.1 for simple classical groups of *bounded* rank.

4.2. *Theorem 1.1 for classical groups: The generic case*

In what follows, we will view our simple classical group S as $L/Z(L)$, where $L=\mathcal{G}^F$ as in Theorem 4.2. We also consider the pair $(\mathcal{G}^*,F^*)$ dual to $(\mathcal{G},F)$ and the group $H:=(\mathcal{G}^*)^{F^*}$ dual to L.

As mentioned above, Theorem 4.2 gives the largest degree of complex irreducible representations of L, a covering group of the finite simple group of Lie type $S=L/Z(L)$, whenever q is large enough. To show that this is also the precise value of $b(S)$, we need Proposition 4.5 (below) which is also of independent interest.

First we prove an auxiliary statement:

LEMMA 4.4. *Let $X\cong\mathbb{Z}^r$ and $w\in\mathrm{GL}(X)\cong\mathrm{GL}_r(\mathbb{Z})$ be an element of finite order. Then the torsion subgroup of the abelian group $X/(w-1)X$ has order at most 2^r.*

Proof. The fixed lattice $Y:=X^{\langle w\rangle}$ is a w-invariant pure submodule of X and $Y/(w-1)Y=Y$ is torsion-free, so we may assume without loss that $Y=0$. Then the characteristic polynomial $f(t)=\det(tI-w)$ of w factorizes as $f(t)=\prod_{j=1}^d(t-\epsilon_j)$, where ϵ_j are nontrivial roots of unity. Setting $V:=X\otimes_{\mathbb{Z}}\mathbb{Q}$ it follows that $f(t)$ is the characteristic polynomial of w on V. Hence $X/(w-1)X$ has order

$$|\det{}_V(w-1)|=|f(1)|=|\prod_{j=1}^r(1-\epsilon_j)|\leq 2^r,$$

as claimed. □

PROPOSITION 4.5. *Let $\mathcal{G}$ be a semisimple algebraic group of rank $r \geq 1$ over the algebraic closure of a finite field and let $F\colon \mathcal{G} \to \mathcal{G}$ be a Steinberg endomorphism with $q > (8.24)^{r+2} r!$. Then for every F-stable maximal torus $\mathcal{T}$ of $\mathcal{G}$, the set of regular elements s of $\mathcal{T}^F$ with $C_{\mathcal{G}^F}(s) = \mathcal{T}^F$ has cardinality greater than $(1 - 1/2^r)|\mathcal{T}^F|$.*

Proof. 1) Let $X \cong \mathbb{Z}^r$ denote the character group of $\mathcal{T}$. Then the Weyl group $W = N_{\mathcal{G}}(\mathcal{T})/\mathcal{T}$ acts on X. By part (3) of the proof of [**16**, Thm. 2.1], the number of elements $t \in \mathcal{T}^F$ which are not regular is at most $2^r r^2 (q+1)^{r-1}$.

2) Next we count the number of regular elements $t \in \mathcal{T}^F$ for which $C_{\mathcal{G}^F}(t) \neq \mathcal{T}^F$. For any such element, there is some $g \in C_{\mathcal{G}^F}(t)$ such that $g \notin \mathcal{T}$. Thus g induces a nontrivial element $w \in W$, which acts on X as an endomorphism of finite order. Since g is F-stable, the actions of w and F on X commute, whence $Y := (w-1)X$ and

$$Y^\perp := \{u \in \mathcal{T} \mid \chi(u) = 1,\ \forall \chi \in Y\}$$

are F-stable. Therefore, the closed connected subgroup $\mathcal{S} := (Y^\perp)^\circ$ of $\mathcal{T}$ is an F-stable torus. Also, since $g^{-1} t g = t$, we have that $t \in (Y^\perp)^F$.

As $Y \leq X \cong \mathbb{Z}^r$ and $w \neq 1$, we can find a basis $(\chi_1, \ldots, \chi_r)$ of X and positive integers $a_1, \ldots, a_d$ (for some $1 \leq d \leq r$) such that $Y = \langle a_1\chi_1, \ldots, a_d\chi_d\rangle_{\mathbb{Z}}$. By the elementary divisor theorem, the torsion subgroup of X/Y has order equal to $\prod_{j=1}^d a_j$. Hence $\prod_{j=1}^d a_j \leq 2^r$ by Lemma 4.4. On the other hand,

$$\varphi\ :\ \mathcal{T} \to \mathrm{GL}_1^r, \quad v \mapsto (\chi_1(v), \ldots, \chi_r(v)),$$

defines an isomorphism between $\mathcal{T}$ and GL_1^r which maps $Y^\perp$ onto the closed subgroup

$$\{(u_1, \ldots, u_r) \mid u_i \in \mathrm{GL}_1,\ u_j^{a_j} = 1,\ \forall 1 \leq j \leq d\}$$

of GL_1^r with connected component

$$\{(u_1, \ldots, u_r) \mid u_i \in \mathrm{GL}_1,\ u_j = 1,\ \forall 1 \leq j \leq d\}.$$

Thus the latter subgroup must be $\varphi(\mathcal{S})$, and so

$$|Y^\perp/\mathcal{S}| \leq \prod_{j=1}^d a_j \leq 2^r. \tag{4.2}$$

Since $\mathcal{S}$ is an F-stable torus of dimension $r - d \leq r-1$, we see that $|\mathcal{S}^F| \leq (q+1)^{r-1}$. But $\mathcal{S}$ is a connected normal subgroup of $Y^\perp$, hence an application of the Lang–Steinberg theorem and (4.2) implies that

$$(Y^\perp)^F/\mathcal{S}^F \cong (Y^\perp/\mathcal{S})^F$$

has order at most 2^r. We have shown that $|(Y^\perp)^F| \leq 2^r(q+1)^{r-1}$.

Recall that $Y = (w-1)X$ and $w \in W$. Since $\mathcal{G}$ is semisimple of rank r, one can check that $|W| \leq (67.5)2^r r!$ (with equality attained for $\mathcal{G}$ of type E_8). Thus the total number of regular elements $t \in \mathcal{T}^F$ with $C_{\mathcal{G}^F}(t) \neq \mathcal{T}^F$ is at most $(67.5)2^{2r} r!(q+1)^{r-1}$.

Note that for $q > (8.24)^{r+2} r!$ we have $2^r r^2 \leq (0.5)2^{2r} r!$ and $(q+1)/(q-1) < 1.0036$, whence

$$\begin{aligned} ((67.5)2^{2r} r! + 2^r r^2)(q+1)^{r-1} &\leq 68 \cdot 2^{2r} r! \cdot (1.0036 \cdot (q-1))^{r-1} \\ &< (67.76) \cdot (4.0144)^r r! (q-1)^{r-1} \\ &< (q-1)^r/2^r. \end{aligned}$$

Since $|\mathcal{T}^F| \geq (q-1)^r$, the proposition follows. □

COROLLARY 4.6. *Let $\mathcal{G}$ be a simple simply connected algebraic group of rank $r \geq 1$ over the algebraic closure of a finite field, $F\colon \mathcal{G} \to \mathcal{G}$ a Steinberg endomorphism with $q > (8.24)^{r+2}r!$, and let $L := \mathcal{G}^F$. Then the bound $|L|_{p'}/|T_0|$ given in Theorem 4.2 actually gives the precise value of $b(L)$ and also $b(S)$, where $S := L/Z(L)$ is the corresponding simple group of Lie type.*

Proof. Let $(\mathcal{G}^*, F^*)$ be dual to $(\mathcal{G}, F)$ and set $H := (\mathcal{G}^*)^{F^*}$. We apply Proposition 4.5 to $\mathcal{G}^*$ with the Steinberg endomorphism F^* and suppose that $q > (8.24)^{r+2}r!$. Let $\mathcal{T}$ be an F-stable maximal torus of $\mathcal{G}$ such that $|\mathcal{T}^F| = |T_0|$, and let $\mathcal{T}^*$ be the F^*-stable maximal torus dual to $\mathcal{T}$. By the choice of $\mathcal{G}^F$, more than $1 - 1/2^r$ of the elements $t \in T^* := (\mathcal{T}^*)^{F^*}$ are regular semisimple with $C_H(t) = T^*$. On the other hand, observe that $[H, H]$ has index at most $|Z(\mathcal{G})| \leq 2^r$ in H, whence at least $1/2^r$ of the elements in T^* belongs to $[H, H]$. It follows that there is some regular semisimple element $s \in T^* \cap [H, H]$ such that $C_H(s) = T^*$. In particular, $|C_H(s)| = |T^*| = |\mathcal{T}^F| = |T_0|$. By [**5**, Theorem 13.23], any character χ in the Lusztig series $\mathcal{E}(L, (s))$ has degree divisible by $|L|_{p'}/|T_0|$. Furthermore, the condition $s \in [H, H]$ guarantees that χ is trivial at $Z(L)$, whence χ can be viewed as an irreducible character of S. $\square$

THEOREM 4.7. *Let S be a finite simple classical group. Suppose that S is not isomorphic to any of the following groups:*

$$\begin{cases} \mathrm{SL}_n(2),\ \mathrm{Sp}_{2n}(2),\ \Omega^{\pm}_{2n}(2), \\ \mathrm{PSL}_n(3) \text{ with } 5 \leq n \leq 14,\ \ \mathrm{PSU}_n(2) \text{ with } 7 \leq n \leq 14, \\ \mathrm{PSp}_{2n}(3) \text{ or } \Omega_{2n+1}(3) \text{ with } 4 \leq n \leq 17,\ \ \mathrm{P}\Omega^{\pm}_{2n}(3) \text{ with } 4 \leq n \leq 30, \\ \mathrm{P}\Omega^{\pm}_8(7),\ \ \mathrm{P}\Omega^{\pm}_{2n}(5) \text{ with } 4 \leq n \leq 6. \end{cases}$$

Then $\varepsilon(S) > 1$.

Proof. 1) First we consider the case $S = \mathrm{PSL}_n(q)$ with $q \geq 3$. Then $\mathcal{G} = \mathrm{SL}_n(\bar{\mathbb{F}}_q)$, $\mathcal{G}^* = \mathrm{PGL}_n(\bar{\mathbb{F}}_q)$, $L = \mathrm{SL}_n(q)$, $H = \mathrm{PGL}_n(q)$, and the maximal tori of minimal order in Theorem 4.2 are the maximally split ones, of order $(q-1)^{n-1}$. Hence

$$b(S) \leq b(L) \leq B/(q-1)^{n-1}, \text{ where } B := |L|_{p'} = (q^2-1)(q^3-1)\cdots(q^n-1).$$

Now we consider a maximal torus T of order $q^{n-1}-1$ in H, with full inverse image $\hat{T} = C_{q^{n-1}-1} \times C_{q-1}$ in $\mathrm{GL}_n(q)$. We will show that, in the generic case, the regular semisimple elements in T will produce enough irreducible characters of S, all of degree less than $b(S)$, and with the sum of squares of their degrees exceeding $b(S)^2$.

Assume $n \geq 4$. A typical element $\hat{s}$ of $\hat{T} \cap L$ is $\mathrm{GL}_n(\bar{\mathbb{F}}_q)$-conjugate to

$$\mathrm{diag}\left(\alpha, \alpha^q, \ldots, \alpha^{q^{n-2}}, \alpha^{(1-q^{n-1})/(q-1)}\right),$$

where $\alpha \in \mathbb{F}^{\times}_{q^{n-1}}$. Let

$$X := \left(\cup_{i=1}^{n-2} \mathbb{F}^{\times}_{q^i} \cup \{x \in \bar{\mathbb{F}}^{\times}_q \mid x^{n(q-1)} = 1\}\right) \cap \mathbb{F}^{\times}_{q^{n-1}}.$$

Also set $m := \lfloor (n-1)/2 \rfloor$. Then for $n \geq 6$ we have $n - m \geq 4$, and so

$$|X| < \sum_{i=0}^{m} q^i + n(q-1) \leq \frac{q^{m+1}-1}{2} + n(q-1) \leq \frac{q^{n-3}-1}{2} + n(q-1) < \frac{q^{n-1}-1}{2}$$

since $q \geq 3$. Direct calculations show that $|X| < (q^{n-1}-1)/2$ also for $n = 4, 5$. Thus there are at least $(q^{n-1}-1)/2$ elements α in $\mathbb{F}^{\times}_{q^{n-1}}$ that do not belong to X. Consider $\hat{s}$ for any such α. Then all the n eigenvalues of $\hat{s}$ are distinct, and exactly one of them (namely $\beta := \alpha^{-(q^{n-1}-1)/(q-1)}$) belongs to $\mathbb{F}_q$. Suppose that $x \in \mathrm{GL}_n(\bar{\mathbb{F}}_q)$ centralizes $\hat{s}$ modulo $Z(\mathrm{GL}_n(\bar{\mathbb{F}}_q))$:

$x\hat{s}x^{-1} = \gamma\hat{s}$. Comparing the determinant, we see that $\gamma^n = 1$. Suppose that for some i with $0 \le i \le n-2$, $\gamma\beta = \alpha^{q^i}$. Then $(\alpha^{q^i})^{n(q-1)} = (\gamma\beta)^{n(q-1)} = 1$, and so $\alpha \in X$, a contradiction. Hence $\gamma\beta = \beta$, i.e. $\gamma = 1$ and x centralizes $\hat{s}$, and clearly $C_{\mathrm{GL}_n(\bar{\mathbb{F}}_q)}(\hat{s})$ is a maximal torus. So if $s \in T$ is the image of $\hat{s}$, then $C_{\mathcal{G}^*}(s) = C_{\mathrm{GL}_n(\bar{\mathbb{F}}_q)}(\hat{s})/Z(\mathrm{GL}_n(\bar{\mathbb{F}}_q))$ is connected and a maximal torus of $\mathcal{G}^*$; in particular, s is regular. Also, $s \in \mathrm{PSL}_n(q) = [H,H]$. Hence each such s defines an irreducible character χ_s of L, of degree $B/|T|$, which is trivial at $Z(L)$. So we can view χ_s as an irreducible character of S. Each such s has at most $q-1$ inverse images $\hat{s} \in \hat{T} \cap L$. Moreover, since $|N_H(T)/T| = n-1$, the H-conjugacy class of s intersects T at $n-1$ elements. We have therefore produced at least $(q^{n-1}-1)/2(q-1)(n-1)$ irreducible characters χ_s of S, each of degree

$$\chi_s(1) = B/|T| = (q^2-1)(q^3-1)\dots(q^{n-2}-1)(q^n-1).$$

Note that $\chi_s(1) < q^{2+3+\dots+(n-2)+n} = \mathrm{St}(1) \le b(S)$. Hence, to show that $\varepsilon(S) > 1$, it suffices to verify that

$$\frac{q^{n-1}-1}{2(q-1)(n-1)} \cdot \left(\frac{B}{q^{n-1}-1}\right)^2 > \left(\frac{B}{(q-1)^{n-1}}\right)^2,$$

equivalently, $(q-1)^{2n-3} > 2(n-1)(q^{n-1}-1)$. The latter inequality holds if $q = 3$ and $n \ge 15$, or if $q = 4$ and $n \ge 5$, or if $q \ge 5$ and $n \ge 4$. It is straightforward to check that $\varepsilon(S) > 1$ when $n = 2, 3$ or $(n,q) = (4,4)$ (using [**9**] for the last case).

2) Next let $S = \mathrm{PSU}_n(q)$ with $n \ge 3$. Then $\mathcal{G} = \mathrm{SL}_n(\bar{\mathbb{F}}_q)$, $\mathcal{G}^* = \mathrm{PGL}_n(\bar{\mathbb{F}}_q)$, $L = \mathrm{SU}_n(q)$, $H = \mathrm{PGU}_n(q)$. The maximal tori of minimal order in Theorem 4.2 have order at least $(q^2-1)^{n/2}/(q+1)$. Hence

$$b(S) \le b(L) \le B(q+1)/(q^2-1)^{n/2}, \text{ where } B := |L|_{p'} = \prod_{i=2}^{n}(q^i - (-1)^i).$$

Now we consider a maximal torus T of order $q^{n-1} - (-1)^{n-1}$ in H, with full inverse image $\hat{T} = C_{q^{n-1}-(-1)^{n-1}} \times C_{q+1}$ in $\mathrm{GU}_n(q)$. We will follow the same approach as in the case of $\mathrm{PSL}_n(q)$.

Assume that $n \ge 4$, and moreover $q \ge 3$ if $4 \le n \le 7$. A typical element $\hat{s}$ of $\hat{T} \cap L$ is $\mathrm{GL}_n(\bar{\mathbb{F}}_q)$-conjugate to

$$\mathrm{diag}\left(\alpha, \alpha^{-q}, \dots, \alpha^{(-q)^{n-2}}, \alpha^{((-q)^{n-1}-1)/(q+1)}\right),$$

where $\alpha \in C_{q^{n-1}-(-1)^{n-1}} < \bar{\mathbb{F}}_q^\times$. Let Y be the set of elements in $C_{q^{n-1}-(-1)^{n-1}}$ that belong to a cyclic subgroup $C_{q^k-(-1)^k}$ of $\bar{\mathbb{F}}_q^\times$ for some $1 \le k < n-1$ or have order dividing $n(q+1)$. Assume $n \ge 9$ and set $m := \lfloor (n-1)/2 \rfloor$. Then $n - m \ge 5$ and so

$$\begin{aligned} |Y| &\le \textstyle\sum_{i=1}^m (q^i - (-1)^i) + n(q+1) \le \sum_{i=0}^m q^i + n(q+1) \\ &= \frac{q^{m+1}-1}{q-1} + n(q+1) \le \frac{q^{n-3}-1}{2} + n(q+1) < \frac{q^{n-1}-1}{2}. \end{aligned}$$

Direct calculations show that $|Y| < (q^{n-1}-1)/2$ also for $4 \le n \le 8$ (recall that we are assuming $q \ge 3$ when $4 \le n \le 7$). Thus there are at least $(q^{n-1} - (-1)^{n-1})/2$ elements of $C_{q^{n-1}-(-1)^{n-1}}$ that do not belong to Y. Consider $\hat{s}$ for any such α. Then all the n eigenvalues of $\hat{s}$ are distinct, and exactly one of them (namely $\alpha^{((-q)^{n-1}-1)/(q+1)}$) belongs to $C_{q+1} < \bar{\mathbb{F}}_q^\times$. Arguing as in the PSL-case, we see that if $x \in \mathrm{GL}_n(\bar{\mathbb{F}}_q)$ centralizes $\hat{s}$ modulo $Z(\mathrm{GL}_n(\bar{\mathbb{F}}_q))$, then x actually centralizes $\hat{s}$, and $C_{\mathrm{GL}_n(\bar{\mathbb{F}}_q)}(\hat{s})$ is a maximal torus. So if $s \in T$ is the image of $\hat{s}$, then $C_{\mathcal{G}^*}(s) = C_{\mathrm{GL}_n(\bar{\mathbb{F}}_q)}(\hat{s})/Z(\mathrm{GL}_n(\bar{\mathbb{F}}_q))$ is connected and a maximal torus of $\mathcal{G}^*$; in particular, s is regular. Also, $s \in \mathrm{PSU}_n(q) = [H,H]$. Hence each such s defines an irreducible character χ_s of L, of degree $B/|T|$, which is trivial at $Z(L)$. So we can view χ_s as an irreducible character of S.

Each such s has at most $q+1$ inverse images $\hat{s} \in \hat{T} \cap L$. Moreover, since $|N_H(T)/T| = n-1$, the H-conjugacy class of s intersects T at $n-1$ elements. We have therefore produced at least $(q^{n-1} - (-1)^{n-1})/2(q+1)(n-1)$ irreducible characters χ_s of S, each of degree

$$\chi_s(1) = \frac{B}{|T|} = \frac{\prod_{i=2}^{n}(q^i - (-1)^i)}{q^{n-1} - (-1)^{n-1}}.$$

Note that $\chi_s(1) < q^{2+3+\ldots+(n-2)+n} = \mathrm{St}(1) \le b(S)$. Hence, to show that $\varepsilon(S) > 1$, it suffices to verify that

$$\frac{q^{n-1} - (-1)^{n-1}}{2(q+1)(n-1)} \cdot \left(\frac{B}{(q^{n-1} - (-1))^{n-1}}\right)^2 > \left(\frac{B(q+1)}{(q^2-1)^{n/2}}\right)^2,$$

equivalently, $(q^2-1)^n > 2(n-1)(q^{n-1} - (-1)^{n-1})(q+1)^3$. The latter inequality holds if $q = 2$ and $n \ge 15$, or if $q = 3$ and $n \ge 6$, or if $q \ge 4$ and $n \ge 4$. It is straightforward to check that $\varepsilon(S) > 1$ when $n = 3$ (and $q \ge 3$), or $(n, q) = (4, 2)$, $(5, 2)$, $(6, 2)$, $(4, 3)$, $(5, 3)$ (using [**17**] in the last case).

3) Here we consider the case $S = \mathrm{PSp}_{2n}(q)$ or $\Omega_{2n+1}(q)$ with $n \ge 2$ and $q \ge 3$ (and q is odd in the Ω-case). Then $L = \mathrm{Sp}_{2n}(q)$, resp. $L = \mathrm{Spin}_{2n+1}(q)$. The maximal tori of minimal order in Theorem 4.2 have order at least $(q-1)^n$. Hence

$$b(S) \le b(L) \le B/(q-1)^n, \text{ where } B := |L|_{p'} = \prod_{i=1}^{n}(q^{2i} - 1).$$

To simplify the computation, we will view S as a normal subgroup of index $\le \kappa := \gcd(2, q-1)$ of the Lie-type group of adjoint type $K := \mathrm{PCSp}_{2n}(q)$, resp. $K := \mathrm{SO}_{2n+1}(q)$. Then any semisimple element in the dual group $K^* = \mathrm{Spin}_{2n+1}(q)$, resp. $\mathrm{Sp}_{2n}(q)$, has connected centralizer (in the underlying algebraic group). Now we consider a maximal torus T of order $q^n - 1$ in K^*, and let X be the set of elements in T of order dividing $q^k \pm 1$ for some k with $1 \le k \le n-1$. Setting $m := \lfloor n/2 \rfloor$ we have

$$|X| \le \sum_{i=1}^{m}((q^i+1) + (q^i-1)) < 2\frac{q^{m+1}-1}{q-1} \le q^{n-1} - 1 < \frac{q^n-1}{3},$$

if $n \ge 3$. One can also check by direct computation that $|X| \le (q^n-1)/2$ if $n = 2$. Hence there are at least $(q^n-1)/2$ elements of T that are regular semisimple. Each such s defines an irreducible character χ_s of K of degree $B/|T|$. Moreover, since $|N_{K^*}(T)/T| = 2n$, the K^*-conjugacy class of s intersects T at $2n$ elements. We have therefore produced at least $(q^n - 1)/4n$ irreducible characters χ_s of K, each of degree

$$\chi_s(1) = \frac{B}{|T|} = \frac{\prod_{i=1}^{n}(q^{2i}-1)}{q^n-1} < q^{n^2} = \mathrm{St}(1) \le b(S).$$

First we consider the characters χ_s which split over S. They exist only when $|K/S| = \kappa = 2$. Then the irreducible constituents of their restrictions to S are all distinct, and the sum of squares of the degrees of the irreducible components of each $(\chi_s)|_S$ is $\chi_s(1)^2/\kappa$. On the other hand, among the χ_s which are irreducible over S, at most κ of them can restrict to the same (given) irreducible character of S. Hence, to show that $\varepsilon(S) > 1$, it suffices to verify that

$$\frac{1}{\kappa} \cdot \frac{q^n-1}{4n} \cdot \left(\frac{B}{q^n-1}\right)^2 > \left(\frac{B}{(q-1)^n}\right)^2,$$

equivalently, $(q-1)^{2n} > 4\kappa n(q^n-1)$. The latter inequality holds if $q = 3$ and $n \ge 18$, or if $q = 4$ and $n \ge 4$, or if $q = 5$ and $n \ge 3$, or if $q \ge 7$ and $n \ge 2$. Using [**4**] and [**9**] one can check that $\varepsilon(S) > 1$ when $(n, q) = (2, 3)$, $(2, 4)$, $(2, 5)$, $(3, 3)$, $(3, 4)$.

4) Finally, we consider the cases $S = \mathrm{P}\Omega^{\pm}_{2n}(q)$, where $n \geq 4$ and $q \geq 3$. We set ϵ to 1 or -1 in the split and non-split cases respectively. Then $L = \mathrm{Spin}^{\pm}_{2n}(q)$, and the maximal tori of minimal order in Theorem 4.2 have order at least $(q-1)^n$. Hence

$$b(S) \leq b(L) \leq B/(q-1)^n, \text{ where } B := |L|_{p'} = (q^n - \epsilon) \cdot \prod_{i=1}^{n-1} (q^{2i} - 1).$$

As in 3), we will view S as a normal subgroup of index $\leq \kappa := \gcd(4, q^n - \epsilon)$ of the Lie-type group of adjoint type $H := P(\mathrm{CO}^{\pm}_{2n}(q)^\circ)$. Then any semisimple element in the dual group L has connected centralizer. Now we consider a maximal torus T of order $q^n - \epsilon$ in L, and let Y be the set of elements in T of order dividing $q^k + 1$ or $q^k - 1$ for some k with $1 \leq k \leq n-1$. As in 3) we see that $|X| \leq (q^n - \epsilon)/3$ since $n \geq 4$. Hence there are at least $2(q^n - \epsilon)/3$ elements of T that are regular semisimple. Each such s defines an irreducible character χ_s of H of degree $B/|T|$. Moreover, since $|N_L(T)/T| = 2n$, the L-conjugacy class of s intersects T at $2n$ elements. We have therefore produced at least $(q^n - \epsilon)/3n$ irreducible characters χ_s of H, each of degree

$$\chi_s(1) = B/|T| = \prod_{i=1}^{n-1} (q^{2i} - 1) < q^{n(n-1)} = \mathrm{St}(1) \leq b(S).$$

The restriction $(\chi_s)|_S$ contains an irreducible constituent ρ_s of degree at least $\chi_s(1)/\kappa$. Conversely, each $\rho \in \mathrm{Irr}(S)$ can lie under at most κ distinct irreducible characters of H. Hence, to show that $\varepsilon(S) > 1$, it suffices to verify that

$$\frac{1}{\kappa} \cdot \frac{q^n - \epsilon}{3n} \cdot \left(\frac{B}{\kappa(q^n - \epsilon 1)} \right)^2 > \left(\frac{B}{(q-1)^n} \right)^2,$$

equivalently, $(q-1)^{2n} > 3\kappa^3 n(q^n - \epsilon)$. The latter inequality holds unless $q = 3$ and $n \leq 30$, or if $q = 5$ and $n \leq 6$, or if $q = 7$ and $n = 4$. □

4.3. *Theorem 1.1 for classical groups over $\mathbb{F}_2$*

Now we handle the remaining infinite families of simple classical groups over $\mathbb{F}_2$.

THEOREM 4.8. *If S is any of the following simple classical groups over $\mathbb{F}_2$:*

$$\mathrm{SL}_n(2), \quad \mathrm{Sp}_{2n}(2)', \quad \Omega^{\epsilon}_{2n}(2),$$

then one of the following statements holds:

(i) *there exists $\psi \in \mathrm{Irr}(S)$ with $81/512 \leq \psi(1)/b(S) < 1$; or*

(ii) $\varepsilon(S) > 9/16$.

In particular, $\varepsilon(S) > 1/40$ in either case.

Proof. The "small" groups $\mathrm{SL}_3(2)$ and $\mathrm{Sp}_4(2)' \cong \mathsf{A}_6$ are easily handled using [**4**]. Also set $q = 2$. In what follows, it is convenient to view the remaining groups S as finite Lie-type groups of *adjoint* type $S^* = (\mathcal{G}^*)^{F^*} = \mathrm{PGL}_n(2)$, $\mathrm{SO}_{2n+1}(2)$, $P(\mathrm{CO}^{\epsilon}_{2n}(2)^\circ)$, respectively, which are isomorphic to S as abstract groups. We will prove the theorem for S^*, using semisimple elements in S (which all have connected centralizer in $\mathcal{G}$ since $\mathcal{G}$ is simply connected) to parameterize Lusztig series for $\mathrm{Irr}(S^*)$.

1) First we consider the case $S = \mathrm{SL}_n(2)$ with $n \geq 4$. Any character $\chi \in \mathrm{Irr}(S^*)$ of largest degree $b(S^*) = b(S)$ can be parametrized by $((s), \phi)$, where (s) is the conjugacy class of a semisimple element $s \in S$ and ϕ is a unipotent character of the centralizer $C := C_S(s)$. Such a centralizer is isomorphic to

$$\mathrm{GL}_{k_1}(2^{d_1}) \times \ldots \times \mathrm{GL}_{k_r}(2^{d_r}),$$

where $k_i, d_i \geq 1$, $k_1d_1 \geq k_2d_2 \geq \ldots \geq k_rd_r$, and $\sum_{i=1}^r k_id_i = n$. Moreover, for each d, the number of indices i such that $d_i = d$ is at most the number of conjugacy classes of semisimple elements in $\mathrm{GL}_{kd}(2)$ with centralizer $\cong \mathrm{GL}_k(2^d)$, i.e. the number of monic irreducible polynomials $f(t)$ of degree d over $\mathbb{F}_2$. Since $\chi(1) = (S : C)_{2'} \cdot \psi(1)$ and $\chi(1) = b(S)$, by Corollary 3.3 ψ must be the Steinberg character St_C of C, and so

$$\psi = \psi_1 \otimes \psi_2 \otimes \ldots \otimes \psi_r,$$

where ψ_i is the Steinberg character of $\mathrm{GL}_{k_i}(q^{d_i})$, of degree $q^{d_ik_i(k_i-1)/2}$.

Observe that $s \neq 1$, i.e. χ is not unipotent. Otherwise $b(S) = \mathrm{St}(1) = q^{n(n-1)/2}$. However, the character $\rho \in \mathrm{Irr}(S^*)$ labeled by $((u), \mathrm{St}_{C_S(u)})$, where $u \in S$ is an element of order 3 with centralizer $C_S(u) \cong C_3 \times \mathrm{GL}_{n-2}(2)$, has degree

$$\frac{(q^n-1)(q^{n-1}-1)}{3} \cdot q^{(n-2)(n-3)/2} > q^{n(n-1)/2} = b(S)$$

as $n \geq 4$, a contradiction.

Next we show that $r > 1$. Assume the contrary: $C \cong \mathrm{GL}_k(q^d)$ with $kd = n$ and $d > 1$. Then by Lemma 4.1(ii) we have

$$\chi(1) = q^{dk(k-1)/2} \cdot \frac{(q-1)(q^2-1)\ldots(q^n-1)}{(q^d-1)(q^{2d}-1)\ldots(q^{kd}-1)} < q^{dk(k-1)/2} \cdot \frac{q^{n(n+1)/2-1}}{\frac{9}{16}q^{dk(k+1)/2}} = \frac{8}{9}q^{n(n-1)/2}$$

as $q = 2$. Thus $\chi(1) < \mathrm{St}(1)$, a contradiction.

Thus we must have that $r \geq 2$. Observe that there is a semisimple element $t \in S$ with centralizer

$$C_S(t) \cong \mathrm{GL}_1(q^{k_1d_1+k_2d_2}) \times \mathrm{GL}_{k_3}(q^{d_3}) \times \ldots \times \mathrm{GL}_{k_r}(q^{d_r}).$$

Choose $\psi \in \mathrm{Irr}(S^*)$ to be labeled by $((t), \mathrm{St}_{C_S(t)})$. Then

$$\frac{\psi(1)}{\chi(1)} = \frac{\prod_{i=1}^{k_1}(q^{id_1}-1) \cdot \prod_{i=1}^{k_2}(q^{id_2}-1)}{q^{d_1k_1(k_1-1)/2} \cdot q^{d_2k_2(k_2-1)/2} \cdot (q^{k_1d_1+k_2d_2}-1)}.$$

By Lemma 4.1(ii), $1 > \prod_{i=1}^{k_j}(q^{id_j}-1)/q^{d_jk_j(k_j+1)/2} > 9/32$ for $j = 1, 2$ (in fact we can replace $9/32$ by $9/16$ if $d_j > 1$). Since $(d_1, d_2) \neq (1, 1)$, it follows that $1 > \psi(1)/\chi(1) > 81/512$.

2) Next we consider the case $S = \mathrm{Sp}_{2n}(2)$ with $n \geq 3$. Since $Z(\mathcal{G}^*) = 1$, by Corollary 14.47 and Proposition 14.42 of [**5**], S^* has a unique Gelfand–Graev character Γ, which is the sum of 2^n *regular* irreducible characters $\chi_{(s)}$. Each such $\chi_{(s)}$ has Lusztig label $((s), \mathrm{St}_{C_S(s)})$, where (s) is any semisimple class in S, see e.g. [**10**].

Note that $\Gamma(1) = |S|_{2'} = \prod_{i=1}^n(2^{2i}-1) > 2^{n(n+1)} \cdot (9/16)$, with the latter inequality following from Lemma 4.1(ii). Hence by the Cauchy–Schwarz inequality we have

$$\sum_{(s)} \chi_{(s)}(1)^2 \geq \frac{(\sum_{(s)}\chi_s(1))^2}{2^n} = \frac{(|S|_{2'})^2}{2^n} > \frac{9}{16} \cdot |S| \geq \frac{9}{16} \cdot b(S)^2.$$

In particular, $\varepsilon(S) > 9/16$ if $b(S)$ is not achieved by any regular character $\chi_{(s)}$. So we will assume that $b(S^*) = b(S)$ is achieved by a regular character $\chi = \chi_{(s)}$.

According to [**26**, Lemma 3.6], $C := C_S(s) = D_1 \times \ldots \times D_r$ is a direct product of groups of the form $\mathrm{GL}_k^\epsilon(q^d)$ (where $\epsilon = +1$ for GL and $\epsilon = -1$ for GU) or $\mathrm{Sp}_{2m}(q)$. Note that, since $q = 2$, C contains at most one factor of the latter form, and no factor of the former form with $(d, \epsilon) = (1, 1)$. First suppose that all of the factors D_i are of the second form. It follows that $s = 1$, $\chi_{(1)} = \mathrm{St}$. Choosing $\psi \in \mathrm{Irr}(S^*)$ to be labeled by $((u), \mathrm{St}_{C_S(u)})$, where $u \in S$ is an element of order 3 with centralizer $C_S(u) \cong C_3 \times \mathrm{Sp}_{2n-2}(2)$, we see that

$$b(S) = \chi(1) = 2^{n^2} > \psi(1) = \frac{(2^{2n}-1)}{3} \cdot 2^{(n-1)^2} > b(S)/2$$

as $n \geq 2$.

Next we consider the case where exactly one of the factors D_i is of the form $\mathrm{GL}_k^\epsilon(q^d)$. Then, since $q = 2$ we must actually have $r \leq 2$, $C = \mathrm{GL}_k^\epsilon(q^d) \times \mathrm{Sp}_{2m}(q)$ with $m := n - kd$, and $(d, \epsilon) \neq (1, 1)$. Hence by Lemma 4.1(ii)

$$\chi(1) = q^{dk(k-1)/2+m^2} \cdot \frac{\prod_{j=m+1}^{n}(q^{2j}-1)}{\prod_{j=1}^{k}(q^{jd}-\epsilon^j)} < q^{dk(k-1)/2+m^2} \cdot \frac{q^{n(n+1)-m(m+1)}}{\frac{9}{16}q^{dk(k+1)/2}} < \frac{16}{9}q^{n^2}.$$

It is easy to check that $\chi(1) \neq q^{n^2}$. Choosing $\psi = \mathrm{St}$, we then have $1 > \psi(1)/b(S) > 9/16$.

Lastly, we consider the case where at least two of the factors D_i are of form $\mathrm{GL}_k^\epsilon(q^d)$:

$$C = \mathrm{GL}_{k_1}^{\epsilon_1}(q^{d_1}) \times \mathrm{GL}_{k_2}^{\epsilon_2}(q^{d_2}) \times \ldots \times \mathrm{GL}_{k_{r-1}}^{\epsilon_{r-1}}(q^{d_{r-1}}) \times \mathrm{Sp}_{2m}(q),$$

where $r - 1 \geq 2$ and m can be zero. We will assume that $k_1d_1 \geq k_2d_2 \geq \ldots \geq k_{r-1}d_{r-1} \geq 1$. Observe that there is a semisimple element $t \in S$ with centralizer

$$C_S(t) \cong \mathrm{GU}_1(q^{k_1d_1+k_2d_2}) \times \mathrm{GL}_{k_3}^{\epsilon_3}(q^{d_3}) \times \ldots \times \mathrm{GL}_{k_{r-1}}^{\epsilon_{r-1}}(q^{d_{r-1}}) \times \mathrm{Sp}_{2m}(q).$$

Choose $\psi \in \mathrm{Irr}(S^*)$ to be labeled by $((t), \mathrm{St}_{C_S(t)})$. Then

$$\frac{\psi(1)}{\chi(1)} = \frac{\prod_{i=1}^{k_1}(q^{id_1}-\epsilon_1^i) \cdot \prod_{i=1}^{k_2}(q^{id_2}-\epsilon_2^i)}{q^{d_1k_1(k_1-1)/2} \cdot q^{d_2k_2(k_2-1)/2} \cdot (q^{k_1d_1+k_2d_2}+1)}.$$

By Lemma 4.1(ii), $\prod_{i=1}^{k_j}(q^{id_j}-\epsilon_j^i)/q^{d_jk_j(k_j+1)/2} > 9/16$ for $j = 1, 2$ since $(d_j, \epsilon_j) \neq (1, 1)$. Furthermore, since $k_1d_1 + k_2d_2 \geq 2$ we have $q^{k_1d_1+k_2d_2} + 1 \leq (5/4)q^{k_1d_1+k_2d_2}$. Thus $\psi(1)/\chi(1) > 81/320$, and so we are done if $\psi(1) \neq \chi(1)$. Suppose that $\psi(1) = \chi(1)$. Then $k_1 = k_2 = 1$, $(d_1, \epsilon_1) = (2, 1)$, and $(d_2, \epsilon_2) = (1, -1)$. In this case we can replace t by a semisimple element t' with

$$C_S(t') \cong \mathrm{GL}_1(q^{k_1d_1+k_2d_2}) \times \mathrm{GL}_{k_3}^{\epsilon_3}(q^{d_3}) \times \ldots \times \mathrm{GL}_{k_{r-1}}^{\epsilon_{r-1}}(q^{d_{r-1}}) \times \mathrm{Sp}_{2m}(q)$$

and repeat the above argument.

3) Finally, let us consider the case $S = \Omega_{2n}^\epsilon(q)$ with $n \geq 4$. Arguing as in the Sp-case, we may assume that $b(S^*) = b(S)$ is attained at a regular character $\chi = \chi_{(s)}$. One can show (see also [**26**, Lemma 3.7]) that

$$C := C_S(s) = K_1 \times H_3 \times \ldots \times H_r$$

where each H_i with $3 \leq i \leq r$ is of the form $\mathrm{GL}_k^\beta(q^d)$ with $\beta = \pm 1$. Furthermore, K_1 has a normal subgroup $H_1 \cong \Omega_{2m}^\pm(q)$ (where m can be zero) such that K_1/H_1 is either trivial, or isomorphic to $\mathrm{GU}_2(2)$. In the latter case, the Steinberg character of the finite connected reductive group K_1 has degree equal to $|K_1|_2 = 2^{m(m-1)+1}$. Thus in either case we may replace $C_S(s)$ by

$$H_1 \times H_2 \times H_3 \times \ldots \times H_r$$

(where each H_i is of the form $\mathrm{GL}_k^\beta(q^d)$ with $\beta = \pm 1$ or $\Omega_{2m}^\pm(q)$, and the latter form can occur for at most one factor H_i), and identify the Steinberg character St_C with $\mathrm{St}_{H_1} \otimes \ldots \otimes \mathrm{St}_{H_r}$.

First we consider the case $s = 1$, i.e. $C = H_1 = S$, and $\chi(1) = \mathrm{St}$. Choosing $\psi \in \mathrm{Irr}(S^*)$ to be labeled by $((u), \mathrm{St}_{C_S(u)})$, where $u \in S$ is an element of order 3 with centralizer $C_S(u) \cong C_3 \times \Omega_{2n-2}^{-\epsilon}(2)$, we see that

$$b(S) = \chi(1) = 2^{n(n-1)} > \psi(1) = \frac{(2^n-\epsilon)(2^{n-1}-\epsilon)}{3} \cdot 2^{(n-1)(n-2)} > b(S)/4$$

as $n \geq 4$.

Next we consider the case where exactly one of the factors H_i is of the form $\mathrm{GL}_k^\alpha(q^d)$. Then, since $q = 2$ we must actually have $r \leq 2$, $C = \mathrm{GL}_k^\alpha(q^d) \times \Omega_{2m}^\beta(q)$ with $m := n - kd$, and

$(d,\alpha) \neq (1,1)$. Hence by Lemma 4.1(ii)

$$\begin{aligned}\chi(1) &= q^{dk(k-1)/2+m(m-1)} \cdot \frac{(q^m+\beta)\cdot\prod_{j=m+1}^{n}(q^{2j}-1)}{(q^n+\epsilon)\cdot\prod_{j=1}^{k}(q^{jd}-\alpha^j)} \\ &< q^{dk(k-1)/2+m(m-1)} \cdot \frac{3}{2}\cdot\frac{16}{15}\cdot\frac{q^{m+n(n+1)-m(m+1)}}{\frac{9}{16}q^{n+dk(k+1)/2}} < \tfrac{128}{45}q^{n(n-1)}.\end{aligned}$$

It is easy to check that $\chi(1) \neq q^{n(n-1)}$. Choosing $\psi = \mathrm{St}$ we then have $1 > \psi(1)/b(S) > 45/128$.

Lastly, we consider the case where at least two of the factors H_i are of the form $\mathrm{GL}_k^{\pm}(q^d)$:

$$C = \mathrm{GL}_{k_1}^{\epsilon_1}(q^{d_1}) \times \mathrm{GL}_{k_2}^{\epsilon_2}(q^{d_2}) \times \ldots \times \mathrm{GL}_{k_{r-1}}^{\epsilon_{r-1}}(q^{d_{r-1}}) \times \Omega_{2m}^{\beta}(q),$$

where $r-1 \geq 2$ and m can be zero. We will assume that $k_1d_1 \geq k_2d_2 \geq \ldots \geq k_{r-1}d_{r-1} \geq 1$. Observe that there is a semisimple element $t \in S$ with centralizer

$$C_S(t) \cong \mathrm{GL}_1^{\alpha}(q^{k_1d_1+k_2d_2}) \times \mathrm{GL}_{k_3}^{\epsilon_3}(q^{d_3}) \times \ldots \times \mathrm{GL}_{k_{r-1}}^{\epsilon_{r-1}}(q^{d_{r-1}}) \times \Omega_{2m}^{\beta}(q)$$

for some $\alpha = \pm 1$. Choose $\psi \in \mathrm{Irr}(S^*)$ to be labeled by $((t), \mathrm{St}_{C_S(t)})$. Then

$$\frac{\psi(1)}{\chi(1)} = \frac{\prod_{i=1}^{k_1}(q^{id_1}-\epsilon_1^i)\cdot\prod_{i=1}^{k_2}(q^{id_2}-\epsilon_2^i)}{q^{d_1k_1(k_1-1)/2}\cdot q^{d_2k_2(k_2-1)/2}\cdot(q^{k_1d_1+k_2d_2}-\alpha)}.$$

By Lemma 4.1(ii), $\prod_{i=1}^{k_j}(q^{id_j}-\epsilon_j^i)/q^{d_jk_j(k_j+1)/2} > 9/16$ for $j=1,2$ since $(d_j,\epsilon_j) \neq (1,1)$. Furthermore, since $k_1d_1+k_2d_2 \geq 2$ we have $q^{k_1d_1+k_2d_2} - \alpha \leq (5/4)q^{k_1d_1+k_2d_2}$. Thus $\psi(1)/\chi(1) > 81/320$, and so we are done if $\psi(1) \neq \chi(1)$. Suppose that $\psi(1) = \chi(1)$. Then $k_1 = k_2 = 1$, which forces $\alpha = \epsilon_1\epsilon_2$, $(d_1,\epsilon_1) = (2,1)$, and $(d_2,\epsilon_2) = (1,-1)$. In this last case we must have that $r = 3$,

$$C = \mathrm{GL}_1(4) \times \mathrm{GU}_1(2) \times \Omega_{2n-6}^{-\epsilon}(2),$$

and

$$\chi(1) = \frac{1}{9}\cdot 2^{(n-3)(n-4)}(2^n-\epsilon)(2^{n-3}-\epsilon)(2^{2n-2}-1)(2^{2n-4}-1).$$

In particular, $1 \neq \chi(1)/\mathrm{St}(1) < 4/3$, and so we are done. □

Corollary 4.9. *$\varepsilon(S) > 2/(120,000!)$ for all finite non-abelian simple groups S. In fact, $\varepsilon(S) > 1/209$ for all but a finite number of finite non-abelian simple groups S.*

Proof. The case of alternating groups follows from Corollary 2.2. If S is an exceptional group of Lie type, then $\varepsilon(S) > 1$ by Proposition 4.3. The same is true for most of the simple classical groups, see Theorem 4.7, as well as for the 26 sporadic simple groups. If S is an exception listed in Theorem 4.7, then either $\varepsilon(S) > 1/40$ by Theorem 4.8, or else S belongs to a finite list of exceptions, for all of which we have $|S| < 3^{2000} < (120,000!)/2$, whence $\varepsilon(S) > 1/|S| > 2/(120,000!)$. □

One can certainly improve on the bound $2/(120,000!)$ by a factor accounting for the small complex representations of A_n (of degree $n-1$, $n(n-3)/2$, etc.) for $n \leq 120,000$, but this is still a very minor improvement.

4.4. *Simple groups of Lie type over sufficiently large fields*

We continue to consider simple groups of Lie type S (defined over a finite field $\mathbb{F}_q$) as $L/Z(L)$, where $L = \mathcal{G}^F$, with $\mathcal{G}$ a simple simply connected algebraic group in characteristic p and $F : \mathcal{G} \to \mathcal{G}$ a Steinberg endomorphism. In this subsection, we show that if we fix $\mathrm{rank}(\mathcal{G})$

and let q tend to infinity, then $\varepsilon(S)$ grows as $q^{\mathrm{rank}(\mathcal{G})}$ (up to a constant). (We are grateful to the referee for suggesting us to establish this kind of asymptotic result.) As in Theorem 4.2, among all F-stable maximal tori of $\mathcal{G}$, choose $\mathcal{T}$ such that $T_0 := \mathcal{T}^F$ has smallest order, and let $W(T_0) := N_{\mathcal{G}}(\mathcal{T})^F/\mathcal{T}^F$. Also, let $d := |Z(\mathcal{G})|$.

THEOREM 4.10. *Keep the above notation, and let $0 < C < 1$ be any constant. Then there is a constant $B = B(r, C)$ depending on r and C such that, if $r := \mathrm{rank}(\mathcal{G})$ is fixed and $q > B$, then*

$$q^{\mathrm{rank}(\mathcal{G})} > \varepsilon(S) \geq q^{\mathrm{rank}(\mathcal{G})} \cdot \left(\frac{C}{d} - \frac{1}{|W(T_0)|}\right).$$

Proof. 1) For the upper bound, observe that $b(S) \geq \mathrm{St}(1) = |L|_p$, whence

$$\varepsilon(S) < \frac{|S|}{b(S)^2} \leq \frac{|L|}{(|L|_p)^2} < q^r.$$

It remains to prove the lower bound. By Corollary 4.6, $b(S) = b(L)$ when q is large enough.

2) Let m denote the number of characters $\gamma \in \mathrm{Irr}(S)$ with $\gamma(1) = b(S)$. Since $|Z(L)| \leq |Z(\mathcal{G})| = d$, we have

$$\varepsilon(S) = \frac{|S|}{b(S)^2} - m \geq \frac{|L|}{db(L)^2} - m = \frac{|L|_p \cdot |T_0|^2}{d|L|_{p'}} - m. \tag{4.3}$$

Choosing $B = B(r, C)$ large enough, we will have that $|T_0| > C^{1/2}q^r$ for $q > B(r, C)$, whence

$$\frac{|L|_p \cdot |T_0|^2}{d|L|_{p'}} > \frac{Cq^r}{d}. \tag{4.4}$$

It remains to bound m.

Let $\gamma \in \mathrm{Irr}(S)$ be such that $\gamma(1) = b(S)$; in particular, $\gamma(1)$ is coprime to p. Now we view $S := [H, H]$ for the corresponding finite Lie-type group H of adjoint type: $H = \mathcal{H}^F$ for a simple algebraic group $\mathcal{H}$ of adjoint type and a Steinberg endomorphism $F : \mathcal{H} \to \mathcal{H}$. Consider any irreducible character $\varphi \in \mathrm{Irr}(H)$ that lies above γ. Since $|H/S|$ is coprime to p, $\varphi(1)$ is also coprime to p. Let $(\mathcal{H}^*, F^*)$ be dual to $(\mathcal{H}, F)$ and let $H^* := (\mathcal{H}^*)^{F^*}$. Suppose that, in Lusztig's parametrization of irreducible characters of H, γ corresponds to a semisimple element $s \in H^*$ and a unipotent character ψ of $C_{H^*}(s)$. Since $\mathcal{H}^*$ is simply connected, $C_{\mathcal{H}^*}(s)$ is a connected reductive group.

Assume that $C_{\mathcal{H}^*}(s)$ has *semisimple* rank at least 1 (equivalently, it is not a torus). Since q is large enough, by [**3**, §13.8 and 13.9], $\psi(1)$ is divisible by p if ψ is not the principal character. But $p \nmid \varphi(1)$, hence $\psi(1) = 1$, and $\varphi(1) = (H : C_{H^*}(s))_{p'}$. Now consider the character $\varphi' \in \mathrm{Irr}(H)$ labeled by s and the Steinberg character of $C_{H^*}(s)$, which has degree divisible by q. Also let $\gamma' \in \mathrm{Irr}(S)$ be any character which lies under φ'. Then $\varphi'(1) \geq q\varphi(1)$, and, since r is bounded and q is large enough, $q > |H/S|$. It follows that

$$\gamma'(1) \geq \frac{\varphi'(1)}{|H/S|} > \varphi(1) \geq \gamma(1) = b(S),$$

a contradiction.

Consequently, $C_{\mathcal{H}^*}(s)$ is a (maximal) torus of $\mathcal{H}^*$, s is regular semisimple, $\psi(1) = 1$, and $\varphi(1) = |H|_{p'}/|C_{H^*}(s)|$. Note that the maximal tori of L and of H^* have the same set of orders; in particular, $|C_{H^*}(s)| \geq |T_0|$. Furthermore, $|H| = |L|$. Since $|L|_{p'}/|T_0| = \gamma(1)$ divides $\varphi(1)$, it follows that $\gamma(1) = \varphi(1)$ and $|C_{H^*}(s)| = |T_0|$. Moreover, since $C_{\mathcal{H}^*}(s)$ is connected, we also see that φ is just the semisimple character labeled by the H^*-conjugacy class of s.

3) We have shown that any $\gamma \in \mathrm{Irr}(S)$ of degree $\gamma(1) = b(S)$ extends to a semisimple character φ labeled by the H^*-conjugacy class of a regular semisimple element $s \in H^*$ with $C_{H^*}(s)$ a

maximal torus of smallest order (equal to $|T_0|$). One can check that such a torus is unique up to conjugacy, both for H and for H^*. Abusing the notation, we will denote the torus of H in duality with $C_{H^*}(s)$ by T_0. Then it has the same Weyl group $W(T_0)$, see e.g. [**20**, Prop. 25.3]. Also, since s is regular semisimple with connected centralizer, such a semisimple character φ is just a Deligne–Lusztig character $\pm R_{T_0,\vartheta}$, where $\vartheta \in \mathrm{Irr}(T_0)$ is in general position. The same is true for all the characters in the $W(T_0)$-orbit of ϑ, and this orbit has length $|W(T_0)|$. Certainly, m does not exceed the number of such characters φ, and the latter does not exceed (in fact equals to) the number of regular orbits of $W(T_0)$ on $\mathrm{Irr}(T_0)$. Hence

$$m \leq \frac{|\,\mathrm{Irr}(T_0)|}{|W(T_0)|} = \frac{|T_0|}{|W(T_0)|} \leq \frac{q^r}{|W(T_0)|}.$$

Together with (4.3) and (4.4), this implies the desired lower bound. □

5. *The Largest Degrees of Simple Groups of Lie Type*

Let L be a finite Lie-type group of simply connected type over $\mathbb{F}_q$. When q is large enough in comparison to the rank of L, Theorem 4.2 gives us the precise value of $b(L)$. However, we do not have a formula for $b(L)$ for small values of q. In the extreme case $L = \mathrm{SL}_n(2)$, there does not even seem to exist a decent upper bound on $b(L)$ in the literature, aside from the trivial bound $b(L) < |L|^{1/2}$. On the other hand, as a polynomial of q, the degree of the Steinberg character St is the same as that of the bound in Theorem 4.2. So it is an interesting question to study the asymptotic of the quantity $c(L) := b(L)/\,\mathrm{St}(1)$. In this section we will prove upper and lower bounds for $c(L)$ for finite classical groups.

5.1. *Groups of type A*

THEOREM 5.1. *Let G be any of the following Lie-type groups of type A: $\mathrm{GL}_n(q)$, $\mathrm{PGL}_n(q)$, $\mathrm{SL}_n(q)$, or $\mathrm{PSL}_n(q)$. Then the following inequalities hold:*

$$\max\left\{1, \frac{1}{4}\left(\log_q((n-1)(1-\frac{1}{q})+q^2)\right)^{3/4}\right\} \leq \frac{b(G)}{q^{n(n-1)/2}} < 13(\log_q(n(q-1)+q))^{2.54}.$$

In particular,

$$\frac{1}{4}\left(\log_q \frac{n+7}{2}\right)^{3/4} < \frac{b(G)}{q^{n(n-1)/2}} < 13(1+\log_q(n+1))^{2.54}.$$

Proof. 1) Since the Steinberg character of $\mathrm{GL}_n(q)$ is trivial at $Z(\mathrm{GL}_n(q))$ and stays irreducible as a character of $\mathrm{PSL}_n(q)$, the inequality $b(G) \geq q^{n(n-1)/2}$ is obvious. Next we prove the upper bound

$$c(G) := \frac{b(G)}{q^{n(n-1)/2}} < 13(\log_q(n(q-1)+q))^{2.54}$$

for $G = \mathrm{GL}_n(q)$, which then also implies the same bound for all other groups of type A. It is not hard to see that the arguments in p. 1 of the proof of Theorem 4.8 also carry over to the case of $G = \mathrm{GL}_n(q)$. It follows that $c(G)$ is just the maximum of

$$P := \frac{\prod_{i=1}^{n}(1-q^{-i})}{\prod_{j=1}^{m}\prod_{i=1}^{k_j}(1-q^{-id_j})},$$

where the maximum is taken over all possible $m, k_j, d_j \geq 1$ with $k_1d_1 \geq \ldots \geq k_md_m$ and $\sum_{j=1}^{m} k_jd_j = n$, and for each $d = 1, 2, \ldots$, the number a_d of the values of j such that d_j equals to d does not exceed the number $\mathfrak{n}_d$ of monic irreducible polynomials $f(t)$ of degree d over $\mathbb{F}_q$; in particular, $a_d < q^d/d$.

By Lemma 4.1(ii), the numerator of P is bounded between $9/32$ and 1 for all $q \geq 2$. It remains to bound (the natural logarithm L of) the denominator of P. By Lemma 4.1(i), $\prod_{i=1}^{\infty}(1-q^{-id_j}) > \exp(-\alpha q^{-d_j})$ with $\alpha = 2\ln(32/9)$. Hence,

$$-L/\alpha < \sum_{j=1}^{m} q^{-d_j} = \sum_{d=1}^{n} a_d q^{-d}.$$

The constraints imply $\sum_{d=1}^{n} da_d \leq n$ and $a_d < q^d/d$. Replacing the a_d with real numbers x_d, we want to maximize $\sum_{d=1}^{\infty} x_d q^{-d}$ subject to the constraints

$$\sum_{d=1}^{\infty} dx_d \leq n, \quad 0 \leq x_d \leq q^d/d.$$

Since the function q^{-t}/t is decreasing on $(0,\infty)$, we see that there exists some d_0 (depending on n) such that the sum is optimized when $x_i = q^i/i$ for all $i < d_0$ and $x_i = 0$ for all $i > d_0$. Thus d_0 is the largest integer such that $\sum_{d=1}^{d_0-1}(q^d/d)d = \sum_{d=1}^{d_0-1} q^d = (q^{d_0}-q)/(q-1)$ does not exceed n, whence

$$d_0 \leq \log_q(n(q-1)+q) < 1 + \log_q(n+1).$$

On the other hand,

$$\sum_{d=1}^{d_0} x_d q^{-d} \leq \sum_{d=1}^{d_0} \frac{1}{d} < 1 + \ln(d_0).$$

Thus $L > -\alpha(1+\ln(d_0))$ and so

$$P < e^{-L} < e^{\alpha(1+\ln(d_0))} = e^{\alpha} d_0^{\alpha} < 13(\log_q(n(q-1)+q))^{2.54}$$

by the choice $\alpha = 2\ln(32/9)$.

2) Now we prove the lower bound

$$c(S) := \frac{b(S)}{q^{n(n-1)/2}} > \frac{1}{4}\left(\log_q((n-1)(1-\frac{1}{q})+q^2)\right)^{3/4}$$

for $S = \mathrm{PSL}_n(q)$, which then also implies the same bound for all other groups of type A. As above, let $\mathfrak{n}_d$ be the number of monic irreducible polynomials $f(t)$ over $\mathbb{F}_q$. Arguing as in p. 1) of the proof of Theorem 4.7, we see that the total number of elements of $\mathbb{F}_{q^d}$ which do not belong to any proper subfield of $\mathbb{F}_{q^d}$ is at least $3q^d/4$ when $d \geq 3$ and at most $q^d - 1$. It follows that for $d \geq 3$ we have

$$\frac{3q^d}{4d} \leq \mathfrak{n}_d < \frac{q^d}{d}. \tag{5.1}$$

Since $b(S) \geq \mathrm{St}(1)$, the claim is obvious if $n \leq q^3$. Hence we may assume that $n \geq q^3+1 \geq 3\mathfrak{n}_3+3$. Let $d^* \geq 3$ be the largest integer such that $m := \sum_{d=3}^{d^*} d\mathfrak{n}_d \leq n-3$. In particular,

$$\sum_{d=3}^{d^*+1} q^d > \sum_{d=3}^{d^*+1} d\mathfrak{n}_d \geq n-2,$$

and so

$$d^*+1 \geq \log_q((n-1)(1-1/q)+q^2). \tag{5.2}$$

Observe that $G_1 := \mathrm{GL}_m(q)$ contains a semisimple element s_1 with

$$C_{G_1}(s_1) = \mathrm{GL}_1(q^3)^{\mathfrak{n}_3} \times \mathrm{GL}_1(q^4)^{\mathfrak{n}_4} \times \ldots \times \mathrm{GL}_1(q^{d^*})^{\mathfrak{n}_{d^*}}.$$

(Indeed, each of the $\mathfrak{n}_d$ monic irreducible polynomials of degree d over $\mathbb{F}_q$ gives us an embedding $\mathrm{GL}_1(q^d) \hookrightarrow \mathrm{GL}_d(q)$.) If $\det(s_1) = 1$, then choose $s := \mathrm{diag}(I_{n-m}, s_1)$ so that

$$C_G(s) = \mathrm{GL}_{n-m}(q) \times C_{G_1}(s_1).$$

Otherwise we choose $s := \mathrm{diag}(I_{n-m-1}, \det(s_1)^{-1}, s_1)$ so that

$$C_G(s) = \mathrm{GL}_{n-m-1}(q) \times \mathrm{GL}_1(q) \times C_{G_1}(s_1).$$

In either case, $\det(s) = 1$ and so $s \in [G, G]$ for $G = \mathrm{GL}_n(q)$. Now consider the (regular) irreducible character χ labeled by $((s), \mathrm{St}_{C_G(s)})$. The inclusion $s \in [G, G]$ implies that χ is trivial at $Z(G)$. Also, our choice of s ensures that s has at most two eigenvalues in $\mathbb{F}_q^\times$: the eigenvalue 1 with multiplicity $\geq n - m - 1 \geq 2$, and at most one more eigenvalue with multiplicity 1. Hence, for any $1 \neq t \in \mathbb{F}_q^\times$, s and st are not conjugate in G. To each such t one can associate a linear character $\hat{t} \in \mathrm{Irr}(G)$ in such a way that the multiplication by $\hat{t}$ yields a bijection between the Lusztig series $\mathcal{E}(G, (s))$ and $\mathcal{E}(G, (st))$ labeled by the conjugacy classes of s and st, cf. [**5**, Prop. 13.30]. Since the distinct Lusztig series are disjoint, we conclude that the number of linear characters $\hat{t} \in \mathrm{Irr}(G)$ such that $\chi\hat{t} = \chi$ is exactly one. It then follows by [**14**, Lemma 3.2(i)] that χ is irreducible over $\mathrm{SL}_n(q)$. Thus we can view χ as an irreducible character of $S = \mathrm{PSL}_n(q)$.

Next, in the case $\det(s_1) = 1$ we have

$$\frac{\chi(1)}{q^{n(n-1)/2}} = \frac{\prod_{i=n-m+1}^{n}(1 - q^{-i})}{\prod_{j=3}^{d^*}(1 - q^{-j})^{\mathfrak{n}_j}},$$

whereas in the case $\det(s_1) \neq 1$ we have that

$$\frac{\chi(1)}{q^{n(n-1)/2}} = \frac{\prod_{i=n-m}^{n}(1 - q^{-i})}{(1 - q^{-1})\prod_{j=3}^{d^*}(1 - q^{-j})^{\mathfrak{n}_j}}.$$

Since $n - m \geq 3$, in either case we have

$$\frac{\chi(1)}{q^{n(n-1)/2}} > \frac{\prod_{i=4}^{\infty}(1 - q^{-i})}{\prod_{j=3}^{d^*}(1 - q^{-j})^{\mathfrak{n}_j}}.$$

By Lemma 4.1(ii), the numerator is at least $(9/16) \cdot (4/3) \cdot (8/7) = 6/7$. To estimate the denominator, observe that $1/(1 - x) > e^x$ for $0 < x < 1$. Applying (5.1) we now see that

$$\begin{aligned} \ln\left(\frac{1}{\prod_{j=3}^{d^*}(1-q^{-j})^{\mathfrak{n}_j}}\right) &> \sum_{j=3}^{d^*} q^{-j}\mathfrak{n}_j \geq \sum_{j=3}^{d^*} \frac{3}{4j} \\ &\geq \tfrac{3}{4}(\ln(d^* + 1) - 1 - \tfrac{1}{2}) = \tfrac{3\ln(d^*+1)}{4} - \tfrac{9}{8}. \end{aligned}$$

Together with (5.2) this implies that

$$\frac{\chi(1)}{q^{n(n-1)/2}} > \frac{6}{7e^{9/8}}\left(\log_q((n-1)(1-\frac{1}{q}) + q^2)\right)^{3/4} > \frac{1}{4}\left(\log_q((n-1)(1-\frac{1}{q}) + q^2)\right)^{3/4}.$$

□

5.2. *Other classical groups*

Abusing the notation, by *a group of type C_n over $\mathbb{F}_q$* we mean any of the following groups: $\mathrm{Sp}_{2n}(q)$ (of simply connected type), $\mathrm{PCSp}_{2n}(q)$ (of adjoint type), or $\mathrm{PSp}_{2n}(q)$ (the simple group, except for a few "small" exceptions). Similarly, by *a group of type B_n over $\mathbb{F}_q$* we mean any of the following group: $\mathrm{Spin}_{2n+1}(q)$ (of simply connected type), $\mathrm{SO}_{2n+1}(q)$ (of adjoint type), or $\Omega_{2n+1}(q)$ (the simple group, except for a few "small" exceptions). By *a group of type D_n or ${}^2D_n$ over $\mathbb{F}_q$* we mean any of the following group: $\mathrm{Spin}^\epsilon_{2n}(q)$ (of simply connected type), $\mathrm{PCO}^\epsilon_{2n}(q)^\circ$ (of adjoint type), $\mathrm{P}\Omega^\epsilon_{2n}(q)$ (the simple group, except for a few "small" exceptions),

$\mathrm{SO}^{\epsilon}_{2n}(q)$, as well as the half-spin group $\mathrm{HS}_{2n}(q)$. We refer the reader to [**3**] for the definition of these finite groups of Lie type.

THEOREM 5.2. *Let G be a group of type B_n, C_n, D_n, or ${}^2D_n$ over $\mathbb{F}_q$. If q is odd, then the following inequalities hold:*

$$\max\left\{1, \frac{1}{5}\left(\log_q \frac{4n+25}{3}\right)^{3/8}\right\} \leq \frac{b(G)}{\mathrm{St}(1)} < 38(1+\log_q(2n+1))^{1.27}.$$

If q is even, then the following inequalities hold:

$$\max\left\{1, \frac{1}{5}\left(\log_q(n+17)\right)^{3/8}\right\} \leq \frac{b(G)}{\mathrm{St}(1)} < 8(1+\log_q(2n+1))^{1.27}.$$

Proof. In all cases, the bound $b(G)/\mathrm{St}(1) \geq 1$ is obvious since St is irreducible over any of the listed possibilities for G.

1) We begin by proving the upper bound in the cases where G is of type B_n, respectively of type D_n or ${}^2D_n$, and q is odd and $n \geq 3$. Let $V = \mathbb{F}_q^{2n+1}$, respectively $V = \mathbb{F}_q^{2n}$, be endowed with a non-degenerate quadratic form. Then it is convenient to work with the *special Clifford group* $G := \Gamma^+(V)$ associated to the quadratic space V, see for instance [**27**]. In particular, G maps onto $\mathrm{SO}(V)$ with kernel C_{q-1}, and contains $\mathrm{Spin}(V)$ as a normal subgroup of index $q-1$. Furthermore, the dual group G^* can be identified with the conformal symplectic group $\mathrm{CSp}_{2n}(q)$ in the B-case, and with the group $\mathrm{CO}(V)^\circ$ in the D-case, cf. [**7**, §3]. Observe that the adjoint group $\mathrm{PCO}(V)^\circ$ contains $\mathrm{PSO}(V)$ as a normal subgroup of index 2. Similarly, the half-spin group contains a quotient of $\mathrm{Spin}(V)$ (by a central subgroup of order 2) as a normal subgroup of index 2. Hence, it suffices to prove the indicated upper bound (with constant 19) for this particular G. Similarly, it will suffice to prove the indicated lower bound (with constant 1/5) for the simple group $S = \mathrm{P}\Omega(V)$.

Let $s \in G^*$ be any semisimple element. Consider for instance the B-case and let $\tau(s) \in \mathbb{F}_q^\times$ denote the factor by which the conformal transformation $s \in \mathrm{CSp}_{2n}(q)$ changes the corresponding symplectic form. Also set $H := \mathrm{Sp}_{2n}(q)$ and denote by $\mathbb{F}_q^{\times 2}$ the set of squares in $\mathbb{F}_q^\times$. Then by [**21**, Lemma 2.4] we have that

$$C := C_{G^*}(s) = C_H(s) \cdot C_{q-1},$$

with

$$C_H(s) = \prod_i \mathrm{GL}^{\epsilon_i}_{k_i}(q^{d_i}) \times \left\{ \begin{array}{lll} \mathrm{Sp}_l(q^2), & \tau(s) \notin \mathbb{F}_q^{\times 2}, & (B1) \\ \mathrm{Sp}_{2k}(q) \times \mathrm{Sp}_{2l-2k}(q), & \tau(s) \in \mathbb{F}_q^{\times 2}, & (B2), \end{array} \right.$$

where $\sum_i k_i d_i = n-l$, $\epsilon_i = \pm 1$, and $0 \leq k \leq l \leq n$ (and we use $(B1)$ and $(B2)$ to label the two subcases which can arise). In the D-case, let $\tau(s) \in \mathbb{F}_q^\times$ denote the factor by which the conformal transformation $s \in \mathrm{CO}(V)^\circ$ changes the corresponding quadratic form; also set $H := \mathrm{SO}(V)$. Then by [**21**, Lemma 2.5] we have that

$$C := C_{G^*}(s) = C_H(s) \cdot C_{q-1},$$

with

$$C_{\mathrm{GO}(V)}(s) \cong \prod_i \mathrm{GL}^{\epsilon_i}_{k_i}(q^{d_i}) \times \left\{ \begin{array}{lll} \mathrm{GO}^{\pm}_l(q^2), & \tau(s) \notin \mathbb{F}_q^{\times 2}, & (D1) \\ \mathrm{GO}^{\pm}_{2k}(q) \times \mathrm{GO}^{\pm}_{2l-2k}(q), & \tau(s) \in \mathbb{F}_q^{\times 2}, & (D2), \end{array} \right.$$

where $\sum_i k_i d_i = n-l$, $\epsilon_i = \pm 1$, and $0 \leq k \leq l \leq n$ (and we use $(D1)$ and $(D2)$ to label the two subcases which can arise).

On the set of monic irreducible polynomials of degree d in $\mathbb{F}_q[t]$ (regardless of whether q is odd or not), one can define the involutive map $f \mapsto \check{f}$ such that $x^d f(1/x)$ is a scalar multiple of $f(x)$. One can show that such an f can satisfy the equality $f = \check{f}$ only when $2|d$ and $\alpha^{q^{d/2}+1} = 1$ for every root α of f. Hence, if $\mathfrak{n}_d^*$ denotes the number of monic irreducible polynomials of degree d over $\mathbb{F}_q$ with $f \neq \check{f}$, then

$$\mathfrak{n}_d^* < \frac{q^d}{d}, \text{ and } \mathfrak{n}_d^* \geq \frac{3q^d}{4d} \text{ if } d \geq 3 \text{ and } q \geq 3, \text{ or if } d \geq 5 \text{ and } q = 2. \tag{5.3}$$

The former inequality is obvious. The latter inequality follows from (5.3) when d is odd (as $\mathfrak{n}_d^* = \mathfrak{n}_d$ in this case), and by direct check when $d = 4$ or $(q, d) = (2, 6)$. Assume $d = 2e \geq 6$ and $(q, d) \neq (2, 6)$. Then the number of elements of $\mathbb{F}_{q^d}$ which belong to a proper subfield of $\mathbb{F}_{q^d}$ or to the subgroup $C_{q^{d/2}+1}$ of $\mathbb{F}_{q^d}^\times$ is at most

$$q^e + \sum_{i=1}^{e} q^i < q^e(q+1) < q^d/4,$$

whence $\mathfrak{n}_d^* > 3q^d/4$ as stated.

In either case, decompose the characteristic polynomial of the transformation s into a product of powers of distinct monic irreducible polynomials over $\mathbb{F}_q$. Then the factor $GL_{k_i}^{\epsilon_i}(q^{d_i})$ in C with $\epsilon_i = 1$, respectively with $\epsilon_i = -1$, corresponds to a factor $(f_i\check{f}_i)^{k_i}$ in this decomposition with $\deg(f_i) = d_i$ and $f_i \neq \check{f}_i$, respectively to a factor $f_i^{k_i}$ in this decomposition with $\deg(f_i) = 2d_i$ and $f_i = \check{f}_i$. In particular, if a_d denotes the number of factors $GL_{k_i}(q_i^d)$ in C with $d_i = d$, then

$$a_d \leq \frac{\mathfrak{n}_d^*}{2} < \frac{q^d}{2d}, \quad \sum_d da_d \leq n. \tag{5.4}$$

Certainly, $\mathrm{St}_C(1) = |C|_p$ for the prime p dividing q. But, since the centralizer of s in the corresponding algebraic group is (most of the time) disconnected, we cannot apply Theorem 1.2 directly to say that $b(G)$ is attained by the regular character $\chi = \chi_{(s)}$ labeled by $((s), \mathrm{St}_C)$. Nevertheless, we claim that the degree of any unipotent character ψ of C is at most $2\kappa \cdot \mathrm{St}_C(1)$ and so $b(G) \leq 2\kappa\chi(1)$, where $\kappa = 2$ in the $(D2)$-case and $\kappa = 1$ otherwise. Indeed, by definition the unipotent character ψ of the (usually disconnected) group C is an irreducible constituent of $\mathrm{Ind}_D^C(\varphi)$ for some unipotent character φ of $D := Z(G^*)C_H(s)$. In turn, φ restricts irreducibly to a unipotent character of the (usually disconnected) group $C_H(s)$. It is easy to see that $C_H(s)$ contains a normal subgroup D_1 of index κ, which is a finite connected group, in fact a direct product of subgroups of form $\mathrm{GL}_{k_i}^{\epsilon_i}(q^{d_i})$, $\mathrm{Sp}_l(q^2)$, $\mathrm{Sp}_l(q)$, $\mathrm{SO}_l^\pm(q^2)$, or $\mathrm{SO}_l^\pm(q)$. Again by definition $\varphi|_{C_H(s)}$ is an irreducible constituent of $\mathrm{Ind}_{D_1}^{C_H(s)}(\varphi_1)$ for some unipotent character φ_1 of D_1. Now we can apply Theorem 1.2 to D_1 to see that $\varphi_1(1) \leq \mathrm{St}_{D_1}(1) = |D_1|_p = |C|_p$. Since $|C/D| = 2$ and $|C_H(s)/D_1| = \kappa$, we conclude that $\psi(1) \leq 2\kappa|C|_p$, as stated.

Observe that $(G^* : C)_{p'} = (H : D_1)_{p'}/\kappa$. We have therefore shown that

$$b(G) = \chi(1) \leq 2(H : D_1)_{p'} \cdot |D_1|_p.$$

By Lemma 4.1(i), (iv), $\prod_{i=1}^\infty (1 - q^{-2i}) > 71/81$ since $q \geq 3$, and $\prod_{i=1}^{k_j}(1 - (\epsilon_j q^{-d_j})^i) > 1$ if $\epsilon_j = -1$. Furthermore, $q^l \pm 1 \geq (2/3)q^l$ and $q^n \pm 1 \leq (28/27)q^n$ since $n, q \geq 3$. Using these estimates, we see that

$$c(G) \leq \frac{A}{\prod_{j\,:\,\epsilon_j=1} \prod_{i=1}^{k_j}(1 - q^{-id_j})}. \tag{5.5}$$

Here, $A = 2 \cdot (28/27) \cdot (81/71) \cdot (3/2)^2 = 378/71$ in the $(D2)$-case. Similarly, $A = 2$ in the $(B1)$-case, $A = 162/71$ in the $(B2)$-case, and $A = 28/9$ in the $(D1)$-case. By Lemma 4.1(i), $c(G) \leq A \cdot \exp(\alpha \sum_d a_d q^{-d})$ with $\alpha = 2\ln(32/9)$, and a_d is subject to the constraints (5.4). Now we can argue as in p. 1) of the proof of Theorem 5.1 to bound $\sum_d a_d q^{-d}$ from above. In particular, we get $\sum_d a_d q^{-d} \leq (1+\ln(d_0))/2$, where d_0 is the largest integer such that $\sum_{d=1}^{d_0-1} d(q^d/2d) \leq n$, i.e.

$$d_0 \leq \log_q(2n(q-1)+q) < 1+\log_q(2n+1).$$

Putting everything together, we obtain

$$c(G) \leq Ae^{\alpha/2}d_0^{\alpha/2} < Ae^{1.27}(1+\log_q(2n+1))^{1.27} \tag{5.6}$$

and so we are done, as $Ae^{1.27} < 19$.

2) Next we briefly discuss how one can prove the upper bound in the remaining cases.

2a) Consider the case G is of type C_n over $\mathbb{F}_q$ with q odd. As above, it suffices to prove the upper bound with the constant 19 for $G = \mathrm{Sp}_{2n}(q)$. In this case, $G^* = \mathrm{SO}_{2n+1}(q)$, and if $s \in G^*$ is a semisimple element, then the structure of $C_{\mathrm{GO}_{2n+1}(q)}(s)$ is as described in the $(D2)$-case. Arguing as above, we arrive at (5.5) and (5.6) with $Ae^{1.27} = 2\cdot(81/71)\cdot(3/2)^2\cdot e^{1.27} < 18.3$.

2b) Next suppose that G is of type C_n over $\mathbb{F}_q$ with q even. In this case, $G^* \cong \mathrm{Sp}_{2n}(q)$, and if $s \in G^*$ is a semisimple element, then the structure of $C_{G^*}(s)$ is as described in the $(B2)$-case with $k=0$. Arguing as above, we arrive at (5.5) and (5.6) with $Ae^{1.27} = e^{1.27} < 3.6$.

2c) Finally, let $G = \Omega^\epsilon_{2n}(q)$ with q even and $n \geq 4$; in particular, $G^* \cong G$. If $s \in G^*$ is a semisimple element, then the structure of $C_{\mathrm{GO}^\epsilon_{2n}(q)}(s)$ is as described in the $(D2)$-case with $k=0$. Arguing as above and using the estimates $q^l \pm 1 \geq q^l/2$ and $q^n \pm 1 \leq (17/16)q^n$, we arrive at (5.5) and (5.6) with $Ae^{1.27} = 2\cdot(17/16)\cdot e^{1.27} < 7.6$ for $q \geq 4$. As in p. 3) of the proof of Theorem 4.8, in the case $q=2$ we need some extra care if $C_G(s)$ contains a factor $K_1 := ((C_3 \times C_3) \times \Omega^\pm_{2r}(2))\cdot 2$, where $C_3 \times C_3$ is the (unique) subgroup of index 2 in $\mathrm{GU}_2(2)$. We claim that we still have the bound $\theta(1) \leq |K_1|_2$ for any unipotent character θ of K_1. Indeed, K_1 is a normal subgroup of index 2 of $\tilde{K}_1 := \mathrm{GU}_2(2) \times \mathrm{GO}^\pm_{2r}(2)$. Now θ is an irreducible constituent of some unipotent character $\tilde{\theta} = \lambda \otimes \mu$ of $\tilde{K}_1$, where $\lambda \in \mathrm{Irr}(\mathrm{GU}_2(2))$ and $\mu \in \mathrm{Irr}(\mathrm{GO}^\pm_{2r}(2))$ are unipotent. It follows that the irreducible constituents of $\theta|_{\Omega^\pm_{2r}(2)}$ are unipotent characters of $H_1 := \Omega^\pm_{2r}(2)$ and so have degree at most $\mathrm{St}_{H_1}(1)$ by Theorem 1.2. But $C_3 \times C_3$ is abelian, so $\theta(1) \leq 2\cdot \mathrm{St}_{H_1}(1) = |K_1|_2$. Now we can proceed as in the case $q \geq 4$.

3) Now we proceed to establish the logarithmic lower bound for the simple groups S of type D_n or ${}^2D_n$ over $\mathbb{F}_q$ with q odd and $n \geq 4$. It is convenient to work instead with $G := \mathrm{SO}^\epsilon_{2n}(q)$, since $G^* \cong G$. Since the lower bound is obvious when $n \leq q^3$, we will assume that $n > q^3 > 3\mathfrak{n}^*_3 + 2$. Let $d^* \geq 3$ be the largest integer such that $m := \sum_{d=3}^{d^*} d(\mathfrak{n}^*_d/2) \leq n-2$. In particular,

$$\sum_{d=3}^{d^*+1} \frac{q^d}{2} > \sum_{d=3}^{d^*+1} d(\mathfrak{n}^*_d/2) \geq n-1,$$

and so

$$d^*+1 \geq \log_q((2n-1)(1-1/q)+q^2). \tag{5.7}$$

Observe that $G_1 := \mathrm{SO}^+_{2m}(q)$ contains a semisimple element s_1 with

$$C_{G_1}(s_1) = \mathrm{GL}_1(q^3)^{\mathfrak{n}^*_3/2} \times \mathrm{GL}_1(q^4)^{\mathfrak{n}^*_4/2} \times \ldots \times \mathrm{GL}_1(q^{d^*})^{\mathfrak{n}^*_{d^*}/2}.$$

(Indeed, each of the $\mathfrak{n}^*_d$ monic irreducible polynomials f of degree d over $\mathbb{F}_q$ with $f \neq \check{f}$ gives us an embedding $\mathrm{GL}_1(q^d) \hookrightarrow \mathrm{SO}^+_{2d}(q)$.) If $s_1 \in \Omega^+_{2m}(q)$, then choose $s := \mathrm{diag}(I_{2n-2m}, s_1)$ so that

$$C_G(s) = \mathrm{SO}^\epsilon_{2n-2m}(q) \times C_{G_1}(s_1).$$

Suppose for the moment that $s_1 \notin \Omega^+_{2m}(q)$. Note that there is some $\delta \in \mathbb{F}_{q^2} \setminus (C_{q+1} \cup \mathbb{F}_q)$ such that $h \neq \check{h}$ for the minimal (monic) polynomial $h \in \mathbb{F}_q[t]$ of δ and moreover the $\mathbb{F}_q$-norm of δ is a non-square in $\mathbb{F}_q^\times$. Hence by [**13**, Lemma 2.7.2], under the embedding $\mathrm{GL}_1(q^2) \hookrightarrow \mathrm{GL}_2(q) \hookrightarrow \mathrm{SO}^+_4(q)$, δ gives rise to an element $s_2 \in \mathrm{SO}^+_4(q) \setminus \Omega^+_4(q)$. Now we choose

$s := \mathrm{diag}(I_{2n-2m-4}, s_2, s_1)$ so that

$$C_G(s) = \mathrm{SO}^{\epsilon}_{2n-2m-4}(q) \times \mathrm{GL}_1(q^2) \times C_{G_1}(s_1).$$

Our construction ensures that $s \in [G,G] = \Omega^{\epsilon}_{2n}(q)$.

Next we consider the (regular) irreducible character ρ labeled by $((s), \mathrm{St}_{C_G(s)})$. The inclusion $s \in [G,G]$ implies that ρ is trivial at $Z(G)$. Since $S = \mathrm{P}\Omega^{\epsilon}_{2n}(q)$ is a normal subgroup of index 2 in $G/Z(G)$, we see that S has an irreducible character χ of degree at least $\rho(1)/2$. Hence in the case $s_1 \in \Omega^{+}_{2m}(q)$ we have

$$\frac{\chi(1)}{\mathrm{St}(1)} \geq \frac{1}{2} \cdot \frac{\prod_{i=n-m}^{n-1}(1-q^{-2i}) \cdot (1-\epsilon q^{-n})}{\prod_{j=3}^{d^*}(1-q^{-j})^{\mathfrak{n}_j^*/2} \cdot (1-\epsilon q^{m-n})},$$

whereas in the case $s_1 \notin \Omega^{+}_{2m}(q)$ we have that

$$\frac{\chi(1)}{\mathrm{St}(1)} \geq \frac{1}{2} \cdot \frac{\prod_{i=n-m-2}^{n-1}(1-q^{-2i}) \cdot (1-\epsilon q^{-n})}{\prod_{j=3}^{d^*}(1-q^{-j})^{\mathfrak{n}_j^*/2} \cdot (1-\epsilon q^{m-n+2}) \cdot (1-q^{-2})}$$

(with the convention that $1-\epsilon q^{m-n+2} = 1$ when $m = n-2$). Observe that $(1-\epsilon q^{-n})/(1-\epsilon q^{-k}) > q/(q+1) \geq 3/4$ for $0 \leq k \leq n$. Furthermore, since $n \geq m+2$ we have

$$\prod_{i=n-m}^{n-1}(1-q^{-2i}) > \frac{\prod_{i=n-m-2}^{n-1}(1-q^{-2i})}{1-q^{-2}} > \prod_{i=2}^{\infty}(1-q^{-2i}) > \frac{71}{72}$$

by Lemma 4.1(i). Thus

$$c(S) \geq \frac{\chi(1)}{\mathrm{St}(1)} > \frac{B}{\prod_{j=3}^{d^*}(1-q^{-j})^{\mathfrak{n}_j^*/2}}, \tag{5.8}$$

with $B = (71/72) \cdot (3/4) \cdot (1/2) = 71/192$. Applying (5.3) we now see that

$$\ln\left(\frac{1}{\prod_{j=3}^{d^*}(1-q^{-j})^{\mathfrak{n}_j^*/2}}\right) > \sum_{j=3}^{d^*} q^{-j}\frac{\mathfrak{n}_j^*}{2} \geq \sum_{j=3}^{d^*}\frac{3}{8j} \geq \frac{3\ln(d^*+1)}{8} - \frac{9}{16}.$$

Together with (5.7) this implies that

$$c(S) > \frac{B}{e^{9/16}}\left(\log_q((2n-1)(1-\frac{1}{q})+q^2)\right)^{3/8} > \frac{B}{e^{9/16}}\left(\log_q\frac{4n+25}{3}\right)^{3/8}, \tag{5.9}$$

and so we are done, since $Be^{-9/16} > 1/5$.

4) We will now briefly discuss how one can prove the lower bound in the remaining cases. Note that the lower bound is obvious when $n \leq q^6$, so we will assume $n > q^6$.

4a) Consider the case G is of type C_n over $\mathbb{F}_q$ with q odd; in particular, $G^* = \mathrm{SO}_{2n+1}(q)$. Choose $d^* \geq 3$ largest possible such that $m := \sum_{d=3}^{d^*} d\mathfrak{n}_d^*/2 \leq n-2$, and so (5.7) holds. Also choose $s_1 \in G_1 := \mathrm{SO}_{2m+1}(q)$ a semisimple element with

$$C_{G_1}(s_1) = \prod_{d=3}^{d^*}\mathrm{GL}_1(q^d)^{\mathfrak{n}_d^*/2}.$$

If $s_1 \in \Omega_{2m+1}(q)$ then we choose $s := \mathrm{diag}(I_{2n-2m+1}, s_1)$, and if $s_1 \notin \Omega_{2m+1}(q)$ then we choose $s := \mathrm{diag}(I_{2n-2m-3}, s_2, s_1)$ where s_2 is defined as in 3). As above, s gives rise to a regular character ρ of G which is trivial at $Z(G)$, so ρ can be viewed as an irreducible character of $S := \mathrm{PSp}_{2n}(q)$. The same arguments as in 3) now show that (5.8) holds with $B = 71/72$, and (5.9) holds with $Be^{-9/16} > 1/2$.

4b) Assume now that $G = \mathrm{SO}_{2n+1}(q)$ with q odd. Then we choose d^* and m as in 4a), and choose $s \in G^* = \mathrm{Sp}_{2n}(q)$ a semisimple element with

$$C_G(s) = \mathrm{Sp}_{2n-2m}(q) \times \prod_{d=3}^{d^*} \mathrm{GL}_1(q^d)^{\mathfrak{n}_d^*/2}.$$

Let χ be an irreducible constituent over $S := \Omega_{2n+1}(q)$ of the regular character labeled by (s). The same arguments as in 3) now show that (5.8) holds with $B = (71/72) \cdot (1/2)$, and (5.9) holds with $Be^{-9/16} > 1/4$.

4c) Next suppose that G is of type C_n over $\mathbb{F}_q$ with q even; in particular, $G^* \cong \mathrm{Sp}_{2n}(q)$. Choose $d^* \geq 5$ largest possible such that $m := \sum_{d=5}^{d^*} d\mathfrak{n}_d^*/2 \leq n$, and so instead of (5.7) we now have

$$d^* + 1 > \log_q((2n+3)(1-1/q) + q^4). \tag{5.10}$$

We can find a semisimple element $s \in G^*$ such that

$$C_{G^*}(s) = \mathrm{Sp}_{2n-2m}(q) \times \prod_{d=3}^{d^*} \mathrm{GL}_1(q^d)^{\mathfrak{n}_d^*/2}.$$

By Lemma 4.1(i), $\prod_{i=1}^{\infty}(1-q^{-2i}) > 11/16$. Considering the regular character ρ labeled by (s), we now obtain

$$c(S) \geq \frac{\rho(1)}{\mathrm{St}(1)} > \frac{B}{\prod_{j=5}^{d^*}(1-q^{-j})^{\mathfrak{n}_j^*/2}}, \tag{5.11}$$

with $B = (11/16)$. Applying (5.3) and arguing as in 3), we arrive at

$$c(S) > \frac{B}{e^{25/32}} \left(\log_q((2n+3)(1-\frac{1}{q}) + q^4)\right)^{3/8} > \frac{B}{e^{25/32}} \left(\log_q(n+17)\right)^{3/8}, \tag{5.12}$$

and so we are done, since $Be^{-25/32} > 0.3$.

4d) Finally, let $G = \Omega_{2n}^{\epsilon}(q)$ with q even and $n \geq 4$; in particular, $G^* \cong G$. Now we choose d^* and m as in 4c), and fix a semisimple element $s \in G^*$ with

$$C_{G^*}(s) = \Omega_{2n-2m}^{\epsilon}(q) \times \prod_{d=3}^{d^*} \mathrm{GL}_1(q^d)^{\mathfrak{n}_d^*/2}.$$

Using the estimate $(1-\epsilon q^{-n})/(1-\epsilon q^{m-n}) > 2/3$ and arguing as in 4c), we see that (5.11) holds with $B = (11/16) \cdot (2/3) = 11/24$. Consequently, (5.12) holds with $Be^{-25/32} > 1/5$. $\square$

Following the same approach, A. Schaeffer has proved:

THEOREM 5.3. [**23**] *Let G be any of the following twisted Lie-type groups of type A: $\mathrm{GU}_n(q)$, $\mathrm{PGU}_n(q)$, $\mathrm{SU}_n(q)$, or $\mathrm{PSU}_n(q)$. Then the following inequalities hold:*

$$\max\left\{1, \frac{1}{4}\left(\log_q((n-1)(1-\frac{1}{q^2}) + q^4)\right)^{2/5}\right\} \leq \frac{b(G)}{q^{n(n-1)/2}} < 2(\log_q(n(q^2-1)+q^2))^{1.27}.$$

5.3. *Proof of Theorem 1.3*

The cases where G is an exceptional group of Lie type follow from the proof of Proposition 4.3. Consider the case G is classical. Then the upper bound follows from Theorems 5.1, 5.2, and 5.3. We need only to add some explanation for the groups of type A, twisted or untwisted. For instance, let G be a group of Lie type in the same isogeny class with $L := \mathrm{SL}_n(q)$. Then

$G \cong (L/Z) \cdot C_d$, where Z is a central subgroup of order d of L, and furthermore the subgroup of all automorphisms of L/Z induced by conjugations by elements in G is contained in $\mathrm{PGL}_n(q)$. Now consider any $\chi \in \mathrm{Irr}(G)$. Let χ_1 be an irreducible constituent of $\chi|_{L/Z}$ viewed as a character of L and let χ_2 be an irreducible constituent of $\mathrm{Ind}_L^H(\chi_1)$, where $H := \mathrm{GL}_n(q)$. Since the quotients $G/(L/Z)$ and H/L are cyclic, we see that $\chi(1)/\chi_1(1)$ is the index (in G) of the inertia group of χ_1 in G, which is at most the index (in H) of the inertia group of χ_1 in H, and the latter index is just $\chi_2(1)/\chi_1(1)$. It follows that $\chi(1) \le \chi_2(1) \le b(H)$. The same argument applies to the twisted case of type A.

For the lower bound, observe that there is some $d_\varepsilon \ge 5$ depending on ε such that

$$\mathfrak{n}_d \ge \mathfrak{n}_d^* > (1-\varepsilon)\frac{q^d}{d}.$$

Choosing $A \le (d_\varepsilon)^{(\varepsilon-1)/\gamma}$ we can guarantee that the lower bound holds for $n \le q^{d_\varepsilon}$. Hence we may assume that $n \ge q^{d_\varepsilon} + 1 \ge d_\varepsilon \mathfrak{n}_{d_\varepsilon} + 3$. Now we can repeat the proofs of Theorems 5.1 and 5.2, replacing the products $\prod_{d=3}^{d^*}$, respectively $\prod_{d=5}^{d^*}$, by $\prod_{d=d_\varepsilon}^{d^*}$. □

5.4. *Proof of Theorem 1.4*

To guarantee the lower bound in the case $\ell = p$ we can take $C \le 1$, since the Steinberg character, being of p-defect 0, is irreducible modulo p. Assume that $\ell \ne p$. As usual, by choosing C small enough we can ignore any finite number of simple groups; also, it suffices to prove the lower bound for the unique non-abelian composition factor S of G. So we will work with $S = G/Z(G)$, where $\mathcal{G}$ is a simple simply connected algebraic group and $G = \mathcal{G}^F$ is the corresponding finite group over $\mathbb{F}_q$. Consider the pair $(\mathcal{G}^*, F^*)$ dual to $(\mathcal{G}, F)$ and the dual group $G^* := (\mathcal{G}^*)^{F^*}$. It is well known that, for $q \ge 5$, $\mathrm{IBr}_\ell(\mathrm{PSL}_2(q))$ contains a character of degree $\ge q-1$, so we may assume that $r := \mathrm{rank}(\mathcal{G}) > 1$. We will show that, with a finite number of exceptions, $[G^*, G^*]$ contains a regular semisimple ℓ'-element s with connected centralizer and such that $C_{G^*}(s)$ is a torus of order at most $2q^r$. For such an s, the corresponding semisimple character $\chi = \chi_s$ can be viewed as an irreducible character of S of degree $|G|_{p'}/|C_{G^*}(s)| > C \cdot |G|_p$ (with $C > 0$ suitably chosen). Moreover, any Brauer character in the ℓ-block of G containing χ has degree divisible by $\chi(1)$ as a consequence of a result of Broué–Michel, see [**11**, Prop. 1]. Hence the reduction modulo ℓ of χ is irreducible and so $b_\ell(S) \ge \chi(1)$.

To find such an s, we will work with two specific tori T_1 and T_2 of G^*. For $G = {}^3D_4(q)$ we can choose $|T_1| = q^4 - q^2 + 1$ and $|T_2| = (q^2 - q + 1)^2$. For $G = \mathrm{SU}_n(q)$, we choose

$$(|T_1|, |T_2|) = \begin{cases} \left(\frac{(q^{n/2}+1)^2}{q+1}, q^{n-1}+1\right) & \text{if } n \equiv 2 \pmod 4, \\ \left(\frac{q^n+1}{q+1}, (q^{(n-1)/2}+1)^2\right) & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

If G is of type B_n or C_n with $2|n$, we can choose

$$(|T_1|, |T_2|) = (q^n + 1, (q^{n/2}+1)^2).$$

For all other G, T_1 and T_2 can be chosen of order indicated in [**19**, Tables 3.5 and 4.2]. We may assume that either q or the rank of G is sufficiently large, so in particular Zsigmondy primes r_i [**29**] exist for the cyclotomic polynomials Φ_{m_i} (in q) of largest possible m_i dividing the orders $|T_i|$. Here $i = 1, 2$, and, furthermore, for $i = 2$ we need to assume that G is not $\mathrm{SL}_3(q)$, $\mathrm{SU}_3(q)$, or $\mathrm{Sp}_4(q) \cong \mathrm{Spin}_5(q)$. According to [**6**], either $r_2 > m_2 + 1$ or r_2^2 divides Φ_{m_2}, again with finitely many exceptions.

Now if $r_1 \ne \ell$, respectively if $\ell = r_1 \ne r_2$ and r_2 is larger than all torsion primes of $\mathcal{G}$ (see e.g. [**20**, Table 2.3] for the list of them), we can choose $s \in T_i$ of prime order r_i, with $i = 1$, respectively $i = 2$, and observe that r_i is coprime to all torsion primes of $\mathcal{G}$ as well as to $|G^*/[G^*, G^*]|$. It follows that $C_{\mathcal{G}^*}(s)$ is connected (cf. [**20**, Prop. 14.20] for instance),

$s \in [G^*, G^*]$, and moreover s can be chosen so that $|C_{G^*}(s)| = |T_i|$. Thus s has the desired properties, and so we are done.

We observe that r_2 can be a torsion prime for $\mathcal{G}$ only when $r_2 = m_2 + 1$ and (G, r_2) is $(\mathrm{SL}_n(q), n)$, or $(\mathrm{SU}_n(q), n)$ with $n \equiv 3 \pmod 4$. In either case we can choose $s \in T_2 \cap [G^*, G^*]$ of order r_2^2. Furthermore, if $G = \mathrm{SL}_3^\epsilon(q)$ with $q \geq 5$, we fix $\alpha \in \mathbb{F}_{q^2}^\times$ of order $q + \epsilon$ and choose $s \in T_2$ with an inverse image $\mathrm{diag}(\alpha, \alpha^{-1}, 1)$ in $\mathcal{G}$. If $G = \mathrm{Sp}_4(q)$ with $q \geq 8$, we fix $\beta \in \mathbb{F}_{q^2}^\times$ of order $q+1$ and choose $s \in T_2$ with an inverse image $\mathrm{diag}(\beta, \beta^{-1}, \beta^2, \beta^{-2})$ in $\mathrm{Sp}_4(\overline{\mathbb{F}}_q) \cong \mathrm{Spin}_5(\overline{\mathbb{F}}_q)$. It remains to show that in these cases the element s has the desired properties. In fact, it suffices to show that $C_{\mathcal{G}^*}(s)$ is a torus. Consider, for instance, the case $G = \mathrm{SL}_n(q)$ (so $r_2 = n$). Then s can be chosen to have an inverse image $\mathrm{diag}(\gamma, \gamma^q, \ldots, \gamma^{q^{n-2}}, 1)$ in the simply connected group $\hat{\mathcal{G}}^*$, where $|\gamma| = n^2$ and $\mathcal{G}^* = \hat{\mathcal{G}}^*/Z(\hat{\mathcal{G}}^*)$. Suppose $x \in \hat{\mathcal{G}}^*$ centralizes g modulo $Z(\hat{\mathcal{G}}^*)$. Then $xgx^{-1} = \delta g$ for some $\delta \in \overline{\mathbb{F}}_q^\times$ with $\delta^n = 1$. It follows that δ is an eigenvalue of g of order dividing n, and so $\delta = 1$. Thus $C_{\mathcal{G}^*}(s)$ equals $C_{\hat{\mathcal{G}}^*}(s)/Z(\hat{\mathcal{G}}^*)$ and so it is a torus. Similar arguments apply to all the remaining cases. □

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