MODULAR IRREDUCIBILITY OF CUSPIDAL UNIPOTENT CHARACTERS

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Abstract. We prove a long-standing conjecture of Geck which predicts that cuspidal unipotent characters remain irreducible after ℓ-reduction. To this end, we construct a progenerator for the category of representations of a finite reductive group coming from generalised Gelfand–Graev representations. This is achieved by showing that cuspidal representations appear in the head of generalised Gelfand–Graev representations attached to cuspidal unipotent classes, as defined and studied in [13].

1. Introduction

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over a finite field of characteristic $p > 0$ with corresponding Frobenius endomorphism F. This paper is devoted to the proof of a long-standing conjecture of Geck regarding cuspidal unipotent characters of the finite reductive group $\mathbf{G}^F$ (see [7, (6.6)]):

Theorem 1. *Assume that p and ℓ are good for $\mathbf{G}$ and that $\ell \nmid p|Z(\mathbf{G})^F/Z^\circ(\mathbf{G})^F|$. Then any cuspidal unipotent character of $\mathbf{G}^F$ remains irreducible under reduction modulo ℓ.*

This property was shown to be of the utmost importance in order to relate Harish-Chandra series in characteristic zero and ℓ, see [7, Prop. 6.5], and for the study of supercuspidal representations, see Hiss [15]. The conjecture had previously only been shown in special cases: for $\mathrm{GU}_n(q)$ by Geck, Hiss and the second author [12], for classical groups at linear primes ℓ by Gruber–Hiss [14] and at the prime $\ell = 2$ by Geck and the second author [13], as well as for some exceptional groups for which the ℓ-modular decomposition matrix is known.

Our strategy for the proof of this conjecture is the construction of a progenerator of the category of representations of $\mathbf{G}^F$ over a field of positive characteristic $\ell \neq p$ (non-defining characteristic). We produce such a progenerator P using generalised Gelfand–Graev representations associated to suitably chosen unipotent classes. We then show that cuspidal unipotent characters occur with multiplicity one in P, which is enough to deduce the truth of Geck's conjecture.

Generalised Gelfand–Graev representations (GGGRs) are a family of projective representations $\{\Gamma_C^G\}$ labelled by unipotent classes C of $G = \mathbf{G}^F$. They are defined whenever the characteristic p is good for $\mathbf{G}$. The sum of all the GGGRs is a progenerator for the representations of G, since the GGGR corresponding to the trivial class is the regular

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representation, hence a progenerator itself. Our construction relies on the work of Geck and the second author [13] who showed that any representation Γ_C^G can be replaced by the Harish-Chandra induction of a GGGR from a suitable Levi subgroup L of G, up to adding and removing GGGRs corresponding to unipotent classes larger than C for the closure ordering. This led them to the notion of *cuspidal classes* for which no such proper Levi subgroup exists. Following their observation, we consider the projective module

$$P = \bigoplus_{(L,C)} R_L^G(\Gamma_C^L)$$

where L runs over the set of 1-split Levi subgroups of G (*i.e.*, Levi complements of rational parabolic subgroups) and C over the set of cuspidal unipotent classes of L.

We show that P is a progenerator (see Corollary 2.3). To this end we adapt the work of Geck–Hézard [11] on a conjecture of Kawanaka to prove that the family of characters of Harish-Chandra induced GGGRs forms a basis of the space of unipotently supported class functions. As a consequence, we obtain that cuspidal modules must appear in the head of GGGRs associated to cuspidal classes. We believe that this should be of considerable interest for studying projective covers of cuspidal modules, as there exist very few cuspidal classes in general. For example, when G is a group of type A, cuspidal classes are regular classes and only usual Gelfand–Graev representations are needed to define P. In this specific case, such a progenerator already appears for example in work of Bonnafé and Rouquier [2, 1].

This paper is organised as follows. Section 2 is devoted to the construction of the progenerator. In Corollary 2.3 we show how to construct it from induced generalised Gelfand–Graev representations. Section 3 contains our main application on the ℓ-reduction of cuspidal unipotent characters, with the proof of Theorem 1.

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2. A progenerator

Let $\mathbf{G}$ be a connected reductive linear algebraic group over $\overline{\mathbb{F}_p}$ and F be a Frobenius endomorphism endowing $\mathbf{G}$ with an $\mathbb{F}_q$-structure. If $\mathbf{H}$ is any F-stable closed subgroup of $\mathbf{G}$, we denote by $H := \mathbf{H}^F$ the finite group of $\mathbb{F}_q$-points in $\mathbf{H}$.

We let $\ell \neq p$ be a prime and let $(K, \mathcal{O}, k)$ denote a splitting ℓ-modular system for G. We will consider representations of G over one of the rings K, $\mathcal{O}$, or k.

Throughout this section, we will always assume that p is good for $\mathbf{G}$. The results on generalised Gelfand–Graev representations that we shall need were originally proved by Kawanaka [16] and Lusztig [20], under some restriction on p and q. This restriction was recently removed by Taylor [21], so that we can work under the assumption that p is good for $\mathbf{G}$. Note that this is already required for the classification of unipotent classes to be independent from p.

2.1. **Unipotent support of unipotent characters.** Given $\rho \in \mathrm{Irr}(G)$ and C an F-stable unipotent class of $\mathbf{G}$, we denote by $\mathrm{AV}(C, \rho) = |C^F|^{-1} \sum_{g \in C^F} \rho(g)$ the average value of ρ on C^F. We say that C is a *unipotent support* of ρ if C has maximal dimension for the property that $\mathrm{AV}(C, \rho) \neq 0$. Geck [9, Thm. 1.4] has shown that whenever p is good for $\mathbf{G}$, any irreducible character ρ of G has a unique unipotent support, which we will denote by C_ρ.

If $\mathrm{AV}(C,\rho) \neq 0$ then there exists $u \in C^F$ such that $\rho(u) \neq 0$. Conversely, if $\rho(u) \neq 0$ then $\mathrm{AV}(C,\rho)$ might be zero. However, when p and q are large, it follows from [20, Thm. 11.2(iv)] that in that case either $C = C_\rho$ or $\dim C < \dim C_\rho$. By [21, Cor. 13.6], this property also holds whenever p is good and $Z(\mathbf{G})$ is connected.

By [20, §11] (see [21, §14] for the extension to any good characteristic), unipotent supports of unipotent characters are special classes. They can be computed as follows: any family $\mathcal{F}$ of the Weyl group of $\mathbf{G}$ contains a unique special representation, which is the image under the Springer correspondence of the trivial local system on a special unipotent class $C_{\mathcal{F}}$. Then this class is the common unipotent support of all the unipotent characters in $\mathcal{F}$.

2.2. **Generalised Gelfand–Graev representations.** Given an F-stable unipotent element $u \in G$, we denote by Γ_u^G, or simply Γ_u, the *generalised Gelfand–Graev representation* associated with u. It is an $\mathcal{O}G$-lattice. The construction is given for example in [16, §3.1.2] (with some extra assumption on p) or in [21, §5]. The first elementary properties that can be deduced are

- if $\ell \neq p$, then Γ_u is a projective $\mathcal{O}G$-module;
- if u and u' are conjugate under G then $\Gamma_u \cong \Gamma_{u'}$.

The character of $K\Gamma_u$ is the *generalised Gelfand–Graev character* associated with u. We denote it by γ_u^G, or simply γ_u. It depends only on the G-conjugacy class of u. When u is a regular unipotent element then γ_u is a usual Gelfand–Graev character as in [6, §14].

Lusztig [20, Thm. 11.2] (see Taylor [21] for the extension to good characteristic) gave a condition on the unipotent support of a character to occur in a generalised Gelfand-Graev character. Namely, given $\rho \in \mathrm{Irr}(G)$ and $\rho^* \in \mathrm{Irr}(G)$ its Alvis-Curtis dual (see [6, §8])

- there exists $u \in C_{\rho^*}^F$ such that $\langle \gamma_u ; \rho \rangle \neq 0$;
- if C is an F-stable unipotent conjugacy class of $\mathbf{G}$ such that $\dim C > \dim C_{\rho^*}$ then $\langle \gamma_u ; \rho \rangle = 0$ for all $u \in C^F$.

2.3. **Cuspidal unipotent classes.** Following Geck and Malle [13] we say that an F-stable unipotent class C of $\mathbf{G}$ is *non-cuspidal* if there exists a 1-split *proper* Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ such that

- $C \cap L \neq \emptyset$;
- for all $u \in C \cap L$, the natural map $C_{\mathbf{L}}(u)/C_{\mathbf{L}}^\circ(u) \longrightarrow C_{\mathbf{G}}(u)/C_{\mathbf{G}}^\circ(u)$ is an isomorphism.

Here recall that a *1-split Levi subgroup* of $(\mathbf{G}, F)$ is by definition an F-stable Levi complement of an F-stable parabolic subgroup of $\mathbf{G}$. If no such proper Levi subgroup exists, we say that the class C is *cuspidal*. Note that cuspidality is preserved under the quotient map $\mathbf{G} \to \mathbf{G}/Z^\circ(\mathbf{G})$. In particular, when $\mathbf{G}$ has connected centre, a unipotent class C is cuspidal if and only if its image in the adjoint quotient $\mathbf{G}_{\mathrm{ad}}$ (with same root system as $\mathbf{G}$) is cuspidal.

2.4. **A progenerator.** Recall that $\mathbf{G}$ is connected reductive in characteristic p and that $\ell \neq p$. In particular every generalised Gelfand–Graev representation of $\mathcal{O}G$ is projective.

When $\mathbf{G} = \mathrm{GL}_n$, it is known from [12, Thm. 7.8] that any cuspidal kG-module N lifts to characteristic zero in a (necessarily) cuspidal KG-module. The latter is a constituent of some Gelfand–Graev representation Γ of G so that N is in the head of $k\Gamma$. We prove an analogue of this result for $\mathbf{G}$ of arbitrary type.

Theorem 2.1. *Assume that p is good for $\mathbf{G}$ and that $\ell \neq p$. Let N be a cuspidal kG-module. Then there exists an F-stable unipotent class C of $\mathbf{G}$ which is cuspidal for $\mathbf{G}_{\mathrm{ad}}$ and $u \in C^F$ such that $\mathrm{Hom}_{kG}(k\Gamma_u, N) \neq 0$.*

This results from the following version of a conjecture by Kawanaka that was proved by Geck and Hézard [11, Thm. 4.5].

Proposition 2.2. *Assume that the centre of $\mathbf{G}$ is connected. Then the $\mathbb{Z}$-module of unipotently supported virtual characters of G is generated by $\{R_L^G(\gamma_u^L)\}$ where $\mathbf{L}$ runs over 1-split Levi subgroups of $\mathbf{G}$, and u over F-stable unipotent elements of cuspidal unipotent classes of $\mathbf{L}$.*

Proof. Let $C_1, \ldots, C_N$ be the F-stable unipotent classes of $\mathbf{G}$, ordered by increasing dimension. For each C_i, we choose a system of representatives $u_{i,1}, u_{i,2}, \ldots$ of the G-orbits in C_i^F. In [11, §4], it is shown, under the assumption that p is large, that to each $u_{i,r}$ one can associate an irreducible character $\rho_{i,r}$ of G such that

(1) $\rho_{i,r}$ has unipotent support C_i;
(2) $\langle D_G(\rho_{i,r}); \gamma_{u_{i,s}} \rangle = \pm\delta_{r,s}$ for all s.

Thus the $\{D_G(\rho_{i,r})\}$ span the $\mathbb{Z}$-module of unipotently supported virtual characters of G. Here D_G denotes the Alvis–Curtis duality on characters (so $D_G(\rho) = \pm\rho^*$ in our earlier notation). Let us explain why the arguments in [11, §4] can be generalised to the case of good characteristic. For a given i, the irreducible characters $\{\rho_{i,1}, \rho_{i,2}, \ldots\}$ are obtained as characters lying in a family of a Lusztig series belonging to an isolated element, and whose associated unipotent support is C_i. When the finite group attached by Lusztig to the family is abelian (resp. isomorphic to $\mathfrak{S}_3$), these characters are constructed using [10, Prop. 6.6] (resp. [10, Prop. 6.7]). As mentioned in [10, §2.4], this requires a generalisation of some of Lusztig's results on generalised Gelfand–Graev representations [20] to the case of good characteristic. This was recently achieved by Taylor [21]. Finally, when the finite group attached to the family is isomorphic to $\mathfrak{S}_4$ or $\mathfrak{S}_5$, then C is a specific special unipotent class of F_4 or E_8. In that case one can use the results in [5, §4] which hold whenever p is good for $\mathbf{G}$, again thanks to [21].

Now let us choose the system of representatives $u_{i,s} \in C_i$ in a minimal Levi subgroup L_i given by [13, Thm. 3.2]. Then property (2) is still satisfied if we replace $\gamma_{u_{i,s}}^G$ by $R_{L_i}^G(\gamma_{u_{i,s}}^{L_i})$, see [13, Cor. 2.7] and [6, Thm. 8.11]. Moreover, for $j < i$ we have $\langle D_G(\rho_{j,r}); R_{L_i}^G(\gamma_{u_{i,s}}^{L_i})\rangle = 0$ since $\rho_{j,r}$ vanishes on the support of $D_G\big(R_{L_i}^G(\gamma_{u_{i,s}}^{L_i})\big)$. Indeed, by [13, Prop. 2.3], this support is contained in the union of unipotent classes C_l satisfying $C_i \subset \overline{C}_l$. But if $\rho_{j,r}(v) \neq 0$ for some $v \in C_l^F$ with $C_i \subset \overline{C}_l$ then either $C_j = C_l$ or $\dim C_l < \dim C_j$ since $Z(\mathbf{G})$ is connected and p is good (see §2.1). Since $\dim C_j \leq \dim C_i$ that would force $C_j = C_l = C_i$. Therefore the matrix

$$\Big(\Big\langle D_G(\rho_{j,r}); R_{L_i}^G(\gamma_{u_{i,s}}^{L_i}) \Big\rangle_G \Big)_{j,r;i,s}$$

is block upper diagonal with identity blocks on the diagonal, hence invertible. Therefore, $\{R_{L_i}^G(\gamma_{u_{i,s}}^{L_i})\}$ also spans the $\mathbb{Z}$-module of unipotently supported virtual characters of G. □

Proof of Theorem 2.1. Let $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ be a regular embedding, compatible with F, that is $\widetilde{\mathbf{G}}$ has connected centre and same derived subgroup as $\mathbf{G}$. Note that this restricts to an isomorphism on the variety of unipotent elements. To avoid any confusion, given a unipotent element u in $\widetilde{G}$ we shall denote by $\widetilde{\Gamma}_u$ (resp. Γ_u) the corresponding generalised Gelfand–Graev representation of $\widetilde{G}$ (resp. G) and by $\widetilde{\gamma}_u$ (resp. γ_u) its character. By construction we have $\widetilde{\Gamma}_u = \mathrm{Ind}_G^{\widetilde{G}}\,\Gamma_u$. Fix a system of coset representatives $g_1, \ldots, g_r$ of $\widetilde{G}/G$. Since $G \trianglelefteq \widetilde{G}$, the Mackey formula yields an isomorphism of functors

$$\mathrm{Res}_G^{\widetilde{G}} \circ \mathrm{Ind}_G^{\widetilde{G}} \simeq \bigoplus_{i=1}^{r} \mathrm{ad}(g_i). \tag{1}$$

In particular $\mathrm{Res}_G^{\widetilde{G}}\,\widetilde{\Gamma}_u \simeq \bigoplus \Gamma_{g_iug_i^{-1}}$ (using that $\mathrm{ad}(g_i)\big(\Gamma_u\big) = \Gamma_{g_iug_i^{-1}}$, see [8, Prop. 2.2]).

Let N be a cuspidal simple kG-module. From (1) we deduce that $\mathrm{Ind}_G^{\widetilde{G}}\,N$ is a cuspidal $k\widetilde{G}$-module. Let M be a simple constituent of the socle of $\mathrm{Ind}_G^{\widetilde{G}}\,N$ and let ψ be its Brauer character. We denote by ψ_{uni} the unipotently supported class function which coincides with ψ on the set of unipotent elements. Then for every unipotent element u contained in a Levi subgroup $\widetilde{L}$ of $\widetilde{G}$ we have

$$\langle R_{\widetilde{L}}^{\widetilde{G}}(\widetilde{\gamma}_u^{\tilde{L}}); \psi_{\mathrm{uni}}\rangle_{\widetilde{G}} = \langle R_{\widetilde{L}}^{\widetilde{G}}(\widetilde{\gamma}_u^{\tilde{L}}); \psi\rangle_{\widetilde{G}},$$

which is zero whenever $\widetilde{L} \neq \widetilde{G}$ since ψ is cuspidal. Together with Proposition 2.2, we deduce that there exists an F-stable cuspidal unipotent class C of $\widetilde{\mathbf{G}}$ and $u \in C^F$ such that $\langle \widetilde{\gamma}_u; \psi\rangle_{\widetilde{G}} \neq 0$. By construction of the generalised Gelfand–Graev characters there exist a unipotent subgroup $U_{1.5}$ of $\widetilde{G}$ and a linear character χ_u of $U_{1.5}$ such that $\widetilde{\gamma}_u = \mathrm{Ind}_{U_{1.5}}^{\widetilde{G}}\,\chi_u$. If k_{χ_u} denotes the 1-dimensional $kU_{1.5}$-module on which $U_{1.5}$ acts by χ_u then since $U_{1.5}$ is a p-group, hence an ℓ'-group, we get

$$\begin{aligned}\dim \mathrm{Hom}_{k\widetilde{G}}(k\widetilde{\Gamma}_u, M) &= \dim \mathrm{Hom}_{kU_{1.5}}(k_{\chi_u}, \mathrm{Res}_{U_{1,5}}^{\widetilde{G}} M) \\ &= \langle \chi_u; \mathrm{Res}_{U_{1.5}}^{\widetilde{G}}\psi\rangle_{U_{1.5}} \\ &= \langle \widetilde{\gamma}_u; \psi\rangle_{\widetilde{G}}\end{aligned}$$

which is non-zero. This proves that there is a surjective $k\widetilde{G}$-homomorphism $k\widetilde{\Gamma}_u \twoheadrightarrow M$. By restriction to G we get surjective kG-homomorphisms $\mathrm{Res}_G^{\widetilde{G}}\,k\widetilde{\Gamma}_u \twoheadrightarrow \mathrm{Res}_G^{\widetilde{G}}\,M \twoheadrightarrow N$. It follows that $\mathrm{Hom}_{kG}(k\Gamma_{g_iug_i^{-1}}, N) \neq 0$ for at least one $i \in \{1, \ldots, r\}$, thus proving the claim. □

Given C an F-stable cuspidal unipotent class of $\mathbf{G}$ we will denote by Γ_C the sum of all the generalised Gelfand–Graev representations corresponding to representatives of G-orbits in C^F. If N is a simple kG-module then there exist a 1-split Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ and a cuspidal kL-module M such that $\mathrm{Hom}_{kG}(R_L^G(M), N) \neq 0$. Consequently, we can build a progenerator of kG using Theorem 2.1 for all 1-split Levi subgroups. For this, define $\mathcal{L}$ to be the set of pairs (L, C) such that $\mathbf{L}$ is a 1-split Levi subgroup of $\mathbf{G}$, and C is an F-stable unipotent class of $\mathbf{L}$ which is cuspidal for $\mathbf{L}_{\mathrm{ad}}$

Corollary 2.3. *Let* $\mathbf{G}$ *be connected reductive with Frobenius map* F *and assume that* p *is good for* $\mathbf{G}$. *Then the module*

$$P = \bigoplus_{(L,C)\in\mathcal{L}} R_L^G(k\Gamma_C^L)$$

is a progenerator of kG.

Remark 2.4. Except for groups of type A, even the multiplicities of unipotent characters in the character of P depend on q (e.g. the multiplicity of the Steinberg character).

Remark 2.5. One can use Theorem 2.1 to reprove that there is at most one unipotent cuspidal module in $\mathrm{GL}_n(q)$ (see for example [4, Thm. 5.21 and Cor. 5.23]). Indeed, the only cuspidal class in $\mathrm{PGL}_n(\overline{\mathbb{F}_p})$ is the regular class, therefore any cuspidal module must appear in the head of the usual Gelfand–Graev representation Γ_u for some (any) regular unipotent element u. Since γ_u has only the Steinberg character as a unipotent constituent, it follows that Γ_u has only one unipotent projective indecomposable summand, therefore at most one unipotent cuspidal module in its head.

For general linear groups again, the progenerator given in Corollary 2.3 involves only parabolic induction of usual Gelfand–Graev representations. This progenerator was already studied in [2] and [1]. Our construction is a natural generalisation to arbitrary finite reductive groups.

3. ℓ-reduction of cuspidal unipotent characters

Recall that $\mathbf{G}$ is a connected reductive group defined over $\mathbb{F}_q$, with corresponding Frobenius endomorphism F, and p, the characteristic of $\mathbb{F}_q$, is good for $\mathbf{G}$. We use the result of the previous section to show that under some mild assumptions on ℓ, cuspidal unipotent characters remain irreducible after ℓ-reduction.

3.1. **Multiplicities in generalised Gelfand–Graev characters.** Recall from §2 that given a unipotent character ρ, any generalised Gelfand–Graev character γ_u with $u \notin C_{\rho^*}$ and $\dim([u]) \geq \dim C_{\rho^*}$ satisfies $\langle \gamma_u; \rho\rangle = 0$. We combine here results from [13, 11] to compute $\langle \gamma_u; \rho\rangle$ when ρ is cuspidal and $u \in C_{\rho^*}$.

Given an F-stable unipotent class C of $\mathbf{G}$, recall that Γ_C is the sum of all the Γ_u's where u runs over a set of representatives of G-orbits in C^F. We will denote by γ_C the character of $K\Gamma_C$.

Proposition 3.1. *Assume that* $\mathbf{G}$ *is simple of adjoint type. Let* ρ *be a cuspidal unipotent character with unipotent support* C_ρ. *Then* $C_\rho = C_{\rho^*}$ *and* $\langle \gamma_{C_\rho}; \rho\rangle = 1$.

Proof. Let $\mathcal{F}$ be the family of unipotent characters containing ρ. Since ρ is cuspidal, we have $\rho = \rho^*$, and hence $C_{\rho^*} = C_\rho$. By [13, Thm. 3.3] C_ρ is a cuspidal class, and for $u \in C_\rho^F$ the finite group $A_G(u)$ is isomorphic to the small finite group associated to $\mathcal{F}$ as in [19, §4]. In particular, the condition $(*)$ in [11, Prop. 2.3] holds for C_ρ (and for C_{ρ^*}).

If $\mathbf{G}$ is of classical type or of type E_7, then the claim follows from [11, Prop. 4.3] since $A_G(u)$ is abelian in that case.

If $\mathbf{G}$ is of exceptional type different from E_7, then $A_G(u)$ equals $\mathfrak{S}_3$ (for types G_2 and E_6), $\mathfrak{S}_4$ (for type F_4) or $\mathfrak{S}_5$ (for type E_8). In that case, by [17, Lemma 1.3.1], there is at most one local system on C_ρ that is not in the image of the Springer correspondence and the multiplicity of ρ in γ_u was first computed by Kawanaka [17]. An explicit formula can also be found in [5, §6]. The assumption (6.1) in loc. cit. is satisfied and the projection of γ_u to $\mathcal{F}$ can be explicitly computed from Lusztig's parametrisation. Namely, let now $u \in C_\rho^F$ be such that F acts trivially on $A = A_G(u)$. Then the conjugacy classes of A are in bijection with the G-orbits in C_ρ^F. Given $a \in A$, we fix a representative $u(a)$ of the corresponding orbit. Then the unipotent characters in $\mathcal{F}$ are parametrised by A-conjugacy classes of pairs (a, χ) where $\chi \in \mathrm{Irr}(C_A(a))$ and according to [5, Thm. 6.5(ii)] the projection of $\gamma_{u(a)}$ to $\mathcal{F}$ is given by

$$\sum_{\chi \in \mathrm{Irr}(C_A(a))} \chi(1)\, \rho^*_{(a,\chi)}.$$

Now the claim follows from the fact that a cuspidal character always corresponds to a pair (a, χ) with $\chi(1) = 1$. This can be checked case-by-case using for example [19, Appendix] (thanks to [18, Thm. 1.15, §1.16], the parametrisation for ${}^2E_6$ can be deduced from the one for E_6 by Ennola duality). □

3.2. **ℓ-reduction.** We can now prove the main application of the construction of our progenerator, viz. Theorem 1, which we restate in a slightly more general form.

Theorem 3.2. *Assume that p is good for $\mathbf{G}$ and that $\ell \neq p$. If either $\mathbf{G}$ is simple of adjoint type, or ℓ is good for $\mathbf{G}$ and does not divide $|Z(\mathbf{G})^F/Z^\circ(\mathbf{G})^F|$, then any cuspidal unipotent character of $\mathbf{G}^F$ remains irreducible under reduction modulo ℓ.*

Proof. First assume that $\mathbf{G}$ is simple of adjoint type. Let ρ be a cuspidal unipotent character of G, and C_ρ be its unipotent support. It is a special and self-dual cuspidal unipotent class. Since ρ is cuspidal, it does not occur in any $R_L^G(\gamma_u^L)$ for proper 1-split Levi subgroups $\mathbf{L}$ of $\mathbf{G}$. In addition, if C is a cuspidal class different from C_ρ then by [13, Thm. 3.3] we have $C_{\rho^*} = C_\rho \subset \overline{C}$ which forces $\langle \gamma_u; \rho\rangle = 0$ for every $u \in C^F$. Finally, it follows from Proposition 3.1 that $\langle \gamma_{C_\rho}; \rho\rangle = 1$. Consequently

$$\Big\langle \sum_{(L,C)\in\mathcal{L}} R_L^G(\gamma_C^L); \rho\Big\rangle = 1$$

which by Corollary 2.3 proves that ρ appears in the character of exactly one projective indecomposable module. In other words, the ℓ-reduction of ρ is irreducible.

In the general case, let $\mathbf{G} \hookrightarrow \widetilde{\mathbf{G}}$ be a regular embedding. Then the restrictions of unipotent characters of $\widetilde{G}$ to G remain irreducible (see [3, Prop. 17.4]). Furthermore, under the assumptions on ℓ, the unipotent characters form a basic set of the union of unipotent blocks [8]. Therefore the unipotent parts of the decomposition matrices of $\widetilde{G}$ and G are equal, so that we can assume without loss of generality that the centre of $\mathbf{G}$ is connected. Now, $Z(\mathbf{G})^F = Z(G)$ and unipotent characters are trivial on the centre. Consequently the unipotent parts of the decomposition matrices of G and of $G/Z(\mathbf{G})^F$ are equal, and we can assume that $\mathbf{G}$ is semisimple of adjoint type. In that case $\mathbf{G}$ is a direct product $\mathbf{G} = \mathbf{G}_1^{n_1} \times \cdots \times \mathbf{G}_r^{n_r}$ where each $\mathbf{G}_i$ is simple of adjoint type, with F cyclically permuting the copies of $\mathbf{G}_i$ in $\mathbf{G}_i^{n_i}$. Now the projection onto the first component $\mathbf{G}_i^{n_i} \to \mathbf{G}_i$ induces a group isomorphism $(\mathbf{G}_i^{n_i})^F \simeq (\mathbf{G}_i)^{F^{n_i}}$ mapping unipotent characters to unipotent characters. Therefore we are reduced to the case that $\mathbf{G}$ is simple of adjoint type, which was treated above. □

By a result of Hiss [15, Prop. 3.3], this has the following consequence which might be of interest for applications to representations of p-adic groups:

Corollary 3.3. *Assume that p and ℓ are good for $\mathbf{G}$ and that $\ell \nmid p|Z(\mathbf{G})^F/Z^\circ(\mathbf{G})^F|$. Then any unipotent supercuspidal simple kG-module is liftable to an $\mathcal{O}G$-lattice.*

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