GENERIC BLOCKS OF FINITE REDUCTIVE GROUPS

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To Charlie Curtis

Contents

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0. Introduction

This work has been motivated by two problems, which interacted with one another: attempts to solve one helped to solve the other one.

The first one originated in [Br1], where is presented a conjecture about the structure of any block with abelian defect of any finite group (see [Br1], 6.1), which states that such a block should have the same type as the corresponding block of the normalizer of its defect group. One of our goals was to check this conjecture in the case of the finite reductive groups $\mathbf{G}^F$ (we denote by $\mathbf{G}$ a connected reductive algebraic group over an algebraic closure of a finite field $\mathbb{F}_q$, by $F\colon \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism defining a rational structure on this finite field, and by $\mathbf{G}^F$ the group of rational points). This is achieved in the present work (see §4.D, 5.24), at least for "large" prime numbers (*i.e.,* in the split case, prime numbers which do not divide the order of the Weyl group of $\mathbf{G}$).

By general properties of isotypies (see [Br1]), our results imply in particular that, for large prime numbers, conjectures as Alperin's weight conjecture or the Alperin–McKay height conjecture are verified for unipotent blocks (and so for "almost" all blocks) of all finite reductive groups.

The second question which motivated this work was to understand better the "generic" aspects of block theory of finite reductive groups.

Throughout the intensive work which has been done recently by many authors about blocks of finite reductive groups (among whom Fong–Srinivasan, Schewe, Cabanes–Enguehard, Hiß, Geck, and the authors), it had become gradually clear that, for large prime numbers ℓ, properties of ℓ–blocks of $\mathbf{G}^F$ do not really depend on the prime number ℓ, but rather depend on the cyclotomic factor of the "polynomial order" of $\mathbf{G}^F$ which is divisible by ℓ (see [BrMa] and §1.A below).

This remark had to be put in the perspective opened by the work of C.W. Curtis, who defined in particular the "generic degrees" of characters of $\mathbf{G}^F$, and subsequently of his student R. Boyce (see [Boy]) who defined characters "with Φ_d–defect zero".

A few experiments convinced us that it was indeed very often possible to replace ℓ by the corresponding Φ_d in many of the congruence relations involved in modular character theory.

One of the aims of the present work is to present further evidence for the existence of a "generic representation theory of generic groups". Using extensively [BrMa] (where are defined what we call here "generic finite reductive groups", as well as their Φ_d–subgroups and their centralizers), we introduce here a formalism which allows us in particular to group the unipotent characters into Φ_d–blocks and to define a suitable notion of "Φ_d–defect group" of a unipotent character (see definition 4.7 below). From many points of view, this Φ_d–defect group (a rational torus in $\mathbf{G}$) behaves like an ordinary defect group. Moreover, if ℓ is a large prime which divides $\Phi_d(q)$, then the actual ℓ–defect group is just the Sylow ℓ–subgroup of the group of rational points of the Φ_d–defect group.

One of the by–products of this approach is indeed the fact that we can treat the case of π–blocks (π a set of large prime numbers which divide the same cyclotomic factor) the same way we would treat ℓ–blocks, and this is what we present here.

With these two questions in mind, we came upon the main theorem of this work (see 3.2). This theorem is certainly the main tool to study our blocks, but it is also interesting in itself. It shows in particular that the usual so–called Harish–Chandra theory of characters, as well as many aspects of the Howlett–Lehrer–Lusztig theory about induction of cuspidal characters, must be viewed as a particular case (the case associated with the first cyclotomic polynomial Φ_1) of a general set of results which hold for every cyclotomic polynomial, provided one replaces the family of "Harish–Chandra" Levi subgroups by the family of centralizers of Φ_d–subgroups (called here d–split subgroups), and the cuspidal characters by the "d–cuspidal characters".

In particular, theorem 3.2 shows that, provided each irreducible unipotent character is equipped with an appropriate sign, Deligne–Lusztig inclusion from d–split subgroups is nothing but the ordinary induction in Weyl–type groups which are independent of q.

1. Notation, prerequisites and complements

A. Generic finite reductive groups.

In §1.A, we recall notation, definitions and some results from [BrMa] *concerning generic finite reductive groups.*

We call "generic finite reductive group" what is called "donnée radicielle complète" in [BrMa]. Let us recall briefly the definition and set our notation.

A root datum of rank r is a quadruple $(X, R, Y, R^\vee)$ such that

(rd.1) X and Y are free $\mathbb{Z}$–modules of rank r, endowed with a duality $X \times Y \to \mathbb{Z}$ denoted by $(x, y) \mapsto <x, y>$.

(rd.2) R and $R^\vee$ are finite subsets of X and Y respectively, endowed with a bijection $R \to R^\vee$ denoted by $\alpha \mapsto \alpha^\vee$.

(rd.3) For $\alpha \in R$ we have $<\alpha, \alpha^\vee> = 2$. Let s_α be the involutive automorphism of X defined by $s_\alpha(x) = x - <x, \alpha^\vee>\alpha$, and let $s^\vee_\alpha$ be its adjoint, an automorphism of Y (one has $s^\vee_\alpha(y) = y - <a, y>\alpha^\vee$). Then $s_\alpha(R) = R$ and $s^\vee_\alpha(R^\vee) = R^\vee$.

The Weyl group of the root datum $(X, R, Y, R^\vee)$ is the subgroup of the group of automorphisms of Y generated by the $s^\vee_\alpha$ for $\alpha \in R$.

The following definition is valid in the general case; for the particular twisted cases ${}^2B_2$, ${}^2F_4$, ${}^2G_2$, *the reader may refer to* [BrMa].

Definition.

- An automorphism of the root datum $(X, R, Y, R^\vee)$ is an automorphism ϕ of Y which stabilizes $R^\vee$ and such that $\phi^\vee$ stabilizes R.
- A *generic finite reductive group* is a pair $\mathbb{G} = (\Gamma_\mathbb{G}, W_\mathbb{G}\phi)$, where $\Gamma_\mathbb{G}$ is a root datum, $W_\mathbb{G}$ is its Weyl group, and ϕ is an automorphism of finite order of $\Gamma_\mathbb{G}$.

Let p be a prime number and let $\bar{\mathbb{F}}_p$ be a chosen algebraic closure of $\mathbb{F}_p$. To a root datum $\Gamma_\mathbb{G}$ is then associated a pair $(\mathbf{G}, \mathbf{T})$ where

- $\mathbf{G}$ is a connected reductive algebraic group over $\bar{\mathbb{F}}_p$,
- $\mathbf{T}$ is a maximal torus of $\mathbf{G}$,

and such a pair is uniquely determined up to inner automorphisms of $\mathbf{G}$ induced by the elements of $\mathbf{T}$.

The isogeny theorem (*cf.* for example [Sp2], 11.4.9) implies the following

(1.1) to a linear map of Y of the form $q\phi$, where q is a power of p and ϕ an automorphism of $\Gamma_{\mathbb{G}}$, is associated an isogeny $F_{q\phi} : \mathbf{G} \to \mathbf{G}$ of the algebraic group $\mathbf{G}$, uniquely determined up to inner automorphisms of $\mathbf{G}$ induced by elements of $\mathbf{T}$.

If $q = 1$, the isogeny F_ϕ is in fact an algebraic automorphism of $\mathbf{G}$; otherwise, the isogeny $F_{q\phi}$ is the Frobenius endomorphism associated to some $\mathbb{F}_q$-structure on $\mathbf{G}$. Note that, since $\mathbf{T}$ is commutative, if ϕ and ϕ' commute then $F_{q\phi}$ commutes with $F_{q'\phi'}$ for any other power q' of p.

Thus, given a generic group $\mathbb{G}$, the choice of a power $q > 1$ of p and of an element ϕ in the coset $W\phi$ determines a triple $(\mathbf{G}, \mathbf{T}, F)$, where $F = F_{q\phi}$ is a Frobenius endomorphism (such a construction is also possible in the twisted cases, where q has to be replaced by an odd power of $\sqrt{2}$ or $\sqrt{3}$, and F is a special isogeny of the corresponding group of type B_2, F_4 or G_2 *cf.* [BrMa], §2). Let us call such a triple a "(q,ϕ)–triple associated to $\mathbb{G}$". As in [BrMa], §2 we will write $\mathbb{G}(q)$ for $\mathbf{G}^F$.

Usual invariants of a generic finite reductive group (*cf.* [BrMa]).

Preliminary remark. What follows is written in the "general case". In order to apply it to the twisted cases (${}^2\mathrm{B}_2$, ${}^2\mathrm{F}_4$, ${}^2\mathrm{G}_2$), one has to perform a few modifications, such as to replace the ring $\mathbb{Z}$ and the field $\mathbb{Q}$ by, respectively, $\mathbb{Z}[\sqrt{2}^{-1}]$ and $\mathbb{Q}(\sqrt{2})$ (for ${}^2\mathrm{B}_2$ and ${}^2\mathrm{F}_4$), or by $\mathbb{Z}[\sqrt{3}^{-1}]$ and $\mathbb{Q}(\sqrt{3})$ (for ${}^2\mathrm{G}_2$). We leave this work to the reader.

Let $\mathbb{G} = ((X, R, Y, R^\vee), W_{\mathbb{G}}\phi)$ be a generic finite reductive group.

We set $V = \mathbb{Q} \otimes Y$ and $W = W_{\mathbb{G}}$. We denote by SV the symmetric algebra of V, by $(SV)^W$ the graded subalgebra of W–invariant elements of SV, and by $\mathfrak{A}_{\mathbb{G}}$ the ideal of $(SV)^W$ consisting of elements without degree zero terms.

• The vector space $\mathfrak{A}_{\mathbb{G}}/\mathfrak{A}_{\mathbb{G}}^2$ has dimension r, and is endowed with an action of the image $\bar\phi$ of ϕ modulo W. We set

$$\varepsilon_{\mathbb{G}} := (-1)^r \det \bar\phi\,.$$

• We set $R\mathbb{G} := SV/(SV\mathfrak{A}_{\mathbb{G}})$. Then $R\mathbb{G}$ is a finite dimensional graded algebra which, as a $\mathbb{Q}W$–module, is isomorphic to the regular representation of W. We denote by $R^n\mathbb{G}$ the subspace of elements of degree n of $R\mathbb{G}$, and by $2N(\mathbb{G})$ the number of elements of the root system R. Then we have

$$R\mathbb{G} = \sum_{n=0}^{N(\mathbb{G})} R^n\mathbb{G}\,.$$

• The polynomial order of $\mathbb{G}$ (denoted by $O_{\mathbb{G}}(x)$ in [BrMa]) is the polynomial denoted here by $|\mathbb{G}|$ and defined by the formula

$$|\mathbb{G}| := \frac{\varepsilon_{\mathbb{G}} x^{N(\mathbb{G})}}{\frac{1}{|W|}\sum_{w\in W} \frac{1}{\det_V(1 - xw\phi)}}\,. \tag{1.2}$$

If (q,ϕ) (whence $(\mathbf{G}, \mathbf{T}, F)$) is chosen, one has $|\mathbf{G}^F| = |\mathbb{G}|(q)$.

We recall now how various definitions are translated in the setting of generic groups.

Tori and Levi subgroups.

- A *generic torus* is a generic group such that $R = R^\vee = \emptyset$ (so $W\phi$ reduces to a single element). Generic subtori of $\mathbb{G}$ (called "sous–données toriques" in [BrMa]) are generic tori of the form $(X', Y', w\phi|_{Y'})$ where $w \in W$ and Y' is a $w\phi$-stable direct factor of Y and X' its dual (a quotient of X).
- *Generic Levi subgroups* of $\mathbb{G}$ (called in [BrMa] "sous–données de Levi", and often abbreviated herein "Levi subgroups") are generic groups of the form $((X, R', Y, R'^\vee), W_{R'}w\phi)$, where $w \in W$, where $R'^\vee$ is a parabolic subsystem of $R^\vee$ which is $w\phi$-stable and where $W_{R'}$ is the Weyl group of R'.

Lifting scalars.

Let Γ be a root datum, and let $\mathbb{G} = (\Gamma, W\phi)$ be an associated generic finite group. Let $a \in \mathbb{N}$.

(1.3) We define the generic finite group $\mathbb{G}^{(a)} = (\Gamma^{(a)}, \phi^{(a)})$ by the following rules:

(ls.1) $\Gamma^{(a)} := \Gamma \times \cdots \times \Gamma$ *(a times),*

(ls.2) $\phi^{(a)}$ *is the product of ϕ (acting diagonally on $\Gamma \times \cdots \times \Gamma$) and the a–cycle which permutes the various factors Γ of $\Gamma^{(a)}$.*

We have $|\mathbb{G}^{(a)}|(x) = |\mathbb{G}|(x^a)$.

Let $(\mathbf{G}, \mathbf{T}, F)$ be a (q, ϕ)–triple associated with $\mathbb{G}$, let $(\mathbf{G}, \mathbf{T}, F_a)$ be a (q^a, ϕ)–triple associated with $\mathbb{G}$, and let $(\mathbf{G}^{(a)}, \mathbf{T}^{(a)}, F^{(a)})$ be a $(q, \phi^{(a)})$–triple associated with $\mathbb{G}^{(a)}$. Then

$$\mathbf{G}^{(a)F^{(a)}} \simeq \mathbf{G}^{F_a} \; . \tag{1.4}$$

In other words, we have $\mathbb{G}^{(a)}(q) \simeq \mathbb{G}(q^a)$.

Changing q into $-q$.

Let $\mathbb{G} = (\Gamma_{\mathbb{G}}, W_{\mathbb{G}}\phi)$ be a generic finite group. We recall (see [BrMa]) that we define the generic finite group $\mathbb{G}^-$ by

$$\mathbb{G}^- := (\Gamma_{\mathbb{G}}, W_{\mathbb{G}}(-\phi)) \; . \tag{1.5}$$

Note that if $-\mathrm{Id} \in W_{\mathbb{G}}$, then $\mathbb{G}^- = \mathbb{G}$.

We have $|\mathbb{G}^-|(x) = (-1)^r|\mathbb{G}|(-x)$.

The radical, the "semi–simple quotient", the adjoint group, the dual group.

If $\mathbb{G} = ((X, R, Y, R^\vee), W\phi)$ is a generic finite reductive group, we denote by $Q(R)$ the $\mathbb{Z}$–submodule of X generated by R.

• The radical of $\mathbb{G}$ is by definition the generic torus defined by

$$\mathrm{Rad}(\mathbb{G}) := ((X/Q(R)^{\perp\perp}, Q(R)^\perp), \phi|_{Q(R)^\perp}) \, .$$

For a triple $(\mathbf{G}, \mathbf{T}, F)$ associated with $\mathbb{G}$, the algebraic group associated with $\mathrm{Rad}(\mathbb{G})$ is $Z^o(\mathbf{G})$ (the connected component of 1 in the center $Z(\mathbf{G})$ of $\mathbf{G}$).

• The generic "semisimple quotient" of $\mathbb{G}$ is "the quotient of $\mathbb{G}$ by its radical $\mathrm{Rad}(\mathbb{G})$", namely

$$\mathbb{G}_{\mathrm{ss}} := ((Q(R)^{\perp\perp}, R, Y/Q(R)^\perp, R^\vee), W\phi)$$

(here "ϕ" stands for the automorphism of $Y/Q(R)^\perp$ induced by ϕ, and $R^\vee$ stands for the image of $R^\vee$ in $Y/Q(R)^\perp$).

For a triple $(\mathbf{G},\mathbf{T},F)$ associated with $\mathbb{G}$, the algebraic group associated with $\mathbb{G}_{\mathrm{ss}}$ is $\mathbf{G}/Z^o(\mathbf{G})$. We have $|\mathbb{G}_{\mathrm{ss}}| = |\mathbb{G}|/|\mathrm{Rad}(\mathbb{G})|$.

• The *dual generic finite group* is defined by

$$\mathbb{G}^* := ((Y,R^\vee,X,R),W\phi^{\vee\,-1})\,,$$

where the automorphism $\phi^\vee$ of X is the adjoint of ϕ, and where W is identified with its contragredient action on X, *i.e.,* W is the group generated by the s_α for $\alpha \in R$ (note that $\mathbb{G}^*$ is covariant in $\mathbb{G}$).

• We denote by $P(R)$ the dual of $Q(R^\vee)$ in $\mathbb{Q}\otimes Q(R)$, *i.e.,* the set of all $v \in \mathbb{Q}\otimes Q(R)$ such that $<v,\alpha^\vee> \in \mathbb{Z}$ for all $\alpha^\vee \in R^\vee$.

The *adjoint generic finite group* of $\mathbb{G}$ is defined by

$$\mathbb{G}_{\mathrm{ad}} := ((Q(R),R,P(R^\vee),R^\vee),W\phi)\,.$$

For a triple $(\mathbf{G},\mathbf{T},\phi)$ associated with $\mathbb{G}$, the algebraic group associated with $\mathbb{G}_{\mathrm{ad}}$ is $\mathbf{G}_{\mathrm{ad}}$.

• We define the generic finite reductive group $D(\mathbb{G})$ by (with obvious abuse of notation)

$$D(\mathbb{G}) := ((X/Q(R^\vee)^\perp,R,Q(R^\vee)^{\perp\perp},R^\vee),W\phi)\,.$$

Thus we have a generic finite torus

$$\mathbb{G}/D(\mathbb{G}) = ((Q(R^\vee)^\perp, Y/Q(R^\vee)^{\perp\perp}), \phi|_{Y/Q(R^\vee)^{\perp\perp}})\,.$$

We have

$$\mathrm{Rad}(\mathbb{G}^*) = (\mathbb{G}/D(\mathbb{G}))^*\,. \tag{1.6}$$

Let $(\mathbf{G},\mathbf{T},F)$ be a (q,ϕ)–triple associated with $\mathbb{G}$. Let $[\mathbf{G},\mathbf{G}]$ be the derived group of $\mathbf{G}$. Then the algebraic group associated with $D(\mathbb{G})$ (resp. $\mathbb{G}/D(\mathbb{G})$) is $[\mathbf{G},\mathbf{G}]$ (resp. $\mathbf{G}/[\mathbf{G},\mathbf{G}]$).

Good primes.

The following definition can be found in [SpSt], 4.1. It depends only on the root system of $\mathbb{G}$.

(*1.7)* *We say that a prime ℓ is good for $\mathbb{G}$ if there is no closed root subsystem R_1 of R such that ℓ divides the order of the torsion subgroup of $Q(R)/Q(R_1)$.*

It results from [SpSt], 4.4, that if ℓ is good for $\mathbb{G}$, then there is no closed root subsystem R_1 of R such that ℓ divides the order of the torsion subgroup of $Q(R^\vee)/Q(R_1^\vee)$.

If ℓ is good for $\mathbb{G}$ it is also good for the dual generic group $\mathbb{G}^*$.

Intersections of Levi subgroups.

Let $\mathbb{L} = ((X,R_{\mathbb{L}},Y,R_{\mathbb{L}}^\vee),W_{\mathbb{L}}v\phi)$ and $\mathbb{M} = ((X,R_{\mathbb{M}},Y,R_{\mathbb{M}}^\vee),W_{\mathbb{M}}w\phi)$ be two Levi subgroups of $\mathbb{G}$.

(*1.8) We say that* $\mathbb{L}\cap\mathbb{M}$ *is defined if*

$$W_{\mathbb{L}}v\phi\cap W_{\mathbb{M}}w\phi\neq\emptyset\,.$$

In that case, choosing an element $u\in W_{\mathbb{L}}v\cap W_{\mathbb{M}}w$*, we define*

$$\mathbb{L}\cap\mathbb{M}:=((X,R_{\mathbb{L}}\cap R_{\mathbb{M}},Y,R_{\mathbb{L}}^{\vee}\cap R_{\mathbb{M}}^{\vee}),(W_{\mathbb{L}}\cap W_{\mathbb{M}})u\phi)\,,$$

Note that $W_{\mathbb{L}}v\phi=W_{\mathbb{L}}u\phi$ and $W_{\mathbb{M}}w\phi=W_{\mathbb{M}}u\phi$, whence $W_{\mathbb{L}}v\phi\cap W_{\mathbb{M}}w\phi=(W_{\mathbb{L}}\cap W_{\mathbb{M}})u\phi$. It is easy to see that $\mathbb{L}\cap\mathbb{M}$ is a well–defined (*i.e.,* independent of the choice of u) Levi subgroup of both $\mathbb{L}$ and $\mathbb{M}$.

By the preceding definition, it is clear that

(*1.9)* $\mathbb{L}\cap\mathbb{M}$ *is defined if and only if there exists a maximal generic torus* $\mathbb{T}$ *of* $\mathbb{G}$ *such that* $\mathbb{T}$ *is contained in both* $\mathbb{L}$ *and* $\mathbb{M}$*. In this case,* $\mathbb{T}$ *is contained in* $\mathbb{L}\cap\mathbb{M}$*.*

Remark. If $(\mathbf{G},\mathbf{T},F)$ is a (q,ϕ)–triple associated to $\mathbb{G}$, we recall (*cf.* [BrMa], 2.1) that there is a well–defined bijection between $W_{\mathbb{G}}$–classes of generic Levi subgroups of $\mathbb{G}$ and $\mathbf{G}^F$–conjugacy classes of F–stable Levi subgroups of $\mathbf{G}$. Then $\mathbb{L}\cap\mathbb{M}$ is defined if and only if there exists $\mathbf{L}$ and $\mathbf{M}$ in the corresponding classes of Levi subgroups in $\mathbf{G}$ such that $\mathbf{L}\cap\mathbf{M}$ contains a maximal torus; in that case, $\mathbf{L}\cap\mathbf{M}$ is a Levi subgroup of $\mathbf{G}$, whose rational conjugacy class corresponds to the $W_{\mathbb{G}}$–conjugacy class of $\mathbb{L}\cap\mathbb{M}$.

Class functions on $\mathbb{G}$.

Let $\mathcal{CF}(\mathbb{G})$ be the space of all W–invariant functions on the coset $W\phi$ with values in $\mathbb{Q}[x]$, called *class functions on* $\mathbb{G}$.

We shall see later on that this space can be "specialized" onto the space of unipotent uniform functions for a chosen (q,ϕ)–triple associated to $\mathbb{G}$.

If α and $\alpha'\in\mathcal{CF}(\mathbb{G})$, we set $<\alpha,\alpha'>_{\mathbb{G}}:=\dfrac{1}{|W|}\sum_{w\in W}\alpha(w\phi)\alpha'(w\phi)\,.$

Induction and restriction.

Let $\mathbb{L}=((X,R_{\mathbb{L}},Y,R_{\mathbb{L}}^{\vee}),W_{\mathbb{L}}w\phi)$ be a Levi subgroup of $\mathbb{G}$, and let $\alpha\in\mathcal{CF}(\mathbb{G})$ and $\beta\in\mathcal{CF}(\mathbb{L})$.

We denote

- by $\mathrm{Res}_{\mathbb{L}}^{\mathbb{G}}\alpha$ the restriction of α to the coset $W_{\mathbb{L}}w\phi$,
- by $\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}\beta$ the class function on $\mathbb{G}$ defined by

$$\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}\beta(u\phi)=\frac{1}{|W_{\mathbb{L}}|}\sum_{v\in W}\tilde{\beta}(vu\phi v^{-1})\quad\text{for }u\phi\in W_{\mathbb{G}}\phi,\tag{1.10}$$

where $\tilde{\beta}(x\phi)=\beta(x\phi)$ if $x\in W_{\mathbb{L}}w$, and $\tilde{\beta}(x\phi)=0$ if $x\notin W_{\mathbb{L}}w$. In other words, we have

$$(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}\beta)(u\phi)=\sum_{v\in W_{\mathbb{G}}/W_{\mathbb{L}},\,{}^v(u\phi)\in W_{\mathbb{L}}w\phi}\beta({}^v(u\phi))\,.\tag{1.11}$$

We have the Frobenius reciprocity:

$$<\alpha,\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}\beta>_{\mathbb{G}}=<\mathrm{Res}_{\mathbb{L}}^{\mathbb{G}}\alpha,\beta>_{\mathbb{L}}\,.\tag{1.12}$$

The Mackey formula.

For two Levi subgroups $\mathbb{L}$ and $\mathbb{M}$ of $\mathbb{G}$, we denote by $\mathcal{S}_{W_\mathbb{G}}(\mathbb{L},\mathbb{M})$ the set of all $w \in W_\mathbb{G}$ such that ${}^w\mathbb{L} \cap \mathbb{M}$ is defined.

It is clear that $W_\mathbb{L}$ acts on $\mathcal{S}_{W_\mathbb{G}}(\mathbb{L},\mathbb{M})$ from the right, while $W_\mathbb{M}$ acts on $\mathcal{S}_{W_\mathbb{G}}(\mathbb{L},\mathbb{M})$ from the left. If we let w run over a chosen double coset $W_\mathbb{M} v W_\mathbb{L}$ for some $v \in \mathcal{S}_{W_\mathbb{G}}(\mathbb{L},\mathbb{M})$, we see that ${}^w\mathbb{L} \cap \mathbb{M}$ is defined up to $W_{{}^w\mathbb{L}}$–conjugation as a subgroup of ${}^w\mathbb{L}$ and up to $W_\mathbb{M}$–conjugation as a subgroup of $\mathbb{M}$, which proves that the operations $\mathrm{Ind}_{\mathbb{M}\cap{}^w\mathbb{L}}^{\mathbb{M}}$ and $\mathrm{Res}_{\mathbb{M}\cap{}^w\mathbb{L}}^{{}^w\mathbb{L}}$ depend only on the double coset of w. This gives sense to the following formula

$$(1.13) \qquad \mathrm{Res}_\mathbb{M}^\mathbb{G} \cdot \mathrm{Ind}_\mathbb{L}^\mathbb{G} = \sum_{w \in W_\mathbb{M} \backslash \mathcal{S}_{W_\mathbb{G}}(\mathbb{L},\mathbb{M})/W_\mathbb{L}} \mathrm{Ind}_{\mathbb{M}\cap{}^w\mathbb{L}}^{\mathbb{M}} \cdot \mathrm{Res}_{\mathbb{M}\cap{}^w\mathbb{L}}^{{}^w\mathbb{L}} \cdot \mathrm{ad}(w)\,,$$

whose proof goes like the proof of the Mackey formula for ordinary induction and restriction (note that $\mathcal{S}_{W_\mathbb{G}}(\mathbb{L},\mathbb{M})$ may be empty).

Some important class functions.

• We denote by $\mathrm{tr}_{R\mathbb{G}}$ the class function on $\mathbb{G}$ defined by the character of the "graded regular representation" $R\mathbb{G}$ (see above). Thus the value of the function $\mathrm{tr}_{R\mathbb{G}}$ on $w\phi$ is $\mathrm{tr}_{R\mathbb{G}}(w\phi) := \sum_{n=0}^{N(\mathbb{G})} \mathrm{tr}(w\phi; R^n\mathbb{G})x^n$.

• We shall consider the following polynomial (which will be identified later on with the "fake degree" $\mathrm{Deg}(\mathbf{\Phi}_\alpha^\mathbb{G})$) :

$$(1.14) \qquad D(\alpha) := <\alpha, \mathrm{tr}_{R\mathbb{G}}>_\mathbb{G} = \sum_{n=0}^{N(\mathbb{G})} \Big(\frac{1}{|W|}\sum_{w\in W} \alpha(w\phi)\mathrm{tr}(w\phi; R^n\mathbb{G})\Big)x^n\,.$$

We have (*cf.* [BrMa], prop. 1.6′)

$$(1.15) \qquad \frac{1}{|W|}\sum_{w\in W} \frac{\alpha(w\phi)}{\det_V(1-xw\phi)} = D(\alpha)\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det_V(1-xw\phi)} \quad ,$$

or, in other words

$$<\alpha, \mathrm{tr}_{SV}>_\mathbb{G} = <\alpha, \mathrm{tr}_{R\mathbb{G}}>_\mathbb{G} <1^\mathbb{G}, \mathrm{tr}_{SV}>_\mathbb{G}\,.$$

where $1^\mathbb{G}$ is the constant function 1 on $W_\mathbb{G}\phi$.

• For a Levi subgroup $\mathbb{L}$ of $\mathbb{G}$, by the Frobenius reciprocity (1.12), we have

$$D(\mathrm{Ind}_\mathbb{L}^\mathbb{G} 1^\mathbb{L}) = <1^\mathbb{L}, \mathrm{Res}_\mathbb{L}^\mathbb{G} \mathrm{tr}_{R\mathbb{G}}>_\mathbb{L} = \sum_{n=0}^{N(\mathbb{G})} \mathrm{tr}(w\phi; (R^n\mathbb{G})^{W_\mathbb{L}})x^n\,.$$

where $W_\mathbb{L} w\phi$ is the coset associated to $\mathbb{L}$ and $(R^n\mathbb{G})^{W_\mathbb{L}}$ are the $W_\mathbb{L}$–invariants in $R^n\mathbb{G}$.

It follows from 1.2, 1.12 and 1.15 (see also [BrMa], prop. 1.7) that

$$(1.16) \qquad |\mathbb{G}|/|\mathbb{L}| = \varepsilon_\mathbb{G}\varepsilon_\mathbb{L} x^{N(\mathbb{G})-N(\mathbb{L})} D(\mathrm{Ind}_\mathbb{L}^\mathbb{G} 1^\mathbb{L})\,;$$

$D(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} 1^{\mathbb{L}})$ will be identified later on with the generic degree $\mathrm{Deg}(R_{\mathbb{L}}^{\mathbb{G}}(\mathbf{1}))$.

• Every element $w\phi \in W\phi$ defines a maximal generic subtorus (or, equivalently, a minimal Levi subgroup) $\mathbb{T}_{w\phi} := ((X,Y), w\phi)$. We have

$$\mathrm{tr}_{R\mathbb{G}}(w\phi) := D(\mathrm{Ind}_{\mathbb{T}_{w\phi}}^{\mathbb{G}} 1^{\mathbb{T}_{w\phi}}) \, .$$

So

$$|\mathbb{G}|/|\mathbb{T}_{w\phi}| = \varepsilon_{\mathbb{G}} \varepsilon_{\mathbb{T}_{w\phi}} x^{N(\mathbb{G})} \mathrm{tr}_{R\mathbb{G}}(w\phi) \, , \tag{1.17}$$

and also $\mathrm{tr}_{R\mathbb{G}}(w\phi) = D(\kappa_{w\phi})$, where $\kappa_{w\phi}$ is $|C_W(w\phi)|$ times the characteristic function of the W–conjugacy class of $w\phi$.

It follows from 1.16 and 1.17 that

$$\mathrm{Res}_{\mathbb{L}}^{\mathbb{G}} \mathrm{tr}_{R\mathbb{G}} = D(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} 1^{\mathbb{L}}) \mathrm{tr}_{R\mathbb{L}} \, . \tag{1.18}$$

(1.19) For $\beta \in \mathcal{CF}(\mathbb{L})$, *we have* $D(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} \beta) = D(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} 1^{\mathbb{L}}) {<}\beta, \mathrm{tr}_{R\mathbb{L}}{>}_{\mathbb{L}}$.

Indeed, $D(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} \beta) = {<}\beta, \mathrm{Res}_{\mathbb{L}}^{\mathbb{G}} \mathrm{tr}_{R\mathbb{G}}{>}_{\mathbb{L}} = D(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} 1^{\mathbb{L}}) {<}\beta, \mathrm{tr}_{R\mathbb{L}}{>}_{\mathbb{L}}$.

Class functions on $\mathbb{G}^-$.

The map $\sigma^{\mathbb{G}} \colon \mathcal{CF}(\mathbb{G}) \to \mathcal{CF}(\mathbb{G}^-)$, given by $\sigma^{\mathbb{G}}\alpha(w(-\phi)) = \alpha(w\phi)$, is an isometry.

The map $\mathbb{L} \mapsto \mathbb{L}^-$ is a $W_{\mathbb{G}}$–equivariant bijection from the set of all generic Levi subgroups of $\mathbb{G}$ onto the set of all generic Levi subgroups of $\mathbb{G}^-$.

Then it is clear that

$$\begin{gathered} \sigma^{\mathbb{G}} \cdot \mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} = \mathrm{Ind}_{\mathbb{L}^-}^{\mathbb{G}^-} \cdot \sigma^{\mathbb{L}} \quad , \quad \sigma^{\mathbb{L}} \cdot \mathrm{Res}_{\mathbb{L}}^{\mathbb{G}} = \mathrm{Res}_{\mathbb{L}^-}^{\mathbb{G}^-} \cdot \sigma^{\mathbb{G}} \, , \\ D(\mathrm{Ind}_{\mathbb{L}^-}^{\mathbb{G}^-} 1^{\mathbb{L}^-})(x) = D(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} 1^{\mathbb{L}})(-x) \; . \end{gathered} \tag{1.20}$$

Class functions on $\mathbb{G}^{(a)}$.

Let $a \in \mathbb{N}$. We have (see 1.3) $W_{\mathbb{G}^{(a)}} = (W_{\mathbb{G}})^a$, and it is clear that the map $(w_1, w_2, \dots, w_a)\phi^{(a)} \mapsto w_1 w_2 \cdots w_a \phi$ defines a bijection between the set of classes of $W_{\mathbb{G}^{(a)}}\phi^{(a)}$ under $W_{\mathbb{G}^{(a)}}$–conjugacy and the set of classes of $W_{\mathbb{G}}\phi$ under $W_{\mathbb{G}}$–conjugacy. Thus it induces an isometry

$$\sigma_{\mathbb{G}}^{(a)} \colon \mathcal{CF}(\mathbb{G}^{(a)}) \xrightarrow{\sim} \mathcal{CF}(\mathbb{G}) \, .$$

1.21. Proposition. *We have*

$$\begin{aligned} \sigma_{\mathbb{G}}^{(a)} \cdot \mathrm{Ind}_{\mathbb{L}^{(a)}}^{\mathbb{G}^{(a)}} &= \mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} \cdot \sigma_{\mathbb{L}}^{(a)} \\ \mathrm{Res}_{\mathbb{L}}^{\mathbb{G}} \cdot \sigma_{\mathbb{G}}^{(a)} &= \sigma_{\mathbb{L}}^{(a)} \cdot \mathrm{Res}_{\mathbb{L}^{(a)}}^{\mathbb{G}^{(a)}} \; . \end{aligned}$$

d–split Levi subgroups.

Let $\mathbb{T} = ((X,Y),\phi)$ be a generic torus. We recall that in this case the polynomial order $|\mathbb{T}|$ is just the characteristic polynomial of ϕ on Y. A generic Φ_d–group (*cf.* [BrMa], §3, abbreviated here by "Φ_d–group" or even by "d-group") is a generic torus whose polynomial order is a power of Φ_d. It is proved in [BrMa], 3.3 that a Φ_d–group is completely specified up to isomorphism by its polynomial order.

Remark. If $(\mathbf{T}, F)$ is an associated algebraic torus, the fact that $\mathbb{T}$ is a Φ_d–group translates to the fact that $\mathbf{T}$ splits "exactly" over $\mathbb{F}_{q^d}$ ("exactly" in the sense that no subtorus splits over a smaller field).

We call *d–split Levi subgroups* of $\mathbb{G}$ the centralizers in $\mathbb{G}$ of the Φ_d–subgroups of $\mathbb{G}$ ("centralizer", as in [BrMa], 1.3 means that their root system consists of the roots orthogonal to the considered subtorus). In particular, the centralizers of the Sylow Φ_d–subgroups are the minimal d–split Levi subgroups.

We shall use the following technical remark.

By [BrMa], proposition 3.8, we know that whenever $\mathbb{L}$ is a minimal d–split Levi subgroup of $\mathbb{G}$, we have $\varepsilon_{\mathbb{G}}\varepsilon_{\mathbb{L}}x^{N(\mathbb{G})-N(\mathbb{L})} \equiv 1 \mod \Phi_d$. It then follows that the preceding congruence holds for every d–split Levi subgroup of $\mathbb{G}$, from which we deduce by 1.16

(*1.22) For any d–split Levi subgroup $\mathbb{L}$ of $\mathbb{G}$, we have*

$$|\mathbb{G}|/|\mathbb{L}| \equiv D(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}} 1^{\mathbb{L}}) \mod \Phi_d\,.$$

The Mackey formula becomes particularly simple when we restrict ourselves to pairs of d–split Levi subgroups.

1.23. Proposition. *Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be two d–split Levi subgroups of $\mathbb{G}$.*

(1) *$\mathbb{M}_1 \cap \mathbb{M}_2$ is defined if and only if $\mathbb{M}_1$ and $\mathbb{M}_2$ contain a common Sylow Φ_d–subgroup of $\mathbb{G}$. In particular $\mathcal{S}_{W_{\mathbb{G}}}(\mathbb{M}_1,\mathbb{M}_2) \neq \emptyset$.*

(2) *If $\mathbb{M}_1 \cap \mathbb{M}_2$ is defined, then it is a d–split Levi subgroup of $\mathbb{G}$.*

(3) *Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be two d–split Levi subgroups containing the minimal d–split Levi subgroup $\mathbb{L}$. Then we have $\mathcal{S}_{W_{\mathbb{G}}}(\mathbb{M}_1,\mathbb{M}_2) = W_{\mathbb{M}_2} N_{W_{\mathbb{G}}}(\mathbb{L}) W_{\mathbb{M}_1}$. In particular, we have*

$$\mathrm{Res}^{\mathbb{G}}_{\mathbb{M}_2} \cdot \mathrm{Ind}^{\mathbb{G}}_{\mathbb{M}_1} = \sum_{w \in W_{\mathbb{M}_2}(\mathbb{L})\backslash W_{\mathbb{G}}(\mathbb{L})/W_{\mathbb{M}_1}(\mathbb{L})} \mathrm{Ind}^{\mathbb{M}_2}_{\mathbb{M}_2 \cap {}^w\mathbb{M}_1} \cdot \mathrm{Res}^{{}^w\mathbb{M}_1}_{\mathbb{M}_2 \cap {}^w\mathbb{M}_1} \cdot \mathrm{ad}(w)\,,$$

where, following the notation of [BrMa], *§1.B, for a Levi subgroup $\mathbb{L}$ of $\mathbb{G}$, we put $W_{\mathbb{G}}(\mathbb{L}) = N_{W_{\mathbb{G}}}(\mathbb{L})/W_{\mathbb{L}}$.*

Proof of 1.23. For a generic Levi subgroup $\mathbb{M}$ of $\mathbb{G}$, let us denote by $Z_d(\mathbb{M})$ the Sylow Φ_d–subgroup of its center. Then $\mathbb{M}$ is d–split if and only if $\mathbb{M} = C_{\mathbb{G}}(Z_d(\mathbb{M}))$.

(1) Suppose first that $\mathbb{M}_1 \cap \mathbb{M}_2$ is defined. By 1.9, there exists a generic maximal torus $\mathbb{T}$ of $\mathbb{G}$ contained in both $\mathbb{M}_1$ and $\mathbb{M}_2$. Then the generic groups $Z_d(\mathbb{M}_1)$ and $Z_d(\mathbb{M}_2)$ are both contained in the Sylow Φ_d–subgroup $\mathbb{T}_d$ of $\mathbb{T}$. Now if $\mathbb{S}$ is a Sylow Φ_d–subgroup of $\mathbb{G}$ containing $\mathbb{T}_d$, we see that $\mathbb{M}_1$ and $\mathbb{M}_2$ both contain $C_{\mathbb{G}}(\mathbb{S})$ hence $\mathbb{S}$. Conversely, if $\mathbb{M}_1$ and $\mathbb{M}_2$ both contain a minimal d–split Levi subgroup, *a fortiori* they both contain a maximal torus $\mathbb{T}$, thus $\mathbb{M}_1 \cap \mathbb{M}_2$ is defined by 1.9.

The last assertion in (1) follows from the Sylow theorems in [BrMa].

(2) Let $\mathbb{T}$ be a common maximal torus for both $\mathbb{M}_1$ and $\mathbb{M}_2$. Then the generic subgroup $Z_d(\mathbb{M}_1).Z_d(\mathbb{M}_2)$ of $\mathbb{T}$ is defined in an obvious way (*cf.* [BrMa], §3.E), and it is then clear that $\mathbb{M}_1 \cap \mathbb{M}_2 = C_{\mathbb{G}}(Z_d(\mathbb{M}_1).Z_d(\mathbb{M}_2))$.

(3) Set $\mathbb{S} = Z_d(\mathbb{L})$ (thus $\mathbb{S}$ is a Sylow Φ_d–subgroup of $\mathbb{G}$). Assume that $w \in \mathcal{S}_{W_{\mathbb{G}}}(\mathbb{M}_1, \mathbb{M}_2)$. Then by what precedes, there exists a Sylow Φ_d–subgroup $\mathbb{S}'$ of $\mathbb{G}$ such that $\mathbb{S}'$ is contained in both ${}^w\mathbb{M}_1$ and $\mathbb{M}_2$. By Sylow theorems (*cf.* [BrMa], th. 3.4), there exist $w_1 \in W_{\mathbb{M}_1}$ and $w_2 \in W_{\mathbb{M}_2}$ such that ${}^{w^{-1}}\mathbb{S}' = {}^{w_1}\mathbb{S}$ and $\mathbb{S}' = {}^{w_2}\mathbb{S}$, from which it follows that $w_2^{-1} w w_1 \in N_{W_{\mathbb{G}}}(\mathbb{S})$. The assertion now follows from the fact that $N_{W_{\mathbb{G}}}(\mathbb{S}) = N_{W_{\mathbb{G}}}(\mathbb{L})$ (*cf.* [BrMa], th. 3.4). □

Given a Levi subgroup
$$((X, R_{\mathbb{L}}, Y, R_{\mathbb{L}}^\vee), W_{\mathbb{L}} w\phi),$$
we define its image in $\mathbb{G}_{\text{ss}}$ (resp. in $\mathbb{G}_{\text{ad}}$) to be $((Q(R)^{\perp\perp}, R_{\mathbb{L}}, Y/Q(R)^{\perp}, R_{\mathbb{L}}^\vee), W_{\mathbb{L}} w\phi)$ (resp. $((Q(R), R_{\mathbb{L}}, P(R^\vee), R_{\mathbb{L}}^\vee), W_{\mathbb{L}} w\phi)$).

1.24. Remark. *A Levi subgoup is d-split if and only if its image in* $\mathbb{G}_{\text{ss}}$ *(resp. in* $\mathbb{G}_{\text{ad}}$*) is d–split.*

B. Generic characters.

Some consequences of Lusztig's results.

In §1.B, we introduce the formalism which is necessary to treat unipotent characters as generic objects, and to prove that related constructions such as Deligne–Lusztig induction are indeed defined at the "generic level".

For chosen q and ϕ, whence a triple $(\mathbf{G}, \mathbf{T}, F)$ (*cf.* §1.A), one has a well–defined bijection between the W–conjugacy classes of the coset $W\phi$ and the $\mathbf{G}^F$–conjugacy classes of F–stable maximal tori of $\mathbf{G}$ (*cf.* [BrMa], 2.1). Thus in particular, for $w\phi \in W\phi$, the virtual Deligne-Lusztig character $R_{\mathbf{T}_{w\phi}}^{\mathbf{G}}(1)$, denoted here $R_{w\phi}^{\mathbf{G}^F}$, is well defined.

The following result is a particular case of a more general theorem which will be proved later (see 1.32).

1.25. Theorem. *(Lusztig) There exists a finite set, denoted by* $\mathrm{Uch}(\mathbb{G})$*, and a map* $\mathrm{Uch}(\mathbb{G}) \to \mathcal{CF}(\mathbb{G})$*, denoted by* $\boldsymbol{\gamma} \mapsto m_{\boldsymbol{\gamma}}$*, with the following property: whenever* $(\mathbf{G}, \mathbf{T}, F)$ *is a* (q, ϕ)*–triple associated to* $\mathbb{G}$*, there is a bijection* $\boldsymbol{\gamma} \mapsto \rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}$ *from* $\mathrm{Uch}(\mathbb{G})$ *onto the set* $\mathcal{E}(\mathbf{G}^F, 1)$ *of unipotent characters of* $\mathbf{G}^F$ *such that for every* $w\phi \in W\phi$*, we have*
$$m_{\boldsymbol{\gamma}}(w\phi) = (\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}, R_{w\phi}^{\mathbf{G}^F})_{\mathbf{G}^F}\,.$$

Thus in particular

(1) for all $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})$, $m_{\boldsymbol{\gamma}}$ takes only integral values,

(2) by the known orthogonality relations on the virtual characters $R_{w\phi}^{\mathbf{G}^F}$, we have

$$\sum_{\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})} m_{\boldsymbol{\gamma}}(w\phi) m_{\boldsymbol{\gamma}}(w'\phi) = \begin{cases} |C_W(w\phi)| & \text{if } w\phi \text{ and } w'\phi \text{ are } W\text{–conjugate}\,, \\ 0 & \text{if not}\,. \end{cases}$$

Sketch of proof of theorem 1.25. Let $\mathrm{Fam}(W)$ be the set of families of ordinary irreducible representations of W. One knows (see [Lu1], chap. 4, and also [DiMi1], §4) how to associate to each element $\mathcal{F} \in \mathrm{Fam}(W)$ a well–defined finite group $\mathcal{G}(\mathcal{F})$.

1. Assume first that ϕ is chosen in $W\phi$. Let $\mathrm{Fam}(W)^\phi$ be the set of all ϕ–invariant families. The choice of ϕ defines, for each $\mathcal{F} \in \mathrm{Fam}(W)^\phi$, an automorphism $\phi_\mathcal{F}$ of $\mathcal{G}(\mathcal{F})$ (see [DiMi1], 4.1). We denote by $\mathrm{Uch}(\mathbb{G}, \phi)$ the union (for all $\mathcal{F} \in \mathrm{Fam}(W)^\phi$) of the sets of $\mathcal{G}(\mathcal{F})$–conjugacy classes of pairs $(g\phi_\mathcal{F}, \chi)$, where $g \in \mathcal{G}(\mathcal{F})$ and $\chi \in \mathrm{Irr}(C_{\mathcal{G}(\mathcal{F})}(g\phi_\mathcal{F}))$.

It is one of Lusztig's main theorems in [Lu1] (*cf.* theorem 4.23) that there is a map $\mathrm{Uch}(\mathbb{G}, \phi) \to \mathcal{CF}(\mathbb{G})$, say $\boldsymbol{\gamma} \mapsto m_{\boldsymbol{\gamma}}$, such that for each choice of a suitable q, there is a bijection $\mathrm{Uch}(\mathbb{G}, \phi) \to \mathcal{E}(\mathbf{G}^F, 1)$ with $m_{\boldsymbol{\gamma}}(w\phi) = \left(\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}, R_{w\phi}^{\mathbf{G}^F}\right)_{\mathbf{G}^F}$.

2. If ϕ' is another element of $W\phi$ (defining a triple $(\mathbf{G}, \mathbf{T}, F')$) there is a well–defined bijection $\sigma_{\phi,\phi'}$ from $\mathrm{Uch}(\mathbb{G}, \phi)$ onto $\mathrm{Uch}(\mathbb{G}, \phi')$, such that

$$\left(\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}, R_{w\phi}^{\mathbf{G}^F}\right)_{\mathbf{G}^F} = \left(\rho_{\sigma_{\phi,\phi'}(\boldsymbol{\gamma})}^{\mathbf{G}^{F'}}, R_{w\phi}^{\mathbf{G}^{F'}}\right)_{\mathbf{G}^{F'}} .$$

The system $(\mathrm{Uch}(\mathbb{G}, \phi), \sigma_{\phi,\phi'})_{\phi,\phi' \in W\phi}$ is projective. We define $\mathrm{Uch}(\mathbb{G})$ as the limit of this system. □

Remark. It results from [Lu1] chap. 4, that the set $\mathrm{Uch}(\mathbb{G})$, as well as the "multiplicities" $m_{\boldsymbol{\gamma}}$, depend only on the action of ϕ on $W_\mathbb{G}$ and on the order of ϕ on X. In particular, to compute the multiplicities we may reduce to the case where $X = Q(R)$ and R is irreducible. Also,

(1.26) any automorphism ϕ' of the generic group $\mathbb{G}$ which commutes with ϕ defines a permutation of the set $\mathrm{Uch}(\mathbb{G})$*, which depends only on the action of ϕ' on $W_\mathbb{G}$ and on the order of ϕ'.*

Remark. We refer to [DiMi1], 6.4, for what follows. The map $\boldsymbol{\gamma} \mapsto \rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}$ is uniquely determined by the condition $(\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}, R_{w\phi}^{\mathbf{G}^F})_{\mathbf{G}^F} = m_{\boldsymbol{\gamma}}(w\phi)$ for all rational unipotent characters $\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}$. This includes all unipotent characters in the case of classical groups. To ensure unicity in the case of exceptional groups for non-rational characters, the following supplementary conditions have to be added:

- To $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})$ there is associated a root of unity $\lambda_{\boldsymbol{\gamma}}$ (cf. [DiMi1], before 6.4 for the definition — we just mention that when $\phi_\mathcal{F}$ acts trivially on $\mathcal{G}(\mathcal{F})$ and $\boldsymbol{\gamma}$ is given by $(g\phi_\mathcal{F}, \chi)$ then $\lambda_{\boldsymbol{\gamma}} = \chi(g)/\chi(1)$). We ask that, if F^δ is the smallest split power of F, the eigenvalues of F^δ associated to $\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}$ in any Deligne-Lusztig variety of $\mathbf{G}$ be equal to $\lambda_{\boldsymbol{\gamma}}$ up to some power of $q^{\delta/2}$.
- To deal with the principal series unipotent characters corresponding to characters χ_q of degree 512 of the Hecke algebra of a group of type E_7 or of degree 4096 of the Hecke algebra of a group of type E_8, we must add another condition: Lusztig ([Lu1], chapter 4) identifies $\mathcal{F}$ to a subset of the couples $(g\phi_\mathcal{F}, \chi)$; the $\boldsymbol{\gamma}$ we are looking at is an element of $\mathcal{F}$ via this identification, which gives us a character $\chi_{\boldsymbol{\gamma}}$ of W. We ask that $\chi_{\boldsymbol{\gamma}}$ be the specialization for $q = 1$ of the character χ_q associated to $\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}$.

In what follows we will assume that these conditions are satisfied. This has the following consequence:

1.27. Proposition. *Let ϕ' be an automorphism of $\Gamma_{\mathbb{G}}$ which commutes with ϕ, and let $U(\phi')$ be the corresponding permutation of* $\mathrm{Uch}(\mathbb{G})$ *(cf. 1.26). Then $F_{q'\phi'}(\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}) = \rho_{U(\phi')(\boldsymbol{\gamma})}^{\mathbf{G}^F}$ for any power q' of p, and for $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})$.*

In particular we get that the action of $F_{q'\phi'}$ on $\mathcal{E}(\mathbf{G}^F, 1)$ does not depend on q'.

Generic unipotent functions, uniform functions.

Generic generalized unipotent functions.

We denote by $\mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$ the free $\mathbb{Q}[x]$–module on $\mathrm{Uch}(\mathbb{G})$, endowed with the quadratic form for which the canonical basis $\{\boldsymbol{\gamma}\}_{\boldsymbol{\gamma}\in\mathrm{Uch}(\mathbb{G})}$ is orthonormal. The scalar product of two elements $\boldsymbol{\psi}$ and $\boldsymbol{\psi}'$ of $\mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$ is denoted by $(\boldsymbol{\psi}, \boldsymbol{\psi}')_{\mathbb{G}}$. The elements of $\mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$ are called the *generic generalized unipotent functions*.

Generic uniform functions.

For $w\phi \in W\phi$, we set $\mathbf{R}_{w\phi}^{\mathbb{G}} := \sum_{\boldsymbol{\gamma}\in\mathrm{Uch}(\mathbb{G})} m_{\boldsymbol{\gamma}}(w\phi)\boldsymbol{\gamma}$. Thus $\mathbf{R}_{w\phi}^{\mathbb{G}}$ depends only on the $W_{\mathbb{G}}$–conjugacy class of $w\phi$, and the system $\{\mathbf{R}_{w\phi}^{\mathbb{G}}\}_{w\phi}$, where $w\phi$ runs over a set of representatives for the $W_{\mathbb{G}}$–conjugacy classes of $W_{\mathbb{G}}\phi$, is an orthogonal system in $\mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$ (by the remark (2) following theorem 1.25).

For all $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})$, we have $m_{\boldsymbol{\gamma}}(w\phi) = (\boldsymbol{\gamma}, \mathbf{R}_{w\phi}^{\mathbb{G}})_{\mathbb{G}}$ and more generally, for $\boldsymbol{\psi} \in \mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$, we set $m_{\boldsymbol{\psi}}(w\phi) = (\boldsymbol{\psi}, \mathbf{R}_{w\phi}^{\mathbb{G}})_{\mathbb{G}}$.

We call *generic uniform functions on* $\mathbb{G}$ the elements of $\mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$ which are $\mathbb{Q}[x]$–linear combinations of the $\mathbf{R}_{w\phi}^{\mathbb{G}}$.

For $\alpha \in \mathcal{CF}(\mathbb{G})$, we set

$$\mathbf{\Phi}_{\alpha}^{\mathbb{G}} := \frac{1}{|W|}\sum_{w\in W} \alpha(w\phi)\mathbf{R}_{w\phi}^{\mathbb{G}}\,, \tag{1.28}$$

so that $\alpha = m_{\mathbf{\Phi}_{\alpha}^{\mathbb{G}}}$. The linear map $\mathbf{\Phi}^{\mathbb{G}} : \mathcal{CF}(\mathbb{G}) \to \mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$ such that $\mathbf{\Phi}^{\mathbb{G}} : \alpha \mapsto \mathbf{\Phi}_{\alpha}^{\mathbb{G}}$ is an isometric embedding whose image is subspace of generic uniform functions.

The orthogonal projection from $\mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$ onto the subspace of uniform functions is denoted $\pi_u^{\mathbb{G}}$. Thus for all $\boldsymbol{\psi} \in \mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$, we have

$$\pi_u^{\mathbb{G}}(\boldsymbol{\psi}) = \frac{1}{|W|}\sum_{w\in W} (\boldsymbol{\psi}, \mathbf{R}_{w\phi}^{\mathbb{G}})_{\mathbb{G}} \cdot \mathbf{R}_{w\phi}^{\mathbb{G}}\,, \tag{1.29}$$

and $\pi_u^{\mathbb{G}}(\boldsymbol{\psi}) = \mathbf{\Phi}_{m_{\boldsymbol{\psi}}}^{\mathbb{G}}$.

Generic degrees.

For all $w \in W_{\mathbb{G}}$, we set (see §1.A) $\mathrm{Deg}(\mathbf{R}_{w\phi}^{\mathbb{G}}) := \mathrm{tr}_{R\mathbb{G}}(w\phi)$. We extend linearly the map Deg to all uniform functions, by setting $\mathrm{Deg}(\mathbf{\Phi}_{\alpha}^{\mathbb{G}}) := <\alpha, \mathrm{tr}_{R\mathbb{G}}>_{\mathbb{G}}$, and then to all of $\mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$ by composition with $\pi_u^{\mathbb{G}}$ (this corresponds to the fact that, for the actual finite reductive groups, the degree of a character equals the degree of its uniform projection). Thus we have

$$\mathrm{Deg}(\boldsymbol{\psi}) := \mathrm{Deg}\pi_u^{\mathbb{G}}(\boldsymbol{\psi}) = \frac{1}{|W|}\sum_{w\in W} m_{\boldsymbol{\psi}}(w\phi)\mathrm{Deg}(\mathbf{R}_{w\phi}^{\mathbb{G}}) = \mathrm{Deg}(\mathbf{\Phi}_{m_{\boldsymbol{\psi}}}^{\mathbb{G}})\,, \tag{1.30}$$

and we see in particular that for $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})$, we have $\mathrm{Deg}(\boldsymbol{\gamma}) \in \dfrac{1}{|W|}\mathbb{Z}[x]$.

We shall need the following more precise statement due to Lusztig:

1.31. Theorem. *There exists an integer c divisible only by bad primes for $\mathbb{G}$ (cf. 1.7) such that* $\operatorname{Deg}(\boldsymbol{\gamma}) \in \frac{1}{c}\mathbb{Z}[x]$.

Proof. Let $\{\tilde{E}\}_{E\in\operatorname{Irr}(W)^\phi}$ be an orthonormal basis formed from the restriction to $W\phi$ of one extension to $W\langle\phi\rangle$ of each ϕ-invariant irreducible character of W. Then $\operatorname{Deg}(\boldsymbol{\Phi}^{\mathbb{G}}_{\tilde{E}}) = \langle\tilde{E}, \operatorname{tr}_{R\mathbb{G}}\rangle$ is in $\mathbb{Z}[x]$, and $\operatorname{Deg}\boldsymbol{\gamma} = \sum_E (\boldsymbol{\Phi}^{\mathbb{G}}_{\tilde{E}}, \boldsymbol{\gamma})_{\mathbb{G}} \operatorname{Deg}(\boldsymbol{\Phi}^{\mathbb{G}}_{\tilde{E}})$; the theorem then follows from the formula given in [Lu1], 4.26 which implies that $(\boldsymbol{\Phi}^{\mathbb{G}}_{\tilde{E}}, \boldsymbol{\gamma})_{\mathbb{G}}$ are rational numbers whose denominators are divisible only by bad primes for $\mathbb{G}$.

□

Remark.

Whenever (q,ϕ) is chosen, we have $\operatorname{Deg}(\boldsymbol{\gamma})(q) = \rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}(1)$.

Indeed, it follows from 1.30 that $\operatorname{Deg}(\boldsymbol{\gamma})(q)$ is equal to the scalar product of $\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}$ with the character $\frac{1}{|W|}\sum_{w\in W} \operatorname{Deg}(R_{w\phi}^{\mathbf{G}^F}(1)) R_{w\phi}^{\mathbf{G}^F}(1)$, which is the unipotent projection of the regular character of $\mathbf{G}^F$.

The polynomial $\operatorname{Deg}(\boldsymbol{\psi})$ is called the *generic degree* of $\boldsymbol{\psi}$.

The generic Deligne-Lusztig induction and restriction.

If $\mathbb{L} = ((X, R_{\mathbb{L}}, Y, R_{\mathbb{L}}^\vee), W_{\mathbb{L}}w\phi)$ is a Levi subgroup of $\mathbb{G}$, a choice of $\phi' \in W_{\mathbb{G}}\phi$ is said to be $\mathbb{L}$–adapted if $\phi' \in W_{\mathbb{L}}w\phi$. The choice of such a ϕ' in the coset $W_{\mathbb{G}}\phi$ and any choice of q determines, as explained below 1.1, a quadruple $(\mathbf{G}, \mathbf{L}, \mathbf{T}, F)$ where $\mathbf{G}$ is a connected reductive algebraic group over $\bar{\mathbb{F}}_p$, $F\colon \mathbf{G}\to\mathbf{G}$ is a surjective endomorphism, $\mathbf{L}$ is an F–stable Levi complement of some parabolic subgroup of $\mathbf{G}$ and $\mathbf{T}$ is an F–stable maximal torus of $\mathbf{L}$.

The following theorem translates into our language the known fact that the Deligne-Lusztig induction is "generic".

1.32. Theorem. *For each Levi subgroup $\mathbb{L}$ of $\mathbb{G}$ there exists a linear map*

$$R_{\mathbb{L}}^{\mathbb{G}}\colon \mathbb{Q}[x]\operatorname{Uch}(\mathbb{L}) \to \mathbb{Q}[x]\operatorname{Uch}(\mathbb{G})$$

with the following properties:

(1) *whenever $\boldsymbol{\lambda} \in \operatorname{Uch}(\mathbb{L})$ and $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda}) = \sum_{\boldsymbol{\gamma}\in\operatorname{Uch}(\mathbb{G})} n_{\boldsymbol{\gamma}}\boldsymbol{\gamma}$, then $n_{\boldsymbol{\gamma}} \in \mathbb{Z}$, and for any $\mathbb{L}$–adapted choice of ϕ, $R_{\mathbf{L}}^{\mathbf{G}}(\rho_{\boldsymbol{\lambda}}^{\mathbf{L}^F}) = \sum_{\boldsymbol{\gamma}\in\operatorname{Uch}(\mathbb{G})} n_{\boldsymbol{\gamma}}\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}$.*

(2) *$R_{\mathbb{L}}^{\mathbb{G}}$ sends the generic uniform functions on $\mathbb{L}$ into the generic uniform functions on $\mathbb{G}$ and induces $\operatorname{Ind}_{\mathbb{L}}^{\mathbb{G}}$ on the level of class functions, i.e., we have $R_{\mathbb{L}}^{\mathbb{G}}\cdot\pi_u^{\mathbb{L}} = \pi_u^{\mathbb{G}}\cdot R_{\mathbb{L}}^{\mathbb{G}}$ and $R_{\mathbb{L}}^{\mathbb{G}}\cdot\boldsymbol{\Phi}^{\mathbb{L}} = \boldsymbol{\Phi}^{\mathbb{G}}\cdot\operatorname{Ind}_{\mathbb{L}}^{\mathbb{G}}$.*

(3) *If $\mathbb{M}$ is a Levi subgroup of $\mathbb{G}$ and if $\mathbb{L}$ is a Levi subgroup of $\mathbb{M}$, then $R_{\mathbb{L}}^{\mathbb{G}} = R_{\mathbb{M}}^{\mathbb{G}}\cdot R_{\mathbb{L}}^{\mathbb{M}}$.*

Sketch of proof of 1.32.

Notice that once (1) is proved, (2) and (3) are just translations of known properties of the map $R_{\mathbf{L}}^{\mathbf{G}}$. We prove (1).

Let $(\mathbf{G}, F)$ be associated with $\mathbb{G}$ as above.

There exists an embedding $\pi : \mathbf{G}\to\tilde{\mathbf{G}}$ where $\tilde{\mathbf{G}}$ has a connected center and $\pi(\mathbf{G})$ contains the derived group of $\tilde{\mathbf{G}}$. Let $\tilde{\mathbf{L}} = \pi(\mathbf{L})$, and let $\pi_{\mathbf{L}}$ be the restriction of π

to $\mathbf{L}$. Since for any $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})$ we have $\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F} = \rho_{\boldsymbol{\gamma}}^{\tilde{\mathbf{G}}^F} \cdot \pi$ (*cf.* [DiMi2], 13.20), and $R_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}(\rho_{\boldsymbol{\gamma}}^{\tilde{\mathbf{L}}^F}) \cdot \pi = R_{\mathbf{L}}^{\mathbf{G}}(\rho_{\boldsymbol{\gamma}}^{\tilde{\mathbf{L}}^F} \cdot \pi_{\mathbf{L}})$ (*cf.* [DiMi2], 13.22), it is enough to prove the theorem for $\tilde{\mathbf{G}}$, *i.e.,* for a group with connected center. Applying the same arguments to the quotient map $\tilde{\mathbf{G}} \to \tilde{\mathbf{G}}_{\mathrm{ad}}$ (where $\tilde{\mathbf{G}}_{\mathrm{ad}}$ is the adjoint group) which has a connected kernel (so [DiMi2], 13.22, still applies), we may assume $\mathbf{G}$ adjoint.

When $\mathbf{G}$ is adjoint, $(\mathbf{G}, F)$ is a direct product of groups of type $(\mathbf{G}_i^{(a_i)}, F^{(a_i)})$ (restriction of scalars from $\mathbb{F}_{q^{a_i}}$ to $\mathbb{F}_q$ of $(\mathbf{G}_i, F)$) where $\mathbf{G}_i$ is simple. Since all our constructions behave nicely with respect to products, we may assume that the group is of the form $(\mathbf{G}^{(a)}, F^{(a)})$ with $\mathbf{G}$ simple.

Now the isomorphisms $\mathbf{G}^{(a)F^{(a)}} \simeq \mathbf{G}^{F_a}$ and $\mathbf{L}^{(a)F^{(a)}} \simeq \mathbf{L}^{F_a}$ (see 1.4) "commute" with $R_{\mathbf{L}}^{\mathbf{G}}$, and map isomorphically $\mathcal{E}(\mathbf{G}^{(a)F^{(a)}}, 1)$ to $\mathcal{E}(\mathbf{G}^{F_a}, 1)$. So we can reduce our problem to the case of $(\mathbf{G}, F_a)$, *i.e.,* to the case where $\mathbf{G}$ is simple. We will use results of Shoji [Sho1], [Sho2] in this case.

Let m be sufficiently divisible so that $(\mathbf{G}, F^m)$ is split, *i.e.,* the generic group $\mathbb{G}_0$ associated to it is such that $\phi = 1$. It follows from 1.27 that there exists a map $\mathrm{U}(\phi)$ on $\mathrm{Uch}(\mathbb{G}_0)$ such that $F(\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^{F^m}}) = \rho_{\mathrm{U}(\phi)(\boldsymbol{\gamma})}^{\mathbf{G}^{F^m}}$. Moreover, the set of fixed points $\mathrm{Uch}(\mathbb{G}_0)^{\mathrm{U}(\phi)}$ has the same cardinality as $\mathrm{Uch}(\mathbb{G})$. In this context, for any $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G}_0)^{\mathrm{U}(\phi)}$, Shoji ([Sho2], 3.1.2 and 3.1.3) defines an element of $\mathbb{Q}\,\mathrm{Uch}(\mathbb{G})$ (that he calls R_x or $R_{\hat{x}}$ depending on the case), which we shall denote here by $\mathrm{Sh}^0\boldsymbol{\gamma}$ (here Sh stands for "Shintani" — see [DiMi1] for the notation used here, such as $\mathrm{Sh}_{F^m/F}$), such that:

- $\boldsymbol{\gamma} \mapsto \mathrm{Sh}^0\boldsymbol{\gamma}$ is an isometry, and the set $\{\mathrm{Sh}^0\boldsymbol{\gamma}\,;\,(\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G}_0)^{\mathrm{U}(\phi)})\}$ spans $\mathbb{Q}\,\mathrm{Uch}(\mathbb{G})$,
- For m divisible enough, there is a well-defined $\mathbf{G}^{F^m}$–class function $\tilde{\rho}_{\boldsymbol{\gamma}}$ on $\mathbf{G}^{F^m}.F$ (whose values are the product of the value of a suitable extension of $\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^{F^m}}$ to the group $\mathbf{G}^{F^m} \cdot <F>$ by a root of unity) such that $\mathrm{Sh}_{F^m/F}(\tilde{\rho}_{\boldsymbol{\gamma}}) = \rho_{\mathrm{Sh}^0\boldsymbol{\gamma}}^{\mathbf{G}^F}$ (where this last function is defined by linearity by extending the notation $\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}$).

We shall give a formula for $R_{\mathbf{L}}^{\mathbf{G}}$ in the basis $\rho_{\mathrm{Sh}^0\boldsymbol{\gamma}}^{\mathbf{G}^F}$ which will make clear it is generic.

Let m be divisible enough so that $(\mathbf{L}, F^m)$ is split and let $\mathbb{L}_0$ be the corresponding generic group. Let $\boldsymbol{\lambda} \in \mathrm{Uch}(\mathbb{L}_0)^{\mathrm{U}(\phi)}$. By Harish-Chandra theory, there exists a Levi subgroup $\mathbf{M}$ of $\mathbf{L}$, such that $(\mathbf{M}, F^m)$ is split, and an element $\boldsymbol{\mu} \in \mathrm{Uch}(\mathbb{M}_0)$ (where $\mathbb{M}_0$ is the generic group corresponding to $(\mathbf{M}, F^m)$) such that $\rho_{\boldsymbol{\mu}}^{\mathbf{M}^{F^m}}$ is cuspidal and $\rho_{\boldsymbol{\lambda}}^{\mathbf{L}^{F^m}}$ is a constituent of $R_{\mathbf{M}}^{\mathbf{L}}(\rho_{\boldsymbol{\mu}}^{\mathbf{M}^{F^m}})$. Furthermore, since $\rho_{\boldsymbol{\lambda}}^{\mathbf{L}^{F^m}}$ is F-stable, we may choose $(\mathbf{M}, \rho_{\boldsymbol{\mu}}^{\mathbf{M}^{F^m}})$ to be F-stable. Let $\mathbb{M}$ be a generic group corresponding to $(\mathbf{M}, F)$ and let $W_{\mathbb{M}}v\phi$ be the corresponding coset. Set $W_{\mathbb{G}_0}(\mathbb{M}_0, \boldsymbol{\mu}) := N_{W_{\mathbb{G}_0}}(\mathbb{M}_0, \boldsymbol{\mu})/W_{\mathbb{M}_0}$. Then the F-stability of $\boldsymbol{\mu}$ implies a natural action of $v\phi$ on $W_{\mathbb{G}_0}(\mathbb{M}_0, \boldsymbol{\mu})$ and on $W_{\mathbb{L}_0}(\mathbb{M}_0, \boldsymbol{\mu})$, and the extension $\tilde{\rho}_{\boldsymbol{\lambda}}$ corresponds naturally to a function $\chi_{\boldsymbol{\lambda}}$ on $W_{\mathbb{L}_0}(\mathbb{M}_0, \boldsymbol{\mu})v\phi$ (an extension of the $v\phi$-invariant irreducible character of $W_{\mathbb{L}_0}(\mathbb{M}_0, \boldsymbol{\mu})$ corresponding to $\rho_{\boldsymbol{\lambda}}^{\mathbf{L}^{F^m}}$ by Howlett–Lehrer–Lusztig theory). Conversely any $W_{\mathbb{L}_0}(\mathbb{M}_0, \boldsymbol{\mu})$-invariant function on $W_{\mathbb{L}_0}(\mathbb{M}_0, \boldsymbol{\mu})v\phi$ corresponds to a linear combination of extensions $\tilde{\rho}_{\boldsymbol{\lambda}}$.

It is proved in [DiMi3], 9.7, that (assuming Shoji's results)

$$R_{\mathbf{L}}^{\mathbf{G}}(\rho_{\mathrm{Sh}^0\boldsymbol{\lambda}}^{\mathbf{L}^F}) = \rho_{\mathrm{Sh}^0\boldsymbol{\lambda}'}^{\mathbf{G}^F}$$

where $\boldsymbol{\lambda}'$ is the linear combination of extensions $\tilde\rho_{\boldsymbol\gamma}$ corresponding to the function $\mathrm{Ind}_{W_{\mathbb{L}_0}(\mathbb{M}_0,\boldsymbol{\mu})v\phi}^{W_{\mathbb{G}_0}(\mathbb{M}_0,\boldsymbol{\mu})v\phi}\chi_{\boldsymbol\lambda}$. $\square$

Note that

• $\mathrm{Uch}(\mathbb{T}_{w\phi})$ consists of only one element denoted by $\mathbf{1}$, and $R_{\mathbb{T}_{w\phi}}^{\mathbb{G}}(\mathbf{1}) = \mathbf{R}_{w\phi}^{\mathbb{G}}$, hence (by 1.32, (3)), for all $w'\phi \in W_{\mathbb{L}}w\phi$, we have $R_{\mathbb{L}}^{\mathbb{G}}(\mathbf{R}_{w'\phi}^{\mathbb{L}}) = \mathbf{R}_{w'\phi}^{\mathbb{G}}$,

• similarly, the image of $1^{\mathbb{L}}$ by the isometric embedding $\boldsymbol{\Phi}^{\mathbb{G}}$ of 1.28 is denoted by $\mathbf{1}^{\mathbb{L}}$ or just $\mathbf{1}$; we have $\mathrm{Deg}(R_{\mathbb{L}}^{\mathbb{G}}(\mathbf{1})) = D(\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}1^{\mathbb{L}})$ (as promised after 1.16).

1.33. Proposition. *For all* $\boldsymbol\zeta \in \mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{L})$, *we have*

$$\mathrm{Deg}R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol\zeta) = \mathrm{Deg}(R_{\mathbb{L}}^{\mathbb{G}}(\mathbf{1}))\mathrm{Deg}(\boldsymbol\zeta)\,.$$

Proof. This is an immediate consequence of 1.19 and of the formula $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\Phi}_\beta^{\mathbb{L}}) = \boldsymbol{\Phi}^{\mathbb{G}}_{\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}\beta}$, for all $\beta \in \mathcal{CF}(\mathbb{L})$. $\square$

Taking the adjoint map of $R_{\mathbb{L}}^{\mathbb{G}}$, we get a map ${}^*R_{\mathbb{L}}^{\mathbb{G}}\colon \mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G}) \to \mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{L})$ which is in particular such that

• for all $\alpha \in \mathcal{CF}(\mathbb{G})$, we have ${}^*R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\Phi}_\alpha^{\mathbb{G}}) = \boldsymbol{\Phi}^{\mathbb{L}}_{\mathrm{Res}_{\mathbb{L}}^{\mathbb{G}}\alpha}$.

• for all $w\phi \in W_{\mathbb{G}}\phi$, we have ${}^*R_{\mathbb{T}_{w\phi}}^{\mathbb{G}}(\boldsymbol\gamma) = m_{\boldsymbol\gamma}(w\phi)\mathbf{1}$.

The following theorem, which extends the Mackey formula (see 1.13) to the preceding context, is a consequence of an unpublished result of Deligne (see [DiMi2], 11.6) and of 1.32, (1) and 1.23, (3).

1.34. Theorem.

(1) *Let* $\mathbb{L}$ *and* $\mathbb{M}$ *be two Levi subgroups of* $\mathbb{G}$. *Then*

$${}^*R_{\mathbb{M}}^{\mathbb{G}} \cdot R_{\mathbb{L}}^{\mathbb{G}} = \sum_{w\in W_{\mathbb{M}}\backslash \mathcal{S}_{W_{\mathbb{G}}}(\mathbb{L},\mathbb{M})/W_{\mathbb{L}}} R_{\mathbb{M}\cap {}^w\mathbb{L}}^{\mathbb{M}} \cdot {}^*R_{\mathbb{M}\cap {}^w\mathbb{L}}^{{}^w\mathbb{L}} \cdot \mathrm{ad}(w)\,.$$

(2) *Suppose that* $\mathbb{M}_1$ *and* $\mathbb{M}_2$ *are two d–split Levi subgroups containing the minimal d–split Levi subgroup* $\mathbb{L}$. *Then*

$${}^*R_{\mathbb{M}_2}^{\mathbb{G}} \cdot R_{\mathbb{M}_1}^{\mathbb{G}} = \sum_{w\in W_{\mathbb{M}_2}(\mathbb{L})\backslash W_{\mathbb{G}}(\mathbb{L})/W_{\mathbb{M}_1}(\mathbb{L})} R_{\mathbb{M}_2\cap {}^w\mathbb{M}_1}^{\mathbb{M}_2} \cdot {}^*R_{\mathbb{M}_2\cap {}^w\mathbb{M}_1}^{{}^w\mathbb{M}_1} \cdot \mathrm{ad}(w)\,.$$

(Indeed, Deligne proved the Mackey formula for actual finite reductive groups under the assumption that q is large. Thus 1.34 results from the genericity of $R_{\mathbb{L}}^{\mathbb{G}}$.)

The arguments at the beginning of the proof of 1.32, together with 1.27, prove the following two propositions that we will use later to reduce some questions to the case of simple groups.

Going to the adjoint group.

1.35. Proposition. *Let* $\mathbb{G} = ((X, R, Y, R^\vee), W\phi)$ *be a generic group, and let* $\mathbb{L} = ((X, R', Y, R'^\vee), W_{R'}w\phi)$ *be a generic Levi subgroup of* $\mathbb{G}$. *The sets* $\mathrm{Uch}(\mathbb{G})$ *and* $\mathrm{Uch}(\mathbb{G}_{\mathrm{ad}})$ *may be identified (cf. remark before 1.26), and similarly the sets* $\mathrm{Uch}(\mathbb{L})$ *and* $\mathrm{Uch}(\overline{\mathbb{L}})$, *where* $\overline{\mathbb{L}}$ *is the generic Levi subgroup of* $\mathbb{G}_{\mathrm{ad}}$ *defined by* $\overline{\mathbb{L}} = ((Q(R), R', P(R^\vee), R'^\vee), W_{R'}w\phi)$. *Then the following diagram is commutative*

$$\begin{array}{ccc} \mathbb{Z}\,\mathrm{Uch}(\mathbb{G}) & \xrightarrow{\sim} & \mathbb{Z}\,\mathrm{Uch}(\mathbb{G}_{\mathrm{ad}}) \\ {\scriptstyle R_{\mathbb{L}}^{\mathbb{G}}}\uparrow & & \uparrow{\scriptstyle R_{\overline{\mathbb{L}}}^{\mathbb{G}_{\mathrm{ad}}}} \\ \mathbb{Z}\,\mathrm{Uch}(\mathbb{L}) & \xrightarrow{\sim} & \mathbb{Z}\,\mathrm{Uch}(\overline{\mathbb{L}}) \end{array}.$$

Remark. If $\mathbf{G}$ is the algebraic group associated to $\mathbb{G}$, and $\pi : \mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$ is the natural isogeny, and if $\mathbf{L}$ is associated to $\mathbb{L}$, then $\pi(\mathbf{L})$ is associated to $\overline{\mathbb{L}}$.

Lifting scalars.

1.36. Proposition. *Let* $\mathbb{G}$ *be a generic finite reductive group and let* $a \in \mathbb{N}$. *For all* $\mathbb{G}$ *there is a natural identification* $\sigma_{\mathbb{G}}^{(a)} : \boldsymbol{\gamma} \mapsto \boldsymbol{\gamma}^{(a)}$ *between* $\mathrm{Uch}(\mathbb{G})$ *and* $\mathrm{Uch}(\mathbb{G}^{(a)})$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathbb{Z}\,\mathrm{Uch}(\mathbb{G}) & \xrightarrow{\sigma_{\mathbb{G}}^{(a)}} & \mathbb{Z}\,\mathrm{Uch}(\mathbb{G}^{(a)}) \\ {\scriptstyle R_{\mathbb{L}}^{\mathbb{G}}}\uparrow & & \uparrow{\scriptstyle R_{\mathbb{L}^{(a)}}^{\mathbb{G}^{(a)}}} \\ \mathbb{Z}\,\mathrm{Uch}(\mathbb{L}) & \xrightarrow{\sigma_{\mathbb{L}}^{(a)}} & \mathbb{Z}\,\mathrm{Uch}(\mathbb{L}^{(a)}) \end{array}.$$

We will use the following result, whose proof is based on an argument given by Geck [Ge] in the case of a torus.

1.37. Theorem. *Let* $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})$ *and* $\mathbb{L}$ *be a Levi subgroup of* $\mathbb{G}$. *Then the polynomial* $\mathrm{Deg}(\boldsymbol{\gamma})$ *divides* $\dfrac{|\mathbb{G}|}{|\mathbb{L}|}\mathrm{Deg}({}^*R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\gamma}))$ *(in* $\mathbb{Q}[x]$*).*

Proof. Let $(\mathbf{G}, \mathbf{L}, \mathbf{T}, F)$ be a quadruple associated to $\mathbb{G}$ and $\mathbb{L}$ for a choice of q. Whenever q is large enough, there exists some regular element $s \in Z^o(\mathbf{L})^F$, *i.e.,* an element such that $C_{\mathbf{G}}(s) = \mathbf{L}$. Then by the "Curtis–type formula" (see, e.g. [DiMi2], 12.5) we have $({}^*R_{\mathbf{L}}^{\mathbf{G}}(\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}))(s) = (\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F})(s)$. On the other hand, since ${}^*R_{\mathbf{L}}^{\mathbf{G}}(\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F})$ is a sum of unipotent characters and $s \in Z(\mathbf{L})$, we have $({}^*R_{\mathbf{L}}^{\mathbf{G}}(\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}))(s) = ({}^*R_{\mathbf{L}}^{\mathbf{G}}(\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}))(1)$. Thus, using that for any s the expression $\dfrac{|\mathbf{G}^F/C_{\mathbf{G}^F}(s)|(\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F})(s)}{(\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F})(1)}$ is an algebraic integer, we get that $\dfrac{|\mathbf{G}^F/\mathbf{L}^F|({}^*R_{\mathbf{L}}^{\mathbf{G}}(\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}))(1)}{(\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F})(1)}$ is an integer. Since this is true for an infinity of values of q, we must have divisibility of the corresponding polynomials in $\mathbb{Q}[x]$, whence the theorem.

□

When $\mathbb{L}$ is a maximal torus $\mathbb{T}_{w\phi}$, we get

1.38.. *Whenever* $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})$ *and* $w\phi \in W_{\mathbb{G}}\phi$ *are such that* $m_{\boldsymbol{\gamma}}(w\phi) \neq 0$*, then the polynomial* $\mathrm{Deg}(\boldsymbol{\gamma})$ *divides* $x^{N(\mathbb{G})}\mathrm{Deg}(\mathbf{R}^{\mathbb{G}}_{w\phi})$ *(in* $\mathbb{Q}[x]$*).*

This was first proved in Boyce [Boy].

2. d–cuspidality and the uniform theory

A. d–cuspidality.

In §2.A, we define the notion of d–cuspidality of unipotent class functions as the generalization of the usual notion of cuspidality (which must be viewed as the 1*-cuspidality).*

Let $\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}$ be a unipotent character of $\mathbf{G}^F$. We say that $\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}$ is *d–cuspidal* if, whenever $\mathbf{T}$ is a maximal F–stable torus of $\mathbf{G}$ such that ${}^*R^{\mathbf{G}}_{\mathbf{T}}(\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}) \neq 0$, then the Sylow Φ_d–subgroup of $\mathbf{T}$ is contained in $Z(\mathbf{G})$. The translation in terms of generic groups (independent of q) is achieved by the following definition.

2.1. Definition.

(1) *A class function* α *on* $\mathbb{G}$ *is d–cuspidal if whenever* $w\phi \in W\phi$ *is such that* $\alpha(w\phi) \neq 0$*, then* $\ker(\Phi_d(w\phi))$ *is orthogonal to all the roots of* $\mathbb{G}$*.*

(2) $\boldsymbol{\psi} \in \mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$ *is said to be d–cuspidal if the corresponding class function* $m_{\boldsymbol{\psi}}$ *is d–cuspidal, i.e., whenever* $\mathbb{T}$ *is a maximal torus of* $\mathbb{G}$ *such that* ${}^*R^{\mathbb{G}}_{\mathbb{T}}(\boldsymbol{\psi}) \neq 0$*, then the Sylow* Φ_d*–subgroup of* $\mathbb{T}$ *is contained in* $\mathrm{Rad}(\mathbb{G})$*.*

Remarks.

1. Note that for $d = 1$ we get the usual notion of cuspidality for the elements of $\mathrm{Uch}(\mathbb{G})$ (*i.e.,* $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})$ is 1-cuspidal if and only if $\rho_{\boldsymbol{\gamma}}^{\mathbf{G}^F}$ is a cuspidal character for any associated group $\mathbf{G}^F$). Indeed, condition (2) above can be reformulated as follows: " ${}^*R^{\mathbb{G}}_{\mathbb{T}}(\boldsymbol{\psi}) = 0$ whenever $\mathbb{T}$ contains some non-central Φ_d-subtorus $\mathbb{S}$". If $\mathbb{L}$ is the centralizer of $\mathbb{S}$ in $\mathbb{G}$, condition (2) becomes: "whenever $\mathbb{L}$ is a proper d-split Levi subgroup of $\mathbb{G}$, the uniform projection of the function ${}^*R^{\mathbb{G}}_{\mathbb{L}}(\boldsymbol{\psi})$ is zero" (in particular, being d-cuspidal is a property of $\pi_u(\boldsymbol{\psi})$). Now, for $d = 1$ and $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})$ the "positivity" of ordinary Harish-Chandra restriction shows that if ${}^*R^{\mathbb{G}}_{\mathbb{L}}(\boldsymbol{\gamma}) \neq 0$, then its degree is non zero, and so its uniform projection cannot vanish; thus we see that $\boldsymbol{\gamma}$ is 1–cuspidal if and only if ${}^*R^{\mathbb{G}}_{\mathbb{L}}(\boldsymbol{\psi}) = 0$ for any proper 1-split Levi subgroup, which is the usual notion of cuspidality.

2. In the case where $\mathbb{G} = \mathbb{GL}_n$, then $W = \mathfrak{S}_n$, $\phi = 1$, and a d–cuspidal class function on $\mathfrak{S}_n$ is a function which vanishes on all elements whose cycle decomposition contains a cycle of length a multiple of d.

Moreover, $\mathrm{Uch}(\mathbb{G})$ may be viewed as the set of partitions of n. Thus it results from the Murnaghan–Nakayama formula (see for example [JaKe]) that $\boldsymbol{\gamma}$ is d–cuspidal if and only if the partition of n corresponding to $\boldsymbol{\gamma}$ has no d–hook (*i.e.,* the d–cuspidal partitions are the d–cores of length n). Thus this paragraph may be considered as the generalization to all Weyl groups of the "yoga" of hooks for the symmetric groups.

2.2. Definition. *Let* $\mathbb{L}$ *be a d–split Levi subgroup of* $\mathbb{G}$*. We denote by* $\mathcal{CF}_d(\mathbb{G},\mathbb{L})$ *the set of all class functions* α *on* $\mathbb{G}$ *with the following property: if* $w\phi \in W_{\mathbb{G}}\phi$ *is such* *that* $\alpha(w\phi) \neq 0$*, then there exists* $w' \in W_{\mathbb{G}}$ *such that* $\ker \Phi_d(w\phi)^{w'}$ *is orthogonal to all roots of* $\mathbb{L}$*.*

Remarks.

- Let $\boldsymbol{\psi} \in \mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$. Then $m_{\boldsymbol{\psi}} \in \mathcal{CF}_d(\mathbb{G},\mathbb{L})$ is equivalent to the following: whenever $\mathbb{T}$ is a maximal torus of $\mathbb{G}$ such that ${}^*R_{\mathbb{T}}^{\mathbb{G}}(\boldsymbol{\psi}) \neq 0$, then the maximal Φ_d–subgroup of $\mathbb{T}$ is contained in a $W_{\mathbb{G}}$–conjugate of $\mathrm{Rad}(\mathbb{L})$.
- $\mathcal{CF}_d(\mathbb{G},\mathbb{G})$ is the set of all d–cuspidal functions,
- if $\mathbb{L}$ is the centralizer of a Sylow Φ_d–subgroup of $\mathbb{G}$, then $\mathcal{CF}_d(\mathbb{G},\mathbb{L}) = \mathcal{CF}(\mathbb{G})$.

2.3. Proposition. *Let* $\mathbb{L}$ *be a* d*–split Levi subgroup of* $\mathbb{G}$*.*

(1) *Let* $\alpha \in \mathcal{CF}_d(\mathbb{G},\mathbb{L})$*. Then* $\mathrm{Res}_{\mathbb{L}}^{\mathbb{G}}\,\alpha$ *is* d*–cuspidal.*

(2) *Let* β *be a* d*–cuspidal class function on* $\mathbb{L}$*. Then* $\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}\beta \in \mathcal{CF}_d(\mathbb{G},\mathbb{L})$*, and*

$$\mathrm{Res}_{\mathbb{L}}^{\mathbb{G}}\,\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}\beta = \sum_{w\in W_{\mathbb{G}}(\mathbb{L})} {}^{w}\beta\,.$$

Proof of 2.3. The first assertion is obvious. The second assertion is an application of formula 1.11. □

Generic degrees and d*–cuspidality.*

For a polynomial $P \in \mathbb{Q}[x]$, we denote by P_d the largest power of the cyclotomic polynomial Φ_d which divides P (*i.e.,* the Φ_d–part of P).

2.4. Proposition. *If* $\boldsymbol{\psi} \in \mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$ *is* d*–cuspidal, then* $|\mathbb{G}_{\mathrm{ss}}|_d$ *divides* $\mathrm{Deg}(\boldsymbol{\psi})$*.*

Proof of 2.4. We have by 1.30

$$\mathrm{Deg}(\boldsymbol{\psi}) = \frac{1}{|W|}\sum_{w\in W} m_{\boldsymbol{\psi}}(w\phi)\mathrm{Deg}(\mathbf{R}_{w\phi}^{\mathbb{G}})\,,$$

where $m_{\boldsymbol{\psi}}$ is d-cuspidal. It follows from definition 2.1 that the preceding sum need only be taken over those w's such that the Sylow Φ_d–subgroup of $\mathbb{T}_{w\phi}$ is contained in $\mathrm{Rad}(\mathbb{G})$, which implies that $|\mathbb{G}_{\mathrm{ss}}|_d$ divides $|\mathbb{G}|/|\mathbb{T}_{w\phi}|$. Since (*cf.* 1.17 and the definition of $\mathbf{R}_{w\phi}^{\mathbb{G}}$) $|\mathbb{G}|/|\mathbb{T}_{w\phi}| = \pm x^{N(\mathbb{G})}\mathrm{Deg}(\mathbf{R}_{w\phi}^{\mathbb{G}})$, we deduce that $|\mathbb{G}_{\mathrm{ss}}|_d$ divides $\mathrm{Deg}(\boldsymbol{\psi})$. □

We may notice the following lemma, which is immediate by 1.29.

2.5. Lemma. *For* $\boldsymbol{\psi} \in \mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$ *we have*

$$\pi_u^{\mathbb{G}}(\boldsymbol{\psi}) = \sum_{[\mathbb{T}]_{W_{\mathbb{G}}}} \frac{1}{|W_{\mathbb{G}}(\mathbb{T})|} R_{\mathbb{T}}^{\mathbb{G}}({}^*R_{\mathbb{T}}^{\mathbb{G}}(\boldsymbol{\psi}))\ .$$

We introduce a new definition which will be technically useful. Let us call "d–anisotropic maximal tori" of $\mathbb{G}$ the maximal tori $\mathbb{T}$ such that the Sylow Φ_d–subgroup of $\mathbb{T}$ is contained in $\mathrm{Rad}(\mathbb{G})$. We denote by $\mathcal{T}_d(\mathbb{G})$ the set of all d–anisotropic maximal tori. For $\boldsymbol{\psi} \in \mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$, we define the "$d$-cuspidal projection" of $\boldsymbol{\psi}$

$$c_d(\boldsymbol{\psi}) := \sum_{[\mathbb{T}\in\mathcal{T}_d(\mathbb{G})]_{W_{\mathbb{G}}}} \frac{1}{|W_{\mathbb{G}}(\mathbb{T})|} R_{\mathbb{T}}^{\mathbb{G}}({}^*R_{\mathbb{T}}^{\mathbb{G}}(\boldsymbol{\psi}))\,, \tag{2.6}$$

so we have by 1.33

$$\text{(2.7)} \qquad \operatorname{Deg}(c_d(\boldsymbol{\psi})) := \sum_{[\mathbb{T}\in\mathcal{T}_d(\mathbb{G})]_{W_\mathbb{G}}} \frac{1}{|W_\mathbb{G}(\mathbb{T})|}\operatorname{Deg}(R_\mathbb{T}^\mathbb{G}(\mathbf{1}))\operatorname{Deg}({}^*R_\mathbb{T}^\mathbb{G}(\boldsymbol{\psi}))\,,$$

which, since $\operatorname{Deg}({}^*R_\mathbb{T}^\mathbb{G}(\boldsymbol{\psi})) = ({}^*R_\mathbb{T}^\mathbb{G}(\boldsymbol{\psi}), \mathbf{1})_\mathbb{T} = (\boldsymbol{\psi}, R_{\mathbb{T}_{w\phi}}^\mathbb{G}(\mathbf{1}))$, can be written:

$$\operatorname{Deg}(c_d(\boldsymbol{\psi})) := \frac{1}{|W|}\sum m_{\boldsymbol{\psi}}(w\phi)\operatorname{Deg}(\mathbf{R}_{w\phi}^\mathbb{G})$$

where the sum is taken over those w's such that the Sylow Φ_d–subgroup of $\mathbb{T}_{w\phi}$ is contained in $\operatorname{Rad}(\mathbb{G})$. It is clear from the above proof of 2.4 that

(2.8) $|\mathbb{G}_{\mathrm{ss}}|_d$ *divides* $\operatorname{Deg}(c_d(\boldsymbol{\psi}))$*, and if* $\boldsymbol{\psi}$ *is* d*–cuspidal, then* $c_d(\boldsymbol{\psi}) = \boldsymbol{\psi}$*.*

The following result is specific to the elements of $\operatorname{Uch}(\mathbb{G})$.

2.9. Proposition. *If* $\boldsymbol{\gamma} \in \operatorname{Uch}(\mathbb{G})$ *and if* $|\mathbb{G}_{\mathrm{ss}}|_d$ *divides* $\operatorname{Deg}(\boldsymbol{\gamma})$*, then* $\boldsymbol{\gamma}$ *is* d*–cuspidal. In particular, for* $\boldsymbol{\gamma} \in \operatorname{Uch}(\mathbb{G})$*, the following statements are equivalent:*

(i) $\boldsymbol{\gamma}$ *is* d*–cuspidal*
(ii) $\operatorname{Deg}(\boldsymbol{\gamma})_d = |\mathbb{G}_{\mathrm{ss}}|_d$*,*
(iii) $\operatorname{Deg}(\boldsymbol{\gamma}) = \operatorname{Deg}(c_d(\boldsymbol{\gamma}))$.

Proof of 2.9. Assume that $\boldsymbol{\gamma} \in \operatorname{Uch}(\mathbb{G})$ is such that $|\mathbb{G}_{\mathrm{ss}}|_d$ divides $\operatorname{Deg}(\boldsymbol{\gamma})$. Whenever $w\phi \in W\phi$ is such that $m_{\boldsymbol{\gamma}}(w\phi) \neq 0$, it follows from remark 1.38 that $(\operatorname{Deg}(\boldsymbol{\gamma}))_d$ divides $\operatorname{Deg}(\mathbf{R}_{w\phi}^\mathbb{G})$. Thus we see that whenever $m_{\boldsymbol{\gamma}}(w\phi) \neq 0$, $|\mathbb{G}_{\mathrm{ss}}|_d$ divides $|\mathbb{G}|/|\mathbb{T}_{w\phi}|$, which proves that $m_{\boldsymbol{\gamma}}$, whence $\boldsymbol{\gamma}$, is d–cuspidal. □

B. Regular unipotent characters.

In §2.B, we give various formulae for the computation of the unipotent part of the regular character — viewed here again in the generic context — and called "regular unipotent character of $\mathbb{G}$*". These formulae will be used later on to prove some divisibility properties for characters of the actual finite groups.*

The unipotent regular character of $\mathbb{G}$ is the element of $\mathbb{Q}[x]\operatorname{Uch}(\mathbb{G})$ defined by the formula

$$\text{(2.10)} \qquad \operatorname{UReg}^\mathbb{G} := \frac{1}{|W|}\sum_{w\in W}\operatorname{Deg}(\mathbf{R}_{w\phi}^\mathbb{G}).\mathbf{R}_{w\phi}^\mathbb{G}\,.$$

Since by 1.30, $\operatorname{Deg}(\boldsymbol{\psi}) = (\boldsymbol{\psi}, \operatorname{UReg}^\mathbb{G})_\mathbb{G}$, we see that

$$\operatorname{UReg}^\mathbb{G} = \sum_{\boldsymbol{\gamma}\in\operatorname{Uch}(\mathbb{G})}\operatorname{Deg}(\boldsymbol{\gamma}).\boldsymbol{\gamma} \quad .$$

We change the notation in formula 2.10 : instead of summing over $w \in W$, we sum over a set $[\mathbb{T}]_{W_\mathbb{G}}$ of representatives for the $W_\mathbb{G}$-classes of maximal tori $\mathbb{T}_{w\phi}$. We recall that $W_\mathbb{G}(\mathbb{T}_{w\phi}) = C_W(w\phi)$ and we get :

$$\text{(2.11)} \qquad \operatorname{UReg}^\mathbb{G} = \sum_{[\mathbb{T}]_{W_\mathbb{G}}}\frac{\operatorname{Deg}(R_\mathbb{T}^\mathbb{G}(\mathbf{1}))}{|W_\mathbb{G}(\mathbb{T})|}R_\mathbb{T}^\mathbb{G}(\mathbf{1})\,,$$

Similarly,

$$\mathrm{c}_d(\mathrm{UReg}^{\mathbb{G}}) = \frac{1}{|W|}\sum \mathrm{Deg}(\mathbf{R}^{\mathbb{G}}_{w\phi}).\mathbf{R}^{\mathbb{G}}_{w\phi}\,,$$

where the sum is taken over those w's such that the Sylow Φ_d–subgroup of $\mathbb{T}_{w\phi}$ is contained in $\mathrm{Rad}(\mathbb{G})$. We have

$$\mathrm{c}_d(\mathrm{UReg}^{\mathbb{G}}) = \sum_{\boldsymbol{\gamma}\in\mathrm{Uch}(\mathbb{G})} \mathrm{Deg}(c_d(\boldsymbol{\gamma})).\boldsymbol{\gamma}\,.$$

Performing a similar change of notation as in 2.10, we get:

$$\text{(2.12)}\qquad \mathrm{c}_d(\mathrm{UReg}^{\mathbb{G}}) = \sum_{[\mathbb{T}\in\mathcal{T}_d(\mathbb{G})]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R^{\mathbb{G}}_{\mathbb{T}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{T})|} R^{\mathbb{G}}_{\mathbb{T}}(\mathbf{1})\,.$$

More generally, we have the following decomposition of uniform functions:

2.13. Proposition. *Let* $\boldsymbol{\psi}\in\mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$. *We have:*

$$\pi^{\mathbb{G}}_u(\boldsymbol{\psi}) = \sum_{[\mathbb{M}\ d\text{–split}]_{W_{\mathbb{G}}}} \frac{1}{W_{\mathbb{G}}(\mathbb{M})} R^{\mathbb{G}}_{\mathbb{M}}(c_d({}^*R^{\mathbb{G}}_{\mathbb{M}}(\boldsymbol{\psi}).$$

where the sum is over a set of representatives for the $W_{\mathbb{G}}$-classes of d-split Levi subgroups of $\mathbb{G}$.

Proof. To prove this, we use the

2.14. Lemma. *Let φ be a $W_{\mathbb{G}}$–stable function on the set of all maximal tori of $\mathbb{G}$. Let $\mathbb{M}$ be a d–split Levi subgroup of $\mathbb{G}$. Then*

$$\sum_{[\mathbb{T}\,;\,\mathbb{T}_d=_{W_{\mathbb{G}}}\mathrm{Rad}(\mathbb{M})_d]_{W_{\mathbb{G}}}} \frac{\varphi(\mathbb{T})}{|W_{\mathbb{G}}(\mathbb{T})|} = \frac{1}{|W_{\mathbb{G}}(\mathbb{M})|}\sum_{[\mathbb{T}\in\mathcal{T}_d(\mathbb{M})]_{W_{\mathbb{M}}}} \frac{\varphi(\mathbb{T})}{|W_{\mathbb{M}}(\mathbb{T})|}\,.$$

Proof of 2.14. Since $\mathbb{M}=C_{\mathbb{G}}(\mathrm{Rad}(\mathbb{M})_d)$, we see that $\mathbb{T}_d=\mathrm{Rad}(\mathbb{M})_d$ if and only if $\mathbb{T}$ is a d–anisotropic maximal torus of $\mathbb{M}$. Moreover, since $N_{W_{\mathbb{G}}}(\mathrm{Rad}(\mathbb{M})_d)=N_{W_{\mathbb{G}}}(\mathbb{M})$, two such tori are $W_{\mathbb{G}}$–conjugate if and only if they are $N_{W_{\mathbb{G}}}(\mathbb{M})$–conjugate. It then follows that

$$\begin{aligned}\sum_{[\mathbb{T}\,;\,\mathbb{T}_d=_{W_{\mathbb{G}}}\mathrm{Rad}(\mathbb{M})_d]_{W_{\mathbb{G}}}} \frac{\varphi(\mathbb{T})}{|W_{\mathbb{G}}(\mathbb{T})|} &= \sum_{[\mathbb{T}\,;\,\mathbb{T}_d=\mathrm{Rad}(\mathbb{M})_d]_{N_{W_{\mathbb{G}}}(\mathbb{M})}} \frac{\varphi(\mathbb{T})}{|W_{\mathbb{G}}(\mathbb{T})|}\\ &= \sum_{[\mathbb{T}\,;\,\mathbb{T}_d=\mathrm{Rad}(\mathbb{M})_d]_{W_{\mathbb{M}}}} \frac{|W_{\mathbb{M}}:N_{W_{\mathbb{M}}}(\mathbb{T})|}{|N_{W_{\mathbb{G}}}(\mathbb{M}):N_{W_{\mathbb{G}}}(\mathbb{T})|}\frac{\varphi(\mathbb{T})}{|W_{\mathbb{G}}(\mathbb{T})|}\\ &= \frac{1}{|W_{\mathbb{G}}(\mathbb{M})|}\sum_{[\mathbb{T}\in\mathcal{T}_d(\mathbb{M})]_{W_{\mathbb{M}}}} \frac{\varphi(\mathbb{T})}{|W_{\mathbb{M}}(\mathbb{T})|}\,.\end{aligned}$$

□

Applying this lemma to $\varphi(\mathbb{T})=R^{\mathbb{G}}_{\mathbb{T}}({}^*R^{\mathbb{G}}_{\mathbb{T}}\boldsymbol{\psi})$, we get from 2.5

$$\pi^{\mathbb{G}}_u = \sum_{[\mathbb{M}\ d\text{–split}]_{W_{\mathbb{G}}}} \frac{1}{|W_{\mathbb{G}}(\mathbb{M})|}\sum_{[\mathbb{T}\in\mathcal{T}_d(\mathbb{M})]_{W_{\mathbb{M}}}} \frac{R^{\mathbb{G}}_{\mathbb{T}}({}^*R^{\mathbb{G}}_{\mathbb{T}}\boldsymbol{\psi})}{|W_{\mathbb{M}}(\mathbb{T})|}$$

which gives the proposition, using the definition of c_d and the equality $R^{\mathbb{G}}_{\mathbb{T}}\cdot{}^*R^{\mathbb{G}}_{\mathbb{T}}\boldsymbol{\psi} = R^{\mathbb{G}}_{\mathbb{M}}\cdot R^{\mathbb{M}}_{\mathbb{T}}\cdot{}^*R^{\mathbb{M}}_{\mathbb{T}}{}^*R^{\mathbb{G}}_{\mathbb{M}}\boldsymbol{\psi}$.

□

2.15. Proposition. *Let* $\mathbb{L}$ *be a generic Levi subgroup of* $\mathbb{G}$*. Then*

$$^*R_{\mathbb{L}}^{\mathbb{G}}(\mathrm{UReg}^{\mathbb{G}}) = \mathrm{Deg}(R_{\mathbb{L}}^{\mathbb{G}}(\mathbf{1}))\mathrm{UReg}^{\mathbb{L}}\ .$$

Proof of 2.15.
For $\boldsymbol{\nu} \in \mathrm{Uch}(\mathbb{L})$, we have

$$\begin{aligned}\Big({}^*R_{\mathbb{L}}^{\mathbb{G}}(\mathrm{UReg}^{\mathbb{G}}), \boldsymbol{\nu}\Big)_{\mathbb{L}} &= \Big(\mathrm{UReg}^{\mathbb{G}}, R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\nu})\Big)_{\mathbb{G}} \\ &= \mathrm{Deg}(R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\nu})) = \mathrm{Deg}(R_{\mathbb{L}}^{\mathbb{G}}(\mathbf{1})).\mathrm{Deg}(\boldsymbol{\nu}) \\ &= \mathrm{Deg}(R_{\mathbb{L}}^{\mathbb{G}}(\mathbf{1}))\Big(\mathrm{UReg}^{\mathbb{L}}, \boldsymbol{\nu}\Big)_{\mathbb{L}}\ .\end{aligned}$$

□

2.16. Proposition. *We have*

$$\mathrm{UReg}^{\mathbb{G}} = \sum_{[\mathbb{M}\ d\text{–split}]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R_{\mathbb{M}}^{\mathbb{G}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{M})|} R_{\mathbb{M}}^{\mathbb{G}}(\mathrm{c}_d(\mathrm{UReg}^{\mathbb{M}}))\,.$$

Proof of 2.16. It is immediate from 2.11, 2.13, 2.15 and the multiplicativity of the degrees ($\mathrm{Deg}(R_{\mathbb{T}}^{\mathbb{G}}(\mathbf{1})) = \mathrm{Deg}(R_{\mathbb{M}}^{\mathbb{G}}(\mathbf{1}))\mathrm{Deg}(R_{\mathbb{T}}^{\mathbb{M}}(\mathbf{1}))$). □

The following statement generalizes proposition 2.4.

2.17. Proposition. *Let* $\boldsymbol{\psi} \in \mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$ *be such that* $m_{\boldsymbol{\psi}} \in \mathcal{CF}_d(\mathbb{G}, \mathbb{L})$*. Then*

(1) $|\mathbb{L}_{\mathrm{ss}}|_d$ *divides both* $\mathrm{Deg}(\boldsymbol{\psi})$ *and* $\mathrm{Deg}({}^*R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\psi}))$*,*
(2) *We have*

$$\frac{\mathrm{Deg}(\boldsymbol{\psi})}{|\mathbb{L}_{\mathrm{ss}}|_d} \equiv \frac{\mathrm{Deg}(R_{\mathbb{L}}^{\mathbb{G}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{L})|}\frac{\mathrm{Deg}({}^*R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\psi}))}{|\mathbb{L}_{\mathrm{ss}}|_d} \mod \Phi_d\,.$$

Proof of 2.17. We know (*cf.* 2.3) that ${}^*R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\psi})$ is d–cuspidal, and it follows from 2.4 that $|\mathbb{L}_{\mathrm{ss}}|_d$ divides $\mathrm{Deg}({}^*R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\psi}))$.
Moreover, we have

$$\mathrm{Deg}(\boldsymbol{\psi}) = (\boldsymbol{\psi}, \mathrm{UReg}^{\mathbb{G}})_{\mathbb{G}} = \sum_{[\mathbb{M}]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R_{\mathbb{M}}^{\mathbb{G}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{M})|}(\boldsymbol{\psi}, R_{\mathbb{M}}^{\mathbb{G}}(\mathrm{c}_d(\mathrm{UReg}^{\mathbb{M}})))_{\mathbb{G}}\,.$$

Since (by 2.12)

$$(\boldsymbol{\psi}, R_{\mathbb{M}}^{\mathbb{G}}(\mathrm{c}_d(\mathrm{UReg}^{\mathbb{M}})))_{\mathbb{G}} = \sum_{[\mathbb{T}\,;\,\mathbb{T}\in\mathcal{T}_d(\mathbb{M})]_{W_{\mathbb{M}}}} \frac{\mathrm{Deg}(R_{\mathbb{T}}^{\mathbb{M}}(\mathbf{1}))}{|W_{\mathbb{M}}(\mathbb{T})|}(\boldsymbol{\psi}, R_{\mathbb{T}}^{\mathbb{G}}(\mathbf{1}))_{\mathbb{G}}\,,$$

we see that this scalar product is zero unless $\mathbb{T}_d \subset_{W_{\mathbb{G}}} \mathrm{Rad}(\mathbb{L})$, whence that Φ_d divides $\dfrac{(\boldsymbol{\psi}, R_{\mathbb{M}}^{\mathbb{G}}(\mathrm{c}_d(\mathrm{UReg}^{\mathbb{M}})))_{\mathbb{G}}}{|\mathbb{L}_{\mathrm{ss}}|_d}$ unless $\mathbb{T}_d =_{W_{\mathbb{G}}} \mathrm{Rad}(\mathbb{L})$. This proves that

$$\frac{(\boldsymbol{\psi}, R_{\mathbb{M}}^{\mathbb{G}}(\mathrm{c}_d(\mathrm{UReg}^{\mathbb{M}})))_{\mathbb{G}}}{|\mathbb{L}_{\mathrm{ss}}|_d} \equiv 0 \mod \Phi_d$$

unless $\mathbb{M} =_{W_{\mathbb{G}}} \mathbb{L}$, from which 2.17 is immediate. □

C. Regular characters.

In §2.C, we define a suitable generic version of some projections of the regular character which occur in block theory (see [BrMi]*), and we establish some technical fromulae.*

Introduction: the regular character of $\mathcal{E}_\pi(\mathbf{G}^F, 1)$.

In what follows we shall give a "generic version" of the series $\mathcal{E}_\pi(\mathbf{G}^F, 1)$ introduced in [BrMi], and we start by computing the regular character of this series in a particular case, in order to justify the notation which will be introduced in the generic case.

Notation and prerequisites.

Let $(\mathbf{G}, \mathbf{T}, F)$ be a (q, ϕ)–triple associated with $\mathbb{G}$.

We recall that there is a natural indexing of the set of minimal closed non trivial unipotent subgroups of $\mathbf{G}$ normalized by $\mathbf{T}$ by the root system R of $\mathbb{G}$. For $\alpha \in R$, we denote by $\mathbf{U}_\alpha$ the corresponding unipotent "root subgroup".

Notice that, since $[\mathbf{G}, \mathbf{G}]$ is connected, we have

$$(\mathbf{G}/[\mathbf{G}, \mathbf{G}])^F = \mathbf{G}^F/[\mathbf{G}, \mathbf{G}]^F \ .$$

2.18. Definition. *Let $\mathbb{K}$ be a characteristic zero field, containing enough roots of unity for $\mathbf{G}^F$.*

(1) *We denote by* $\mathrm{AbIrr}(\mathbf{G}^F)$ *the group of $\mathbb{K}$–characters of the abelian group $\mathbf{G}^F/[\mathbf{G}, \mathbf{G}]^F$, considered as characters of $\mathbf{G}^F$. We denote by* $\mathrm{AbReg}^{\mathbf{G}^F} = \sum_{\theta \in \mathrm{AbIrr}(\mathbf{G}^F)} \theta$ *the corresponding regular character.*

(2) *For a set of prime numbers π, we denote by* $\mathrm{Ab}_\pi \mathrm{Irr}(\mathbf{G}^F)$ *the subgroup of* $\mathrm{AbIrr}(\mathbf{G}^F)$ *consisting of characters whose order is a π–number. We denote by* $\mathrm{Ab}_\pi \mathrm{Reg}^{\mathbf{G}^F} = \sum_{\theta \in \mathrm{Ab}_\pi \mathrm{Irr}(\mathbf{G}^F)} \theta$ *the corresponding regular character.*

Notice that the notation "$\mathrm{AbIrr}(\mathbf{G}^F)$" is slightly abusive, since $\mathbf{G}^F/[\mathbf{G}, \mathbf{G}]^F$ depends on $\mathbf{G}$ and F, and not only on $\mathbf{G}^F$.

Whenever $\mathbf{M}$ is a Levi subgroup of $\mathbf{G}$, we have $\mathbf{M} = Z^o(\mathbf{M})[\mathbf{M}, \mathbf{M}]$. In particular for a Levi subgroup $\mathbf{L}$ of $\mathbf{M}$ we have $\mathbf{M} = \mathbf{L}[\mathbf{M}, \mathbf{M}]$, and $\mathbf{M}/[\mathbf{M}, \mathbf{M}] = \mathbf{L}/\mathbf{L} \cap [\mathbf{M}, \mathbf{M}]$. Thus $\mathbf{M}^F/[\mathbf{M}, \mathbf{M}]^F$ is a quotient of $\mathbf{L}^F/[\mathbf{L}, \mathbf{L}]^F$, and the restriction of characters induces inclusions

$$\mathrm{AbIrr}(\mathbf{G}^F) \subseteq \mathrm{AbIrr}(\mathbf{M}^F) \subseteq \mathrm{AbIrr}(\mathbf{L}^F)\,.$$

From now on, we assume that

(2.19) π is a set of good prime numbers for $\mathbb{G}$.

Centralizers of π–elements.

Since $\mathbf{T} = \bar{\mathbb{F}}_p^\times \otimes Y$, the group $\bar{\mathbb{F}}_p^\times \otimes X$ is identified with $\mathrm{Hom}(\mathbf{T}, \bar{\mathbb{F}}_p^\times)$. For $s \in \mathbf{T}$ and $\chi \in X$, we denote by $\chi(s)$ the corresponding element of $\bar{\mathbb{F}}_p^\times$.

Let $s \in \mathbf{T}$. We set

$$R_s := \{\alpha \ ; \ (\alpha \in R)(\alpha(s) = 1)\} \ , \qquad (2.20)$$

and we denote by $\mathbf{G}(s)$ the connected centralizer $C^o_{\mathbf{G}}(s)$ of s in $\mathbf{G}$.

2.21. Proposition. *Let s be a π–element of* $\mathbf{T}$*.*

(1) *R_s is a parabolic subsystem of R.*

(2) *$\mathbf{G}(s)$ is the Levi subgroup of $\mathbf{G}$ generated by $\mathbf{T}$ and by the U_α for $\alpha \in R_s$.*

Proof of 2.21.

(1) *cf.* [GeHi], 2.1.

(2) See for example [BoTi], 3.4, and also [DiMi2], 2.3. □

Centralizers of π–characters.

Let θ be a character of $\mathbf{T}^F$, whose order is divisible only by primes in π : we then say that θ is a π–character of $\mathbf{T}^F$.

Suppose chosen once for all an isomorphism $\bar{\mathbb{F}}_p^\times \xrightarrow{\sim} (\mathbb{Q}/\mathbb{Z})_{p'}$. Then it follows from [DeLu], (5.2.3), that we have an exact sequence

$$\{0\} \longrightarrow Y \overset{F-1}{\longrightarrow} Y \longrightarrow \mathbf{T}^F \longrightarrow \{1\} \ , \tag{2.22}$$

which allows us to view $\mathbf{T}^F$ as a quotient of Y, and thus to define the image $\theta(y)$ of an element $y \in Y$ through a character θ of $\mathbf{T}^F$. For such a character, we then define the closed subsystem

$$R_\theta := \{\alpha \ ; (\alpha \in R)(\theta(\alpha^\vee) = 1)\} \ , \tag{2.23}$$

and the subgroup $\mathbf{G}(\mathbf{T}, \theta)$ of $\mathbf{G}$ by

$$\mathbf{G}(\mathbf{T}, \theta) := \mathbf{T}.<\mathbf{U}_\alpha>_{\alpha \in R_\theta} \ .$$

2.24. Proposition. *Let θ be a π–character of* $\mathbf{T}^F$*.*

(1) *The subsystem R_θ is parabolic in R.*

(2) *The group $\mathbf{G}(\mathbf{T}, \theta)$ is the largest F–stable Levi subgroup $\mathbf{M}$ of $\mathbf{G}$ such that*

- $\mathbf{T} \subset \mathbf{M}$*,*
- *θ belongs to the image of* $\mathrm{AbIrr}(\mathbf{M}^F)$ *in* $\mathrm{AbIrr}(\mathbf{T}^F)$*.*

Proof of 2.24.

(1) We may assume that $\mathbf{T}$ is part of a (q, ϕ)–triple associated to $\mathbb{G}$ and let $(\mathbf{G}^*, \mathbf{T}^*, F^*)$ be a $(q, \phi^{\vee -1})$–triple associated with $\mathbb{G}^*$. We choose an isomorphism

$$\mathbf{T}^{*F^*} \xrightarrow{\sim} \mathrm{Irr}(\mathbf{T}^F) \ . \tag{*}$$

The reciprocal image of θ through the isomorphism (*) is a π–element $s \in \mathbf{T}^{*F^*}$. The system R_θ is then the dual of $R_s^\vee$ (see 2.20), hence is parabolic by 2.21, (1).

(2) The second assertion of 2.24 results from [DeLu], proposition 5.11, (i), and from the fact that a closed connected subgroup $\mathbf{H}$ of $\mathbf{G}$ containing $\mathbf{T}$ is generated by $\mathbf{T}$ and by $\{\mathbf{U}_\alpha \ ; (\alpha \in R)(\mathbf{U}_\alpha \subset \mathbf{H})\}$ (see [BoTi], 3.4). □

We set $\mathbf{G}^F(\mathbf{T}, \theta) := \mathbf{G}(\mathbf{T}, \theta)^F$.

Following Lusztig's notation, we denote by $\mathcal{E}(\mathbf{G}^F, 1)$ the set of unipotent characters of $\mathbf{G}^F$. For any $\rho \in \mathcal{E}(\mathbf{G}^F(\mathbf{T}, \theta), 1)$ we then have (see [Lu2]) a well defined irreducible character of $\mathbf{G}^F$, defined by the formula

$$\chi^{\mathbf{G}^F}_{(\mathbf{G}(\mathbf{T},\theta),\theta,\rho)} := \varepsilon_{\mathbf{G}} \varepsilon_{\mathbf{G}(\mathbf{T},\theta)} R^{\mathbf{G}}_{\mathbf{G}(\mathbf{T},\theta)}(\theta\rho)$$

(this is a particular case of the "Jordan decomposition of characters").

Two characters $\chi^{\mathbf{G}^F}_{(\mathbf{G}(\mathbf{T},\theta),\theta,\rho)}$ and $\chi^{\mathbf{G}^F}_{(\mathbf{G}(\mathbf{T}',\theta'),\theta',\rho')}$ are equal if and only if the triples $(\mathbf{G}(\mathbf{T},\theta),\theta,\rho)$ and $(\mathbf{G}(\mathbf{T}',\theta'),\theta',\rho)$ are $\mathbf{G}^F$–conjugate (see [Lu2], and also [DeLu], 5.20).

A particular case.

We assume now that

(*2.25) Every π–element of $\mathbf{G}^F$ is $\mathbf{G}^F$–conjugate to an element of $\mathbf{T}^F$.*

We shall see in §4 that the preceding hypothesis is satisfied if $\pi = \{\ell\}$ for a "large" prime number ℓ ("d-adapted" in the sense of §4) and $\mathbf{T}$ contains a Sylow Φ_d-subgroup of $\mathbf{G}$.

It is then easy to see that the regular character of the series $\mathcal{E}_\pi(\mathbf{G}^F,1)$ (see [BrMi], above 2.1) is given by the following formula:

$$\text{(2.26)} \qquad \mathrm{Reg}_\pi^{\mathbf{G}^F} := \sum_{[(\mathbf{G}(\mathbf{T},\theta),\theta,\rho)]_{\mathbf{G}^F}} \chi^{\mathbf{G}^F}_{(\mathbf{G}(\mathbf{T},\theta),\theta,\rho)}(1) . \chi^{\mathbf{G}^F}_{(\mathbf{G}(\mathbf{T},\theta),\theta,\rho)} \,.$$

For an F–stable Levi subgroup $\mathbf{M}$ of $\mathbf{G}$ containing $\mathbf{T}$, we define

$$\text{(2.27)} \qquad \Theta^{\mathbf{M}^F}_{\mathbf{G},\pi} := \sum_{\{\theta\,;\,(\theta\in \mathrm{Ab}_\pi \,\mathrm{Irr}(\mathbf{M}^F))(\mathbf{M}=\mathbf{G}(\mathbf{T},\theta))\}} \theta \,.$$

2.28. Lemma. *Under the preceding hypothesis, we have*

$$\begin{aligned} \mathrm{Reg}_\pi^{\mathbf{G}^F} &= \sum_{[\mathbf{M}]_{\mathbf{G}^F}} \frac{\mathrm{Deg}(R_{\mathbf{M}}^{\mathbf{G}}(1))}{|W_{\mathbf{G}^F}(\mathbf{M})|} R_{\mathbf{M}}^{\mathbf{G}}(\Theta^{\mathbf{M}^F}_{\mathbf{G},\pi}.\,\mathrm{UReg}^{\mathbf{M}^F}) \\ &= \sum_{[(\mathbf{M},\boldsymbol{\mu})]_{\mathbf{G}^F}} \frac{\mathrm{Deg}(R_{\mathbf{M}}^{\mathbf{G}}(\boldsymbol{\mu}))}{|W_{\mathbf{G}^F}(\mathbf{M},\boldsymbol{\mu})|} R_{\mathbf{M}}^{\mathbf{G}}(\Theta^{\mathbf{M}^F}_{\mathbf{G},\pi}.\boldsymbol{\mu})\,, \end{aligned}$$

where $\mathbf{M}$ runs over representatives of $\mathbf{G}^F$-conjugacy classes of F-stable Levi subgroups of $\mathbf{G}$, where $\boldsymbol{\mu}$ runs over the set $\mathcal{E}(\mathbf{M}^F,1)$ of irreducible unipotent characters of $\mathbf{M}^F$, where

$$\mathrm{UReg}^{\mathbf{M}^F} := \sum_{\boldsymbol{\mu}\in\mathcal{E}(\mathbf{M}^F,1)} \boldsymbol{\mu}(1)\boldsymbol{\mu}$$

and

$$W_{\mathbf{G}^F}(\mathbf{M}) := N_{\mathbf{G}^F}(\mathbf{M})/\mathbf{M}^F \quad , \quad W_{\mathbf{G}^F}(\mathbf{M},\boldsymbol{\mu}) := N_{\mathbf{G}^F}(\mathbf{M},\boldsymbol{\mu})/\mathbf{M}^F \quad .$$

Proof of 2.28. Left to the reader (it is immediate once the notation is understood). □

We note that the sum in the above lemma is over the F–stable Levi subgroups $\mathbf{M}$ of $\mathbf{G}$ for which there exists an element $\theta \in \mathrm{Ab}_\pi \,\mathrm{Irr}(\mathbf{T}^F)$ such that $\mathbf{M}$ is $\mathbf{G}^F$–conjugate to $\mathbf{G}(\mathbf{T},\theta)$. These will be generically the "π-split" Levi subgroups of $\mathbf{G}$.

The generic formalism.

Lemma 2.28 justifies the introduction of the following formalism, meant to define a "generic" version of the character $\mathrm{Reg}_\pi^{\mathbf{G}^F}$.

• The group $W_{\mathbb{G}}$ acts naturally on the set of pairs $(\mathbb{M},\boldsymbol{\mu})$ (extending remark 1.26 to isomorphisms), where $\mathbb{M}$ is a d–split generic Levi subgroup of $\mathbb{G}$ and where $\boldsymbol{\mu}\in\mathrm{Uch}(\mathbb{M})$; we set $W_{\mathbb{G}}(\mathbb{M},\boldsymbol{\mu}) = N_{W_{\mathbb{G}}}(\mathbb{M},\boldsymbol{\mu})/W_{\mathbb{M}}$. We denote by $\mathcal{R}_d(\mathbb{G})$ the free $\mathbb{Q}[x]$–module on the set of $W_{\mathbb{G}}$-conjugacy classes of such pairs, with basis denoted by $\{\chi^{\mathbb{G}}_{(\mathbb{M},\boldsymbol{\mu})}\}$.

We define (*cf.* 2.28)

$$\mathrm{Reg}_d^{\mathbb{G}} := \sum_{[(\mathbb{M},\boldsymbol{\mu})]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R^{\mathbb{G}}_{\mathbb{M}}(\boldsymbol{\mu}))}{|W_{\mathbb{G}}(\mathbb{M},\boldsymbol{\mu})|}.\chi^{\mathbb{G}}_{(\mathbb{M},\boldsymbol{\mu})}\,. \tag{2.29}$$

• For any generic Levi subgroup $\mathbb{N}$ of $\mathbb{G}$, we denote by $\mathcal{A}_{d,\mathbb{G}}(\mathbb{N})$ the free $\mathbb{Q}[x]$–module on the set of pairs $(\mathbb{M},\boldsymbol{\nu})$ such that $\mathbb{M}$ is a d–split Levi subgroup of $\mathbb{G}$ containing $\mathbb{N}$ and $\boldsymbol{\nu}\in\mathrm{Uch}(\mathbb{N})$, with basis denoted by $\{\boldsymbol{\Theta}^{\mathbb{N}}_{\mathbb{G},\mathbb{M}}\otimes\boldsymbol{\nu}\}$.

We set $\boldsymbol{\Theta}^{\mathbb{N}}_{\mathbb{G},\mathbb{M}} := \boldsymbol{\Theta}^{\mathbb{N}}_{\mathbb{G},\mathbb{M}}\otimes\mathbf{1}$. For a d–split generic Levi subgroup $\mathbb{M}$ of $\mathbb{G}$, we set $\boldsymbol{\Theta}^{\mathbb{M}}_{\mathbb{G}} := \boldsymbol{\Theta}^{\mathbb{M}}_{\mathbb{G},\mathbb{M}}$.

Remark. $\boldsymbol{\Theta}^{\mathbb{N}}_{\mathbb{G},\mathbb{M}}\otimes\boldsymbol{\nu}$ should be viewed as the formal version of $\mathrm{Res}^{\mathbf{M}^F}_{\mathbf{N}^F}(\Theta^{\mathbf{M}^F}_{\mathbf{G},\pi})\rho^{\mathbf{N}^F}_{\boldsymbol{\nu}}$.

• We define $R^{\mathbb{G}}_{\mathbb{N}} : \mathcal{A}_{d,\mathbb{G}}(\mathbb{N})\to\mathcal{R}_d(\mathbb{G})$ by the formulae

$$\begin{cases} R^{\mathbb{G}}_{\mathbb{M}}(\boldsymbol{\Theta}^{\mathbb{M}}_{\mathbb{G}}\otimes\boldsymbol{\mu}) = \chi^{\mathbb{G}}_{(\mathbb{M},\boldsymbol{\mu})} & \text{for } \mathbb{N}=\mathbb{M}, \text{ a } d\text{–split Levi subgroup of } \mathbb{G},\\ R^{\mathbb{G}}_{\mathbb{N}}(\boldsymbol{\Theta}^{\mathbb{N}}_{\mathbb{G},\mathbb{M}}\otimes\boldsymbol{\nu}) = R^{\mathbb{G}}_{\mathbb{M}}(\boldsymbol{\Theta}^{\mathbb{M}}_{\mathbb{G}}\otimes R^{\mathbb{M}}_{\mathbb{N}}(\boldsymbol{\nu})) & \text{for } \mathbb{N}\subset\mathbb{M},\ \mathbb{M}\ d\text{–split.}\end{cases} \tag{2.30}$$

We shall see that the map $R^{\mathbb{G}}_{\mathbb{L}}$ of 1.32 identifies to a special case of this one.

• We set

$$\mathrm{Ab}_d\,\mathrm{Reg}^{\mathbb{N}} := \sum_{\{\mathbb{M}\,;\,(\mathbb{N}\preccurlyeq\mathbb{M})(\mathbb{M}\ d\text{–split})\}} \boldsymbol{\Theta}^{\mathbb{N}}_{\mathbb{G},\mathbb{M}}\,. \tag{2.31}$$

[Notice that $\mathrm{Ab}_d\,\mathrm{Reg}^{\mathbb{N}}$ should be viewed as the generic version of $\mathrm{Ab}_\pi\,\mathrm{Reg}^{\mathbf{N}^F} := \sum_{\theta\in\mathrm{Ab}_\pi\,\mathrm{Irr}(\mathbf{N}^F)}\theta$.]

With this notation, it follows from definition 2.29 that

$$\begin{aligned}\mathrm{Reg}_d^{\mathbb{G}} &= \sum_{[(\mathbb{M},\boldsymbol{\mu})]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R^{\mathbb{G}}_{\mathbb{M}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{M},\boldsymbol{\mu})|} R^{\mathbb{G}}_{\mathbb{M}}(\boldsymbol{\Theta}^{\mathbb{M}}_{\mathbb{G}}\otimes\mathrm{Deg}(\boldsymbol{\mu}).\boldsymbol{\mu})\\ &= \sum_{[\mathbb{M}]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R^{\mathbb{G}}_{\mathbb{M}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{M})|} R^{\mathbb{G}}_{\mathbb{M}}(\boldsymbol{\Theta}^{\mathbb{M}}_{\mathbb{G}}\otimes\mathrm{UReg}^{\mathbb{M}})\,.\end{aligned} \tag{2.32}$$

The following theorem is the analogue of 2.16.

2.33. Theorem.

(1)

$$\mathrm{Reg}_d^{\mathbb{G}} = \sum_{[\mathbb{T}]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R_{\mathbb{T}}^{\mathbb{G}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{T})|} R_{\mathbb{T}}^{\mathbb{G}}(\mathrm{Ab}_d \, \mathrm{Reg}^{\mathbb{T}}) \, ,$$

where $\mathbb{T}$ *runs over the set of maximal tori of* $\mathbb{G}$*.*

(2)

$$\begin{aligned}\mathrm{Reg}_d^{\mathbb{G}} &= \sum_{[(\mathbb{M},\boldsymbol{\mu})]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R_{\mathbb{M}}^{\mathbb{G}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{M},\boldsymbol{\mu})|} R_{\mathbb{M}}^{\mathbb{G}}(\mathrm{Ab}_d \, \mathrm{Reg}^{\mathbb{M}} \otimes \mathrm{Deg}(c_d(\boldsymbol{\mu})).\boldsymbol{\mu}) \\ &= \sum_{[\mathbb{M}]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R_{\mathbb{M}}^{\mathbb{G}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{M})|} R_{\mathbb{M}}^{\mathbb{G}}(\mathrm{Ab}_d \, \mathrm{Reg}^{\mathbb{M}} \otimes \mathrm{c}_d(\mathrm{UReg}^{\mathbb{M}})) \, ,\end{aligned}$$

where $\mathbb{M}$ *runs over the set of* d*–split Levi subgroups of* $\mathbb{G}$*.*

Proof of 2.33.

(1) By 2.32, we see that

$$\mathrm{Reg}_d^{\mathbb{G}} = \frac{1}{|W_{\mathbb{G}}|} \sum_{\mathbb{M}} |W_{\mathbb{M}}| \mathrm{Deg}(R_{\mathbb{M}}^{\mathbb{G}}(\mathbf{1})) R_{\mathbb{M}}^{\mathbb{G}}(\boldsymbol{\Theta}_{\mathbb{G}}^{\mathbb{M}} \otimes \mathrm{UReg}^{\mathbb{M}}) \, .$$

Also, by 2.11, we see that

$$\mathrm{UReg}^{\mathbb{M}} = \frac{1}{|W_{\mathbb{M}}|} \sum_{\mathbb{T} \subseteq \mathbb{M}} \mathrm{Deg}(R_{\mathbb{T}}^{\mathbb{M}}(\mathbf{1})) R_{\mathbb{T}}^{\mathbb{M}}(\mathbf{1}) \, .$$

Thus we get, using 2.30,

$$\mathrm{Reg}_d^{\mathbb{G}} = \frac{1}{|W_{\mathbb{G}}|} \sum_{\mathbb{T} \subseteq \mathbb{G}} \mathrm{Deg}(R_{\mathbb{T}}^{\mathbb{G}}(\mathbf{1})) \Bigg(\sum_{\{\mathbb{M}\,;\,\mathbb{T} \subseteq \mathbb{M}\}} R_{\mathbb{T}}^{\mathbb{G}}(\boldsymbol{\Theta}_{\mathbb{G},\mathbb{M}}^{\mathbb{T}} \otimes \mathbf{1}) \Bigg)$$

and the result follows from 2.31.

(2) We extend c_d of 2.7 to $\mathrm{Reg}_d^{\mathbb{G}}$ by setting

$$\mathrm{c}_d(\mathrm{Reg}_d^{\mathbb{G}}) := \sum_{[\mathbb{T} \in \mathcal{T}_d(\mathbb{G})]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R_{\mathbb{T}}^{\mathbb{G}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{T})|} R_{\mathbb{T}}^{\mathbb{G}}(\mathrm{Ab}_d \, \mathrm{Reg}^{\mathbb{T}}) \, .$$

By lemma 2.14 applied with $\varphi(\mathbb{T}) = \mathrm{Deg}(R_{\mathbb{T}}^{\mathbb{G}}(\mathbf{1})) R_{\mathbb{T}}^{\mathbb{G}}(\mathrm{Ab}_d \, \mathrm{Reg}^{\mathbb{T}})$, we see that

$$\mathrm{Reg}_d^{\mathbb{G}} = \sum_{[\mathbb{M}\,;\,\mathbb{M}\ d\text{–split}]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R_{\mathbb{M}}^{\mathbb{G}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{M})|} R_{\mathbb{M}}^{\mathbb{G}}(\mathrm{c}_d(\mathrm{Reg}_d^{\mathbb{M}})) \quad .$$

Thus it suffices to prove the following lemma.

2.34. Lemma. *We have* $\mathrm{c}_d(\mathrm{Reg}_d^{\mathbb{G}}) = \mathrm{Ab}_d \mathrm{Reg}^{\mathbb{G}} \otimes \mathrm{c}_d(\mathrm{UReg}^{\mathbb{G}})$.

Proof of 2.34. We have

$$\mathrm{Ab}_d \mathrm{Reg}^{\mathbb{G}} \otimes \mathrm{c}_d(\mathrm{UReg}^{\mathbb{G}}) = \sum_{[\mathbb{T}\in\mathcal{T}_d(\mathbb{G})]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R_{\mathbb{T}}^{\mathbb{G}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{T})|} \mathrm{Ab}_d \mathrm{Reg}^{\mathbb{G}} \otimes R_{\mathbb{T}}^{\mathbb{G}}(\mathbf{1}) \,,$$

and it suffices to check that, for $\mathbb{T} \in \mathcal{T}_d(\mathbb{G})$, we have

$$\mathrm{Ab}_d \mathrm{Reg}^{\mathbb{G}} \otimes R_{\mathbb{T}}^{\mathbb{G}}(\mathbf{1}) = R_{\mathbb{T}}^{\mathbb{G}}(\mathrm{Ab}_d \mathrm{Reg}^{\mathbb{T}}) \,,$$

which is immediate. □

□

In order to define a generic version of the adjoint ${}^*R_{\mathbb{N}}^{\mathbb{G}}$ of $R_{\mathbb{N}}^{\mathbb{G}}$ which may be "specialized" to the ordinary ${}^*R_{\mathbf{N}}^{\mathbf{G}}$, we must introduce more notation.

Let $\mathcal{R}_{d,\mathbb{G}}(\mathbb{N})$ be the free $\mathbb{Q}[x]$–module on the set of $W_{\mathbb{N}}$–conjugacy classes of pairs $(\mathbb{M}, \boldsymbol{\delta})$, where

- $\mathbb{M}$ is a d–split Levi subgroup of $\mathbb{G}$ such that $\mathbb{M} \cap \mathbb{N}$ is defined,
- $\boldsymbol{\delta} \in \mathrm{Uch}(\mathbb{M} \cap \mathbb{N})$,

with basis denoted by $\{\chi_{(\mathbb{M},\boldsymbol{\delta})}^{\mathbb{N}}\}$.

Remark. We must see $\chi_{(\mathbb{M},\boldsymbol{\delta})}^{\mathbb{N}}$ as $R_{\mathbb{M}\cap\mathbb{N}}^{\mathbb{N}}(\boldsymbol{\Theta}_{\mathbb{G},\mathbb{M}}^{\mathbb{M}\cap\mathbb{N}} \otimes \boldsymbol{\delta})$.

We define $R_{\mathbb{N}}^{\mathbb{G}} \colon \mathcal{R}_{d,\mathbb{G}}(\mathbb{N}) \to \mathcal{R}_d(\mathbb{G})$ by the formula

$$R_{\mathbb{N}}^{\mathbb{G}}(\chi_{(\mathbb{M},\boldsymbol{\delta})}^{\mathbb{N}}) := R_{\mathbb{M}}^{\mathbb{G}}(\boldsymbol{\Theta}_{\mathbb{G}}^{\mathbb{M}} \otimes \boldsymbol{\delta}) \,, \tag{2.35}$$

and then we define ${}^*R_{\mathbb{N}}^{\mathbb{G}} \colon \mathcal{R}_d(\mathbb{G}) \to \mathcal{R}_{d,\mathbb{G}}(\mathbb{N})$ in such a way as to fit with the Mackey formula, *i.e.,* for any d–split pair $(\mathbb{M}, \boldsymbol{\mu})$ of $\mathbb{G}$, we set

$${}^*R_{\mathbb{N}}^{\mathbb{G}}(\chi_{(\mathbb{M},\boldsymbol{\mu})}^{\mathbb{N}}) := \sum_{W_{\mathbb{N}}\backslash\mathcal{S}_{W_{\mathbb{G}}}(\mathbb{M},\mathbb{N})/W_{\mathbb{M}}} R_{\mathbb{N}\cap{}^w\mathbb{M}}^{\mathbb{N}} \left(\boldsymbol{\Theta}_{\mathbb{G},{}^w\mathbb{M}}^{{}^w\mathbb{M}\cap\mathbb{N}} \otimes {}^*R_{{}^w\mathbb{M}\cap\mathbb{N}}^{{}^w\mathbb{M}}({}^w\boldsymbol{\mu})\right) . \tag{2.36}$$

2.37. Theorem. *Let* $\mathbb{N}$ *be a generic Levi subgroup of* $\mathbb{G}$*. Then*

$${}^*R_{\mathbb{N}}^{\mathbb{G}}(\mathrm{Reg}_d^{\mathbb{G}}) = \mathrm{Deg}(R_{\mathbb{N}}^{\mathbb{G}}(\mathbf{1}))\mathrm{Reg}_d^{\mathbb{N}} \;.$$

Sketch of proof of 2.37.

Since, by 2.33,

$$\mathrm{Reg}_d^{\mathbb{G}} = \sum_{[\mathbb{T}]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R_{\mathbb{T}}^{\mathbb{G}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{T})|} R_{\mathbb{T}}^{\mathbb{G}}(\mathrm{Ab}_d \mathrm{Reg}^{\mathbb{T}}) \,,$$

and since $\mathrm{Ab}_d \mathrm{Reg}^{\mathbb{T}} = \mathrm{Reg}_d^{\mathbb{T}}$, it follows that

$${}^*R_{\mathbb{T}}^{\mathbb{G}}(\mathrm{Reg}_d^{\mathbb{G}}) = \mathrm{Deg}(R_{\mathbb{T}}^{\mathbb{G}}(\mathbf{1}))\mathrm{Reg}_d^{\mathbb{T}}$$

and also that

$${}^*R_{\mathbb{N}}^{\mathbb{G}}(\mathrm{Reg}_d^{\mathbb{G}}) = \sum_{[\{\mathbb{T}\,;\,\mathbb{T}\subseteq\mathbb{N}\}]_{W_{\mathbb{N}}}} R_{\mathbb{T}}^{\mathbb{N}}({}^*R_{\mathbb{T}}^{\mathbb{N}}({}^*R_{\mathbb{N}}^{\mathbb{G}}(\mathrm{Reg}_d^{\mathbb{G}}))) \,.$$

Thus the result follows from the transitivity of both $R_{\mathbb{N}}^{\mathbb{G}}$ and ${}^*R_{\mathbb{N}}^{\mathbb{G}}$. □

D. First applications to actual finite reductive groups.

In §2.D, as a first application, we prove that the specialization of our previous regular characters behaves as the regular character of a "Φ_d–idempotent". The actual application to true ℓ–idempotents will be given in §5.

Let $(\mathbf{G},\mathbf{T},F)$ be a (q,ϕ)–triple associated with $\mathbb{G}$.

Following the notation introduced at the beginning of §2.C, let $(\mathbf{G}/[\mathbf{G},\mathbf{G}])_{d'}$ be the product of all the Sylow Φ_e–subgroups ($e\neq d$) of the torus $\mathbf{G}/[\mathbf{G},\mathbf{G}]$. We then denote by $\mathrm{Ab}_d\,\mathrm{Irr}(\mathbf{G}^F)$ the group of all characters of $\mathbf{G}^F/[\mathbf{G},\mathbf{G}]^F$ (viewed as characters of $\mathbf{G}^F$) whose kernel contains the group of rational points of $(\mathbf{G}/[\mathbf{G},\mathbf{G}])_{d'}$. We denote by $\mathrm{Ab}_d\,\mathrm{Reg}^{\mathbf{G}^F} := \sum_{\theta\in\mathrm{Ab}_d\,\mathrm{Irr}(\mathbf{G}^F)}\theta$ the corresponding regular character.

2.38. Definition–Proposition.

(1) Let $\mathbf{N}$ be an F–stable Levi subgroup of $\mathbf{G}$. The class function $\Theta_{\mathbf{G}}^{\mathbf{N}^F}$ on $\mathbf{N}^F$ is defined inductively by the formula

$$\sum_{\{\mathbf{M}\,;\,(\mathbf{M}\ d\text{–split})(\mathbf{N}\subseteq\mathbf{M})\}}\mathrm{Res}_{\mathbf{N}^F}^{\mathbf{M}^F}(\Theta_{\mathbf{G}}^{\mathbf{M}^F}) = \mathrm{Ab}_d\,\mathrm{Reg}^{\mathbf{N}^F}\,.$$

(2) (Equivalent definition) For an F–stable Levi subgroup $\mathbf{N}$ of $\mathbf{G}$, we define

$$\mathrm{Irr}(\Theta_{\mathbf{G}}^{\mathbf{N}^F}) := \mathrm{Ab}_d\,\mathrm{Irr}(\mathbf{N}^F) - \bigcup_{\{\mathbf{M}\,;\,(\mathbf{M}\ d\text{–split})(\mathbf{N}\subset\mathbf{M})\}}\mathrm{Ab}_d\,\mathrm{Irr}(\mathbf{M}^F)\,.$$

Then we have $\Theta_{\mathbf{G}}^{\mathbf{N}^F} = \sum_{\theta\in\mathrm{Irr}(\Theta_{\mathbf{G}}^{\mathbf{N}^F})}\theta$.

For a unipotent character $\boldsymbol{\mu}$ of $\mathbf{M}^F$, we set $W_{\mathbf{G}^F}(\mathbf{M},\boldsymbol{\mu}) := N_{\mathbf{G}^F}(\mathbf{M},\boldsymbol{\mu})/\mathbf{M}^F$ and (with obvious notation)

$$\mathrm{Reg}_d^{\mathbf{G}^F} := \sum_{[(\mathbf{M},\boldsymbol{\mu})]_{\mathbf{G}^F}}\frac{\mathrm{Deg}R_{\mathbf{M}}^{\mathbf{G}}(\boldsymbol{\mu})}{|W_{\mathbf{G}^F}(\mathbf{M},\boldsymbol{\mu})|}R_{\mathbf{M}}^{\mathbf{G}}(\Theta_{\mathbf{G}}^{\mathbf{M}^F}.\boldsymbol{\mu})\,.$$

The following result is a kind of "generic version" of [BrMi], thm. 2.2.

2.39. Theorem. *All the values of $|W_{\mathbf{G}}|\,\mathrm{Reg}_d^{\mathbf{G}^F}$ are divisible by $|\mathbb{G}|_d(q)$.*

Proof of 2.39. This depends on the following general property of the Deligne–Lusztig induction.

2.40. Proposition. *Let $\mathbf{L}$ be an F–stable Levi subgroup of $\mathbf{G}$. If ζ is a class function on $\mathbf{L}^F$ which takes integral values, then the function $|\mathbf{G}^F|_p R_{\mathbf{L}}^{\mathbf{G}}(\zeta)$ takes integral values.*

Proof of 2.40. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ such that $\mathbf{L}$ is a Levi complement of $\mathbf{P}$, and let $\mathbf{U}$ be the unipotent radical of $\mathbf{P}$. We denote by $\mathcal{V}_{\mathbf{U}}$ the associated Deligne–Lusztig variety, on which the finite group $\mathbf{G}^F$ acts by left multiplication, while the finite group $\mathbf{L}^F$ acts by right multiplication. We denote by $\Lambda_{\mathbf{U}}$ the class function on $\mathbf{G}^F\times\mathbf{L}^F$ defined by the formula

$$\Lambda_{\mathbf{U}}(g,l) := \sum_i \mathrm{tr}((g,l^{-1})\,;\,H^i(\mathcal{V}_{\mathbf{U}},\mathbb{Q}_\ell))\,.$$

It is known (*cf.* [DeLu], prop. 3.3) that the function $\Lambda_{\mathbf{U}}$ takes integral values, which are independent of ℓ ($\ell \neq p$). Moreover, the map $R_{\mathbf{L}}^{\mathbf{G}}$ is defined by $\Lambda_{\mathbf{U}}$ as follows (*cf.* for example [Br1], §1). For a class function ζ on $\mathbf{L}^F$, the value of $R_{\mathbf{L}}^{\mathbf{G}}(\zeta)$ at $g \in \mathbf{G}^F$ is given by the following formula:

$$R_{\mathbf{L}}^{\mathbf{G}}(\zeta)(g) = (\Lambda_{\mathbf{U}}(g,\cdot),\zeta)_{\mathbf{L}^F}$$

where we denote by $\Lambda(g,\cdot)$ the class function on $\mathbf{L}^F$ whose value at $l \in \mathbf{L}^F$ is $\Lambda(g,l)$.

By [Br1], 4.1, it suffices then to prove that $\Lambda_{\mathbf{U}}$ is "ℓ–perfect" for every $\ell \neq p$. This follows from [Br1], prop. 2.1 (see also [DeLu], prop. 3.5). □

We can now prove Theorem 2.39.

By 2.33 and 1.27 (which shows that we may identify $W_{\mathbb{G}}(\mathbb{M},\boldsymbol{\mu})$ and $W_{\mathbf{G}}(\mathbf{M},\rho_{\boldsymbol{\mu}}^{\mathbf{M}^F})$), we see (note that our notation has been chosen so that everything specializes nicely) that

$$\text{(2.41)} \qquad \operatorname{Reg}_d^{\mathbf{G}^F} = \sum_{[(\mathbf{M},\boldsymbol{\mu})]_{\mathbf{G}^F}} \frac{\operatorname{Deg} R_{\mathbf{M}}^{\mathbf{G}}(1)}{|W_{\mathbf{G}^F}(\mathbf{M},\boldsymbol{\mu})|} R_{\mathbf{M}}^{\mathbf{G}}(\operatorname{Ab}_d \operatorname{Reg}^{\mathbf{M}^F}.\operatorname{Deg}(c_d(\boldsymbol{\mu}))(q).\boldsymbol{\mu}) \;.$$

It suffices to prove the following lemma.

2.42. Lemma. *Whenever* $\mathbf{M}$ *is a d–split Levi subgroup of* $\mathbf{G}$ *and* $\boldsymbol{\mu}$ *is a unipotent character of* $\mathbf{M}^F$, $|W_{\mathbf{M}}|\dfrac{\operatorname{Deg} R_{\mathbf{M}}^{\mathbf{G}}(1)}{|W_{\mathbf{G}^F}(\mathbf{M},\boldsymbol{\mu})|} R_{\mathbf{M}}^{\mathbf{G}}(\operatorname{Ab}_d \operatorname{Reg}^{\mathbf{M}^F}.\operatorname{Deg}(c_d(\boldsymbol{\mu}))(q).\boldsymbol{\mu})$ *takes values which are divisible by* $|\mathbb{G}|_d(q)$.

Proof of 2.42.

• On one hand, since $W_{\mathbf{G}^F}(\mathbf{M},\boldsymbol{\mu}) = N_{\mathbf{G}^F}(\mathbf{M},\boldsymbol{\mu})/\mathbf{M}^F$, we have

$$\frac{\operatorname{Deg} R_{\mathbf{M}}^{\mathbf{G}}(1)}{|W_{\mathbf{G}^F}(\mathbf{M},\boldsymbol{\mu})|} = \frac{\varepsilon_{\mathbf{G}}\varepsilon_{\mathbf{M}}}{|W_{\mathbf{G}^F}(\mathbf{M},\boldsymbol{\mu})|_p} |\mathbf{G}^F : N_{\mathbf{G}^F}(\mathbf{M},\boldsymbol{\mu})|_{p'} \;.$$

• On the other hand, by 2.8, we see that $|W_{\mathbf{M}}|\operatorname{Deg}(c_d(\boldsymbol{\mu}))(q)$ is divisible by $|\mathbb{M}_{\mathrm{ss}}|_d(q)$ (since $|W_{\mathbf{M}}|\operatorname{Deg}(c_d(\boldsymbol{\mu}))$ and $|\mathbb{M}_{\mathrm{ss}}|_d$ are both polynomials in $\mathbb{Z}[X]$, the second one monic), and thus $|W_{\mathbf{M}}|\operatorname{Ab}_d \operatorname{Reg}^{\mathbf{M}^F}.\operatorname{Deg}(c_d(\boldsymbol{\mu}))(q).\boldsymbol{\mu}$ takes values divisible by $|\mathbb{M}|_d(q)$.

Since $|\mathbb{M}|_d(q) = |\mathbb{G}|_d(q)$, we see that 2.39 results from 2.40. □

□

3. Generalized Harish–Chandra theory

In §3.A, in order to decompose our regular characters, we define partitions of the set of unipotent characters in a way which generalizes Harish–Chandra and Howlett–Lehrer–Lusztig theories.

A. The fundamental theorem.

We call *d–split pairs* of $\mathbb{G}$ the pairs $(\mathbb{M},\boldsymbol{\mu})$ where $\mathbb{M}$ is the centralizer in $\mathbb{G}$ of a Φ_d–torus of $\mathbb{G}$, and where $\boldsymbol{\mu} \in \operatorname{Uch}(\mathbb{M})$. We say that such a pair is *d–cuspidal* if $\boldsymbol{\mu}$ is d–cuspidal.

3.1. Definition. *Let* $(\mathbb{M}_1, \boldsymbol{\mu}_1)$ *and* $(\mathbb{M}_2, \boldsymbol{\mu}_2)$ *be two d–split pairs. We write*

$$(\mathbb{M}_1, \boldsymbol{\mu}_1) \preccurlyeq (\mathbb{M}_2, \boldsymbol{\mu}_2)$$

if

(1) $\mathbb{M}_1$ *is a Levi subgroup of* $\mathbb{M}_2$,
(2) $(R_{\mathbb{M}_1}^{\mathbb{M}_2}(\boldsymbol{\mu}_1), \boldsymbol{\mu}_2)_{\mathbb{G}} \neq 0$.

The Weyl group $W_{\mathbb{G}}$ of $\mathbb{G}$ acts on the set of d–split pairs of $\mathbb{G}$ and it stabilizes the relation $\preccurlyeq$. If $(\mathbb{M}_1, \boldsymbol{\mu}_1)$ and $(\mathbb{M}_2, \boldsymbol{\mu}_2)$ are two d–split pairs, we set $(\mathbb{M}_1, \boldsymbol{\mu}_1) \preccurlyeq_{W_{\mathbb{G}}} (\mathbb{M}_2, \boldsymbol{\mu}_2)$ if and only if there exists $w \in W_{\mathbb{G}}$ such that $(\mathbb{M}_1, \boldsymbol{\mu}_1)^w \preccurlyeq (\mathbb{M}_2, \boldsymbol{\mu}_2)$.

For a d–cuspidal pair $(\mathbb{L}, \boldsymbol{\lambda})$ of $\mathbb{G}$, we denote by $\mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \boldsymbol{\lambda}))$ the set of $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})$ such that $(\mathbb{L}, \boldsymbol{\lambda}) \preccurlyeq (\mathbb{G}, \boldsymbol{\gamma})$. Note that $\mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \boldsymbol{\lambda}))$ depends only on the $W_{\mathbb{G}}$–conjugacy class of $(\mathbb{L}, \boldsymbol{\lambda})$.

The following statement is fundamental. It expresses that Deligne–Lusztig induction can be viewed as ordinary induction through generalized Weyl groups, provided each unipotent character is equipped with the appropriate sign, and if we work only with *d–split* Levi subgroups of $\mathbb{G}$.

3.2. Fundamental Theorem.

(1) *For each d, the sets* $\mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \boldsymbol{\lambda}))$ *(where* $(\mathbb{L}, \boldsymbol{\lambda})$ *runs over a complete set of representatives of the* $W_{\mathbb{G}}$*–conjugacy classes of d–cuspidal pairs of* $\mathbb{G}$*) partition* $\mathrm{Uch}(\mathbb{G})$.
(2) *There exists a collection of isometries*

$$I_{(\mathbb{L},\boldsymbol{\lambda})}^{\mathbb{M}} : \mathbb{Z}\mathrm{Irr}(W_{\mathbb{M}}(\mathbb{L}, \boldsymbol{\lambda})) \to \mathbb{Z}\,\mathrm{Uch}(\mathbb{M}, (\mathbb{L}, \boldsymbol{\lambda}))\,,$$

where

- $\mathbb{M}$ *runs over the set of all d–split generic subgroups of* $\mathbb{G}$,
- $(\mathbb{L}, \boldsymbol{\lambda})$ *runs over the set of d–cuspidal pairs of* $\mathbb{M}$,

such that

(a) for all $\mathbb{M}$ *and all* $(\mathbb{L}, \boldsymbol{\lambda})$, *we have*

$$R_{\mathbb{M}}^{\mathbb{G}} \cdot I_{(\mathbb{L},\boldsymbol{\lambda})}^{\mathbb{M}} = I_{(\mathbb{L},\boldsymbol{\lambda})}^{\mathbb{G}} \cdot \mathrm{Ind}_{W_{\mathbb{M}}(\mathbb{L},\boldsymbol{\lambda})}^{W_{\mathbb{G}}(\mathbb{L},\boldsymbol{\lambda})}\,.$$

(b) The collection $\left(I_{(\mathbb{L},\boldsymbol{\lambda})}^{\mathbb{M}}\right)_{\mathbb{M},(\mathbb{L},\boldsymbol{\lambda})}$ *is stable under the conjugation action by* $W_{\mathbb{G}}$.

(c) $I_{(\mathbb{L},\boldsymbol{\lambda})}^{\mathbb{L}}$ *maps the trivial character of the trivial group* $W_{\mathbb{L}}(\mathbb{L}, \boldsymbol{\lambda})$ *to* $\boldsymbol{\lambda}$.

We believe that 3.2 will have a general and nice proof, involving in particular a generalization of the notion of Hecke algebras (*cf.* [Br1] last paragraph, and [Br2]) and a suitable generalization of Howlett–Lehrer theory (which must be understood as adapted to the case $d = 1$ — see below) to a "d–Howlett–Lehrer theory". Unfortunately such a theory seems out of reach for us at the moment.

Indeed, the case $d = 1$ is known: the first assertion is the so–called Harish–Chandra theory, and the second assertion can be deduced from theorem 5.9 in [HoLe].

We provide a case by case proof of the fundamental theorem. The case of type A will be treated first; for the classical groups, we give a rough sketch (we refer to [En2] for a more detailed proof), and a case-by-case analysis for the exceptional groups using tables given in the appendix.

It will be convenient to prove 3.2 together with the following result, stating a general "Ennola"–duality via $\sigma^{\mathbb{G}}$ for Deligne–Lusztig induction:

3.3. Theorem[1] **(***"Ennola"–duality for* $R_{\mathbb{L}}^{\mathbb{G}}$**).** *There exists a bijective isometry* $\sigma^{\mathbb{G}}$: $\mathrm{Uch}(\mathbb{G}) \to \mathrm{Uch}(\mathbb{G}^-)$ *such that the following diagram is commutative :*

$$\begin{array}{ccc} \mathbb{Z}\,\mathrm{Uch}(\mathbb{G}) & \xrightarrow{\sigma^{\mathbb{G}}} & \mathbb{Z}\,\mathrm{Uch}(\mathbb{G}^-) \\ {\scriptstyle R_{\mathbb{L}}^{\mathbb{G}}}\big\uparrow & & \big\uparrow{\scriptstyle R_{\mathbb{L}^-}^{\mathbb{G}^-}} \\ \mathbb{Z}\,\mathrm{Uch}(\mathbb{L}) & \xrightarrow{\sigma^{\mathbb{L}}} & \mathbb{Z}\,\mathrm{Uch}(\mathbb{L}^-) \end{array} .$$

We define a sign $\varepsilon_{\boldsymbol{\gamma}}$ and $\boldsymbol{\gamma}^-$ by $\sigma^{\mathbb{G}}(\boldsymbol{\gamma}) = \varepsilon_{\boldsymbol{\gamma}}\boldsymbol{\gamma}^-$ where $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})$ and $\boldsymbol{\gamma}^- \in \mathrm{Uch}(\mathbb{G}^-)$. Note that the $\sigma^{\mathbb{G}}$ defined above extends the one of 1.20 via the isometry of $\mathcal{CF}(\mathbb{G})$ with the subspace of uniform functions of $\mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$.

Proof of 3.2 and 3.3. First note that it suffices to prove both statements in the case where $\mathbb{G}$ is simple. Namely, by 1.35 and remark 1.24, we may reduce to the case of adjoint generic groups, which are direct products of simple generic groups $\mathbb{G}^{(a)}$. Since $R_{\mathbb{L}}^{\mathbb{G}}$ is compatible with products, we may hence assume that we have just one factor $\mathbb{G}^{(a)}$, and finally, by 1.36, that $a = 1$.

The case of the type A.

Since the property we want to prove concerns only the unipotent characters, we may as well assume that we are dealing with the general linear or unitary group.

Let us denote by $\mathbb{GL}_n$ the generic finite group corresponding to the general linear groups of rank n. Thus $\mathbb{GL}_n = ((\mathbb{Z}^n, R, \mathbb{Z}^n, R^\vee), \mathfrak{S}_n)$ with $R = R^\vee = \{(e_i - e_j)\,;\,(1 \leq i \neq j \leq n)\}$, where $\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{Z}^n$.

Then the generic finite group corresponding to the unitary groups of rank n is $\mathbb{GU}_n := \mathbb{GL}_n^-$, where $\mathbb{GL}_n^- = ((\mathbb{Z}^n, R, \mathbb{Z}^n, R^\vee), \mathfrak{S}_n.(-\mathrm{Id}))$ (*cf.* [BrMa], §2).

If we have shown 3.3, then it suffices to prove 3.2 for $\mathbb{GL}_n$. But 3.3 is well known in the case of type A. Namely, by [LuSr] (see for example [DiMi2], 15.4) if we take as reference torus the "diagonal" maximal torus of $\mathbf{GL}_n(\bar{\mathbb{F}}_q)$ (which is stable by both the standard and the "twisted" Frobenius endomorphism), the generic irreducible unipotent characters of $\mathbb{GL}_n$ are the functions

$$\boldsymbol{\Phi}_\chi^{\mathbb{GL}_n} := \frac{1}{n!}\sum_{w\in\mathfrak{S}_n} \chi(w) R_w^{\mathbb{GL}_n} \quad \text{for } \chi \in \mathrm{Irr}(\mathfrak{S}_n)\,,$$

while the generic irreducible unipotent characters of $\mathbb{GU}_n$ are the $\varepsilon_\chi \boldsymbol{\Phi}_\chi^{\mathbb{GU}_n}$ where $\varepsilon_\chi = \pm 1$ and

$$\boldsymbol{\Phi}_\chi^{\mathbb{GU}_n} := \frac{1}{n!}\sum_{w\in\mathfrak{S}_n} \chi(w) R_w^{\mathbb{GU}_n} \quad \text{for } \chi \in \mathrm{Irr}(\mathfrak{S}_n)\,.$$

[1]At present, when $\mathbb{G}$ contains a factor of exceptional type, we have only verified this result when $\mathbb{L}$ is d–split for some d.

Identifying $\mathrm{Uch}(\mathbb{G})$ and $\mathrm{Uch}(\mathbb{G}^-)$ with $\mathrm{Irr}(\mathfrak{S}_n)$, we then see that the map $\chi \mapsto \varepsilon_\chi \chi$ is the map $\sigma_\mathbb{G}$, which shows 3.3.

The natural bijection $\mathbb{M} \mapsto \mathbb{M}^-$ between the set of generic Levi subgroups of $\mathbb{GL}_n$ and the set of generic Levi subgroups of $\mathbb{GU}_n$ is such that the $\Phi_d(x)$–subgroups of $\mathbb{GL}_n$ correspond to the $\Phi_d(-x)$–subgroups of $\mathbb{GU}_n$ (recall that, for $d > 2$, $\Phi_d(-x) = \Phi_{2d}(x)$, $\Phi_{d/2}(x)$, or $\Phi_d(x)$ if d is respectively odd, congruent to 2 modulo 4, or divisible by 4). So to prove 3.2 for type A, we may as well assume that $\mathbb{G} = \mathbb{GL}_n$, which we do from now on.

We identify $\mathcal{CF}(\mathbb{G})$ and the subspace of uniform functions of $\mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$ via the map $\alpha \mapsto \mathbf{\Phi}_\alpha^\mathbb{G}$ (see 1.28). Thus the map $R_\mathbb{L}^\mathbb{G}$ is identified to the induction $\mathrm{Ind}_\mathbb{L}^\mathbb{G}$ on class functions (see 1.32, (2)) and ${}^*R_\mathbb{L}^\mathbb{G}$ to $\mathrm{Res}_\mathbb{L}^\mathbb{G}$.

Let $\boldsymbol{\gamma}$ be an irreducible character of $\mathfrak{S}_n$ and let $w \in \mathfrak{S}_n$. Then ${}^*R_{\mathbb{T}_w}^\mathbb{G}(\boldsymbol{\gamma}) = \boldsymbol{\gamma}(w)\mathbf{1}$ and (*cf.* remark 2 after 2.1) $\boldsymbol{\gamma}$ is d–cuspidal if and only if the partition of n corresponding to γ is a d-core.

Moreover, a d–split Levi subgroup of $\mathbb{G}$ is of type $\mathbb{GL}_{n_1}^{(d)} \times \mathbb{GL}_{n_2}^{(d)} \times \cdots \times \mathbb{GL}_{n_a}^{(d)} \times \mathbb{GL}_s$ (*cf.* 1.3). It is then easy to see that a d–cuspidal pair of $\mathbb{G}$ is a pair $(\mathbb{L}, \boldsymbol{\lambda})$ such that $\mathbb{L}$ is the direct product $\mathbb{T} \times \mathbb{GL}_r$ where

- $\mathbb{T}$ is a torus such that $|\mathbb{T}| = (x^d - 1)^a$,
- $\boldsymbol{\lambda}$ (with the appropriate identifications) is a character of $\mathfrak{S}_r$ corresponding to a d–core of cardinal r.

Thus $n = ad + r$, and $\mathbb{L} = (\Gamma_\mathbb{L}, W_\mathbb{L} v)$, where

- $\Gamma_\mathbb{L}$ is a suitable root datum,
- $W_\mathbb{L} \simeq \mathfrak{S}_r$,
- v is an element of cycle type d^a of $\mathfrak{S}_{ad}$

(here we identify $\mathfrak{S}_{ad}$ with the obvious subgroup of the Young subgroup $\mathfrak{S}_{ad} \times \mathfrak{S}_r$ in $\mathfrak{S}_n$).

We have $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \boldsymbol{\lambda}))$ if and only if $\boldsymbol{\lambda}$ is a d–core of $\boldsymbol{\gamma}$. This proves in particular the first assertion of 3.2.

A d–split Levi subgroup $\mathbb{M}$ containing $\mathbb{L}$ is of type $\mathbb{GL}_{n_1}^{(d)} \times \cdots \times \mathbb{GL}_{n_a}^{(d)} \times \mathbb{GL}_s$, where $s \geq r$. We can compute $R_\mathbb{M}^\mathbb{G}$ by computing successively $R_\mathbb{M}^{\mathbb{M}_1}$, then $R_{\mathbb{M}_1}^{\mathbb{M}_2}$, etc., where $\mathbb{M}_1$ is of type $\mathbb{GL}_{n_1}^{(d)} \times \cdots \times \mathbb{GL}_{n_{a-1}}^{(d)} \times \mathbb{GL}_{dn_a+s}$, $\mathbb{M}_2$ is of type $\mathbb{GL}_{n_1}^{(d)} \times \cdots \times \mathbb{GL}_{n_{a-2}}^{(d)} \times \mathbb{GL}_{dn_{a-1}+dn_a+s}$, and so we see that it is enough to assume (since $R_\mathbb{L}^\mathbb{G}$ behaves nicely with respect to products)

$$\mathbb{G} = \mathbb{GL}_n \quad \text{and} \quad \mathbb{M} = \mathbb{GL}_b^{(d)} \times \mathbb{GL}_s$$

with $\mathbb{L} \subseteq \mathbb{M}$, which we do from now on. Thus $n = ad + r = bd + s$ with $r \leq s$, $\mathbb{G} = (\Gamma_\mathbb{G}, W_\mathbb{G})$ with $W_\mathbb{G} \simeq \mathfrak{S}_n$, and $\mathbb{M} = (\Gamma_\mathbb{M}, W_\mathbb{M} w)$ with

- $W_\mathbb{M} \simeq (\mathfrak{S}_b)^d \times \mathfrak{S}_s$, a Young subgroup of $\mathfrak{S}_n$
- w is an element of cycle type d^b in $\mathfrak{S}_{bd}$ (note that $W_\mathbb{M} w = W_\mathbb{M} v$).

The class functions on $\mathbb{G}$ are the usual class functions on $\mathfrak{S}_n$, while the set $\mathcal{CF}(\mathbb{M})$ may be identified with the set of usual class functions on $\mathfrak{S}_b \times \mathfrak{S}_s$. Indeed, every $\beta \in \mathcal{CF}(\mathbb{M})$ (*i.e.,* β is a $W_\mathbb{M}$–stable function on $W_\mathbb{M} w$) defines a class function β_b on $\mathfrak{S}_b$ as follows: for $(\sigma_1, \sigma_2, \ldots, \sigma_d) \in (\mathfrak{S}_b)^d$, we set $\beta_b(\sigma_1\sigma_2 \cdots \sigma_d) := \beta((\sigma_1, \sigma_2, \ldots, \sigma_d).w)$.

We have

$$W_{\mathbb{G}}(\mathbb{L},\boldsymbol{\lambda}) = C_{\mathfrak{S}_{ad+r}}(v)/\mathfrak{S}_r \simeq ((\mathbb{Z}/d\mathbb{Z})\wr\mathfrak{S}_a)\,,$$

and

$$W_{\mathbb{M}}(\mathbb{L},\boldsymbol{\lambda}) \simeq \mathfrak{S}_b \times ((\mathbb{Z}/d\mathbb{Z})\wr\mathfrak{S}_{a-b})\,.$$

With the preceding identifications, the description of $I^{\mathbb{G}}_{(\mathbb{L},\boldsymbol{\lambda})}$ and $I^{\mathbb{M}}_{(\mathbb{L},\boldsymbol{\lambda})}$ amounts to the description of corresponding maps $I^m_{d,\boldsymbol{\lambda}}$ $(m = n, s)$, where

$$I^n_{d,\boldsymbol{\lambda}}\colon \mathbb{Z}\mathrm{Irr}((\mathbb{Z}/d\mathbb{Z})\wr\mathfrak{S}_a) \to \mathbb{Z}\mathrm{Irr}(\mathfrak{S}_n,(d,\boldsymbol{\lambda}))\,,$$
$$I^s_{d,\boldsymbol{\lambda}}\colon \mathbb{Z}\mathrm{Irr}((\mathbb{Z}/d\mathbb{Z})\wr\mathfrak{S}_{a-b}) \to \mathbb{Z}\mathrm{Irr}(\mathfrak{S}_s,(d,\boldsymbol{\lambda}))\,,$$

where $\mathrm{Irr}(\mathfrak{S}_m,(d,\boldsymbol{\lambda}))$ denotes the set of irreducible characters of $\mathfrak{S}_m$ (partitions of m) with d–core $\boldsymbol{\lambda}$. We want these maps to be such that the following diagram is commutative :

$$\text{(3.4)}\qquad \begin{array}{ccc} \mathbb{Z}\mathrm{Irr}((\mathbb{Z}/d\mathbb{Z})\wr\mathfrak{S}_a)) & \xrightarrow{I^n_{d,\boldsymbol{\lambda}}} & \mathbb{Z}\mathrm{Irr}(\mathfrak{S}_n) \\ \downarrow & & \downarrow \\ \mathbb{Z}\mathrm{Irr}(\mathfrak{S}_b\times(\mathbb{Z}/d\mathbb{Z})\wr\mathfrak{S}_{a-b}) & \xrightarrow{\mathrm{Id}_{\mathfrak{S}_b}\otimes I^s_{d,\boldsymbol{\lambda}}} & \mathbb{Z}\mathrm{Irr}(\mathfrak{S}_b\times\mathfrak{S}_s) \end{array}$$

where the vertical arrows are the restriction maps.

With appropriate choices (see [Os]), an irreducible character of $(\mathbb{Z}/d\mathbb{Z})\wr\mathfrak{S}_a$ (resp. of $(\mathbb{Z}/d\mathbb{Z})\wr\mathfrak{S}_{a-b}$) is defined by a family of partitions $(\pi_i \vdash a_i)_{i=1,\dots,d}$ where $\sum_{i=1}^d a_i = a$ (resp. $\sum_{i=1}^d a_i = a-b$). Such a collection may be interpreted as the "d–quotient" of a partition π of n (resp. of s) with d–core $\boldsymbol{\lambda}$, and defines a sign ε_π (*cf.* for example [En1], §3) in a way which is "compatible" with the Murnaghan–Nakayama formula, and this defines the maps $I^m_{d,\boldsymbol{\lambda}}$.

Assume first that $a = b$. In this case 3.4 is equivalent to the following property :

For all $\zeta \in \mathrm{Irr}((\mathbb{Z}/d\mathbb{Z})\wr\mathfrak{S}_a)$, for all $(\sigma_1,\sigma_2,\dots,\sigma_d) \in (\mathfrak{S}_a)^d$ and for all $\sigma \in \mathfrak{S}_r$, we have

$$I^n_{d,\boldsymbol{\lambda}}(\zeta)((\sigma_1,\sigma_2,\dots,\sigma_d).w.\sigma) = \zeta(\sigma_1\sigma_2\cdots\sigma_d)\boldsymbol{\lambda}(\sigma)\,.$$

This equality results from the Murnaghan–Nakayama formula for wreath products (see [Os] or also [Rou]) and from the fact that the cycle type of $((\sigma_1,\sigma_2,\dots,\sigma_d).w) \in \mathfrak{S}_{ad}$ is d times the cycle type of $\sigma_1\sigma_2\cdots\sigma_d \in \mathfrak{S}_a$.

The general case $a > b$ is analogous, and is translated as follows.

We let w be as above. Then for $\zeta \in \mathrm{Irr}((\mathbb{Z}/d\mathbb{Z})\wr\mathfrak{S}_a)$ the value of $I^n_{d,\boldsymbol{\lambda}}(\zeta)$ on $((\mathfrak{S}_b)^d\times(\mathfrak{S}_{a-b})^d\times\mathfrak{S}_r).w$ is given as follows:

For $(\sigma_1,\sigma_2,\dots,\sigma_d,\sigma'_1,\sigma'_2,\dots,\sigma'_d) \in (\mathfrak{S}_b)^d\times(\mathfrak{S}_{a-b})^d$ and $\sigma\in\mathfrak{S}_r$ we have

$$I^n_{d,\boldsymbol{\lambda}}(\zeta)((\sigma_1,\sigma_2,\dots,\sigma_d,\sigma'_1,\sigma'_2,\dots,\sigma'_d).w.\sigma) = \zeta((\sigma_1\sigma_2\cdots\sigma_d)(\sigma'_1\sigma'_2\cdots\sigma'_d))\boldsymbol{\lambda}(\sigma)\ .$$

Let now w' be an element of cycle type $(d)^{a-b}$ in $\mathfrak{S}_{(a-b)d}$, and let ζ_b denote the restriction of ζ to $\mathfrak{S}_b\times(\mathbb{Z}/d\mathbb{Z})\wr\mathfrak{S}_{a-b}$. Then the value of $(\mathrm{Id}_{\mathfrak{S}_b}\otimes I^s_{d,\boldsymbol{\lambda}})(\zeta_b)$ on $(\mathfrak{S}_b\times(\mathfrak{S}_{a-b})^d\times\mathfrak{S}_r).w'$ is given as follows:

For $(\sigma_1,\sigma_2,\dots,\sigma_d,\sigma'_1,\sigma'_2,\dots,\sigma'_d) \in (\mathfrak{S}_b)^d\times(\mathfrak{S}_{a-b})^d$ and $\sigma\in\mathfrak{S}_r$ we have

$$(\mathrm{Id}_{\mathfrak{S}_b}\otimes I^s_{d,\boldsymbol{\lambda}})(\zeta_b)((\sigma_1\sigma_2\cdots\sigma_d)(\sigma'_1,\sigma'_2,\dots,\sigma'_d).w'.\sigma) = \zeta((\sigma_1\sigma_2\cdots\sigma_d)(\sigma'_1\sigma'_2\cdots\sigma'_d))\boldsymbol{\lambda}(\sigma)\ .$$

It is clear that the preceding formulae prove the commutativity of 3.4.

Note also that statement (b) of the theorem holds trivially since all automorphisms of $\mathbb{GL}_n$ fix all unipotent characters.

The case of classical groups.

We use Asai's results on the decomposition of the Lusztig functor $R_{\mathbb{L}}^{\mathbb{G}}$ as cited in the paper [FoSr] of Fong and Srinivasan.

Let $\mathbb{G}$ be a generic group of type B_l, C_l, D_l or ${}^2D_l$. Lusztig has shown that the unipotent characters $\mathrm{Uch}(\mathbb{G})$ of these groups may be parametrized by means of so called symbols. A *symbol* is an unordered set $\{S,T\}$ of two strictly increasing sequences $S = x_1 < \cdots < x_a$, $T = y_1 < \cdots < y_b$ of nonnegative integers, usually denoted as

$$\Lambda = \begin{pmatrix} x_1 & \ldots & x_a \\ y_1 & \ldots & y_b \end{pmatrix}.$$

Two symbols are considered to be equivalent if they can be transformed into each other by a sequence of steps

$$\begin{pmatrix} x_1 & \ldots & x_a \\ y_1 & \ldots & y_b \end{pmatrix} \sim \begin{pmatrix} 0 & x_1+1 & \ldots & x_a+1 \\ 0 & y_1+1 & \ldots & y_b+1 \end{pmatrix}.$$

The *rank* of a symbol Λ is

$$\mathrm{rank}(\Lambda) := \sum x_i + \sum y_i - \left[\left(\frac{a+b-1}{2} \right)^2 \right],$$

and $|a-b|$ is called the *defect* of Λ. The unipotent characters of groups of type B_l and C_l are in bijection with the equivalence classes of symbols of rank l and odd defect. The unipotent characters of groups of type D_l are parametrized by (classes of) symbols of rank l and defect divisible by 4, except that if the two rows of Λ are equal, two unipotent characters correspond to the same symbol; such symbols are called *degenerate*. Finally the unipotent characters of groups of type ${}^2D_l$ are parametrized by symbols Λ of rank l and defect congruent $2 \pmod 4$ (*cf.* [Ca], 13.8).

Let us first assume that the integer d of the theorem is odd. We will need the decomposition of $R_{\mathbb{L}}^{\mathbb{G}}$ in the particular case of a Levi subgroup $\mathbb{L} = \mathbb{T} \times \mathbb{H}$, where $\mathbb{H}$ is a generic classical group of the same type as $\mathbb{G}$ (or possibly $\mathbb{G}^-$ in the case of even orthogonal groups) and $\mathbb{T}$ is a torus such that $|\mathbb{T}| = x^f - 1$ (we will always have $d|f$). In this particular situation a unipotent character of $\mathbb{H}$ and hence of $\mathbb{L}$ is indexed by a symbol $\Lambda = (S,T)$. The decomposition of $R_{\mathbb{L}}^{\mathbb{G}}(\gamma_\Lambda)$ is explicitly described by a simple formula of Asai given in (3.1) of [FoSr]. Namely,

$$(3.5) \quad R_{\mathbb{L}}^{\mathbb{G}}(\gamma_\Lambda) = \sum_{\substack{x \in S \\ x+f \notin S}} (-1)^{M_{S,x}} \gamma_{S-\{x\}\cup\{x+f\},T} + \sum_{\substack{y \in T \\ y+f \notin T}} (-1)^{M_{T,y}} \gamma_{S,T-\{y\}\cup\{y+f\}},$$

where $M_{S,x} := |\{s \in S \mid x < s < x+f\}|$. Also by convention if a degenerate symbol occurs on the right hand side of the formula, then it is understood that both unipotent characters corresponding to it appear in the sum. The formula above has a convenient interpretation in terms of f-hooks: The symbols occurring as indices of unipotent characters on the right hand side are precisely those which are obtained from Λ by adding an f-hook. (The concept of hooks, cohooks, cores and cocores of symbols is explained in detail in [Ol] and [FoSr], for example.)

The proof will be in several steps. We first describe the d-cuspidal pairs. Each symbol Λ has a uniquely determined d-core. We claim that the d-cuspidal unipotent characters $\boldsymbol{\gamma}_\Lambda$ of $\mathbb{G}$ are precisely those where Λ itself is a d-core. Indeed it is easily seen that the generic d-split subgroups $\mathbb{L}$ of $\mathbb{G}$ all have type

$$\mathbb{GL}_{n_1}^{(d)} \times \cdots \times \mathbb{GL}_{n_a}^{(d)} \times \mathbb{H},$$

where $\mathbb{H}$ has the same type as $\mathbb{G}$ (or possibly $\mathbb{G}^-$). Now by 1.36 the d-cuspidal unipotent characters of $\mathbb{GL}_n^{(d)}$ are in natural bijection with the 1-cuspidal characters of $\mathbb{GL}_n$, and as we already saw above such characters exist only if $n = 1$. Hence d-split subgroups $\mathbb{L}$ can have d-cuspidal unipotent characters only if $\mathbb{L}$ is of type

$$\mathbb{T} \times \mathbb{H}, \qquad \text{where } |\mathbb{T}| = (x^d - 1)^a, \tag{3.6}$$

where $\mathbb{H}$ has the type of $\mathbb{G}^{\pm}$. In turn, if $a > 0$ by induction these d-cuspidal characters have the form $\boldsymbol{\lambda} = 1 \times \cdots \times 1 \times \boldsymbol{\gamma}_\Lambda$, where Λ is a d-core. Now $R_\mathbb{L}^\mathbb{G}$ behaves well with respect to direct products. So the constituents of $R_\mathbb{L}^\mathbb{G}(\boldsymbol{\lambda})$ for $(\mathbb{L}, \boldsymbol{\lambda})$ as above may be computed by repeated application of Asai's formula 3.5 in the particular case $f = d$, using transitivity. Clearly all constituents of $R_\mathbb{L}^\mathbb{G}(\boldsymbol{\lambda})$ have the same d-core Λ, so we obtain a partition of the unipotent characters of $\mathbb{G}$ into series, such that two characters lie in the same series if and only if their associated symbols have the same d-core. This completes the description of the d-cuspidal pairs and of the d-series and proves 3.2(1).

We next show that the series are disjoint even with respect to induction from intermediate d-split Levi subgroups. Obviously by induction it suffices to do this for maximal subgroups $\mathbb{M}$ (see the reasoning for type A). Up to $\mathbb{G}$-conjugacy any maximal d-split generic subgroup of $\mathbb{G}$ containing $\mathbb{L}$ has the form

$$\mathbb{M} = \mathbb{GL}_b^{(d)} \times \mathbb{H}' \tag{3.7}$$

where $b \leq a$ and $\mathbb{H}'$ has the same type as $\mathbb{G}^{\pm}$. Although the formula of Asai doesn't explicitly yield the decomposition of $R_\mathbb{M}^\mathbb{G}(\boldsymbol{\mu})$ for $\boldsymbol{\mu}$ a unipotent character of $\mathbb{M}$, we can still deduce it. Namely, the unipotent characters of $\mathbb{GL}_b^{(d)}$ are indexed by the irreducible characters χ of $\mathfrak{S}_b = W(\mathbb{GL}_b)$. The unipotent character of $\mathbb{GL}_b$ corresponding to χ is

$$\boldsymbol{\gamma}_\chi = \frac{1}{|\mathfrak{S}_b|} \sum_{w \in \mathfrak{S}_b} \chi(w) R_{\mathbb{T}_w}^{\mathbb{GL}}(\mathbf{1}), \tag{3.8}$$

where $|\mathbb{T}_w| = \prod_i (x^{db_i} - 1)$ when w has cycle type $\prod_i (b_i)$. It is now clear by repeated application of Asai's formula for $f = db_i$ that the constituents of $R_\mathbb{M}^\mathbb{G}(\boldsymbol{\mu})$ are indexed by symbols obtained from Λ by adding db_i-hooks, and since the addition of a db_i-hook may be mimicked by the addition of b_i d-hooks, $R_\mathbb{M}^\mathbb{G}$ preserves the series defined above, which proves disjointness.

Thus it remains to establish the existence of the collection of maps $I_{(\mathbb{L},\boldsymbol{\lambda})}^\mathbb{M}$. As in the case of type A above it is sufficient to do this for $\mathbb{M}$ maximal. Let $\mathbb{L}$ be a fixed d-split subgroup of $\mathbb{G}$ as in 3.6 with $\mathbb{H}$ of rank r, so that $l = ad + r$, and $\boldsymbol{\lambda} = \boldsymbol{\gamma}_\Lambda$ a d-cuspidal character of $\mathbb{H}$. Then $\mathbb{M}$ has the form 3.7. Let us assume for the moment that $\mathbb{G}$ has type B_l or C_l. Then $W_\mathbb{G}(\mathbb{L}, \boldsymbol{\lambda}) \cong (\mathbb{Z}/2d\mathbb{Z}) \wr \mathfrak{S}_a$ (where implicitly we have already made all the identifications as in the proof for type A). The characters of this wreath product are in bijection with $2d$-tuples of partitions $(\omega_i \vdash a_i)$ with $\sum a_i = a$. As we have seen, the unipotent characters in $\mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \boldsymbol{\lambda}))$ are parametrized by symbols Ψ with given d-core Λ and hence weight a. Since a symbol $\Lambda = \{S, T\}$ is uniquely determined (up to equivalence) by its defect and by the two partitions represented by the β-numbers S, T, we get a bijection from the set of symbols with a given d-core onto the set of pairs of partitions of a_1, a_2 with prescribed d-cores and $a_1 + a_2 = a$ (we get ordered pairs since the defect of Λ is odd). These in turn are in bijection with the set of pairs of d-quotients (see [JaKe]), hence with $2d$-tuples of partitions $(\omega_i \vdash a_i)$ of $a = \sum a_i$. This defines a bijection between $\mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \boldsymbol{\lambda}))$ and $\mathrm{Irr}(W_\mathbb{G}, (\mathbb{L}, \boldsymbol{\lambda}))$.

For $\mathbb{M}$ as above we also have $W_\mathbb{M}(\mathbb{L}, \boldsymbol{\lambda}) \cong \mathfrak{S}_b \times ((\mathbb{Z}/2d\mathbb{Z}) \wr \mathfrak{S}_{a-b})$. The set $\mathrm{Uch}(\mathbb{M}, (\mathbb{L}, \boldsymbol{\lambda}))$ is parametrized by pairs (ω, Γ) consisting of a partition $\omega \vdash b$ and a symbol Γ for $\mathbb{H}'$ with d-core Λ. We have an obvious bijection from these to $\mathrm{Irr}(W_\mathbb{M}(\mathbb{L}, \boldsymbol{\lambda}))$.

We now have to compare the decomposition of $R_\mathbb{M}^\mathbb{G}(\boldsymbol{\mu})$, for $\boldsymbol{\mu}$ a constituent of $R_\mathbb{L}^\mathbb{M}(\boldsymbol{\lambda})$, with the corresponding decomposition of induced characters in the Weyl group. We first note that by (3B) of [FoSr] there is associated a well defined sign ϵ to each unipotent character of $\mathbb{G}$ so that no cancelling takes place in the induction.

We make again use of the decomposition 3.8 for the unipotent characters of $\mathbb{GL}_b$. The different tori $\mathbb{T}_w$ appearing in the sum correspond to the one element subsets $\{w\}$. Induction to $W_\mathbb{G}(\mathbb{L}, \boldsymbol{\lambda})$ just yields the characteristic function on the corresponding class, so that the multiplicity of any irreducible character χ of $W_\mathbb{G}(\mathbb{L}, \boldsymbol{\lambda})$ is just equal to the value $\chi(w)$. This value may be computed by the generalized Murnaghan-Nakayama formula of Osima [Os] for wreath products, by successively removing hooks of length n_i from w of cycle type $\prod_i n_i$. On the other hand, the multiplicity of a unipotent character of $\mathbb{G}$ in $R_{\mathbb{T}_w}^\mathbb{G}(\mathbf{1})$ is seen by 3.5 to be obtained (up to sign) by the same procedure of adding hooks of length dn_i. Adding a hook of length dn_i is equivalent to adding a hook of length n_i to one of the partitions of the d-quotient. Hence the multiplicities on both sides agree and the assertion of 3.2 follows for types B and C.

In types D and ${}^2D$, two complications arise. Firstly, degenerate symbols may occur in 3.5, and secondly, sometimes the normalizer $W_\mathbb{G}(\mathbb{L}, \boldsymbol{\lambda})$ is not the full wreath product. It will become clear presently that both problems are intimately linked. But if both do not occur, the above proof carries through precisely as for types B and C.

As the d-core of a degenerate symbol is easily seen to be also degenerate, the first complication occurs when the cuspidal character $\boldsymbol{\lambda}$ is indexed by a degenerate symbol. But also, the normalizer $W_\mathbb{G}(\mathbb{L}, \boldsymbol{\lambda})$ is not the full wreath product precisely when Λ is degenerate. So it remains to check the existence of $I_{(\mathbb{L}, \boldsymbol{\lambda})}^\mathbb{M}$ in this case. Then $W_\mathbb{G}(\mathbb{L}, \boldsymbol{\lambda})$ is a subgroup of index two in $(\mathbb{Z}/2d\mathbb{Z}) \wr \mathfrak{S}_a$, generated by the elements of $\mathfrak{S}_a$ and the even elements (products of two elements) in $(\mathbb{Z}/2d\mathbb{Z})^a$. It hence suffices to see that precisely those characters χ of $(\mathbb{Z}/2d\mathbb{Z}) \wr \mathfrak{S}_a$ indexed by $2d$-tuples of partitions $(\omega_i \vdash a_i)$ of $a = \sum a_i$ with $a_i = a_{i+d}$ and $\omega_i = \omega_{i+d}$, $1 \le i \le d$, split when restricted to the subgroup described above. This in turn is equivalent to saying that precisely those χ vanish on all odd elements of $(\mathbb{Z}/2d\mathbb{Z}) \wr \mathfrak{S}_a$. This final fact follows from the

Murnaghan-Nakayama formula in [Os], and the proof for odd d is completed.

It remains to treat the case when d is even. This resembles very much the previous case, so we do not go into any details. (Indeed, the particular case where d is twice an odd number will follow immediately from 3.3, which is proved below.) Let $\mathbb{L}$ be a generic Levi subgroup of $\mathbb{G}$ of type $\mathbb{T}\times\mathbb{H}$, where $\mathbb{H}$ is a generic classical group of the same type as $\mathbb{G}$ (or possibly $\mathbb{G}^-$) and $\mathbb{T}$ is a torus such that $|\mathbb{T}|=x^f+1$. Again, the decomposition of $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\gamma}_\Lambda)$, where $\boldsymbol{\gamma}_\Lambda$ denotes the unipotent character of $\mathbb{L}$ indexed by the symbol Λ, was explicitly given in [FoSr], (3.2). Namely, $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\gamma}_\Lambda)$ equals

(3.9)
$$\sum_{\substack{x\in S\\ x+f\notin T}}(-1)^{N_{S,x}+N_{T,x+f}}\boldsymbol{\gamma}_{S-\{x\},T\cup\{x+f\}}+\sum_{\substack{y\in T\\ y+f\notin S}}(-1)^{N_{S,y+f}+N_{T,y}}\boldsymbol{\gamma}_{S\cup\{y+f\},T-\{y\}}.$$

Here $N_{S,x}:=|\{s\in S\mid s<x\}|$, and the same convention concerning degenerate symbols as in 3.5 applies. The symbols occurring as indices of unipotent characters on the right hand side are precisely those which are obtained from Λ by adding an f-cohook. Each symbol has a well defined f-cocore, and arguments as in the previous part now show that the d-cuspidal pairs of $\mathbb{G}$ are the $(\mathbb{L},\boldsymbol{\lambda})$ where $\boldsymbol{\lambda}$ is a unipotent character of the form $1\times\cdots\times 1\times\boldsymbol{\gamma}_\Lambda$ of

$$\mathbb{L}=\mathbb{T}\times\mathbb{H},\qquad \text{where } |\mathbb{T}|=(x^e+1)^a,\qquad e:=d/2,$$

and where Λ is an e-cocore. As above this is an easy consequence of 3.9. In types B_l and C_l we have $W_{\mathbb{G}}(\mathbb{L},\boldsymbol{\lambda})\cong Z_{2e}\wr\mathfrak{S}_a$. The same holds for D_l and ${}^2D_l$, except that when Λ is degenerate, we only obtain the subgroup of even elements as previously. It is well known that for symbols the mechanism of d–hooks when d is odd is completely parallel to that of $d/2$–cohooks when d is even. Since this same parallelity also holds for Asai's formula 3.9, the previous arguments may be applied with only minor changes. A more detailed account is contained in [En2].

We next prove the general Ennola–duality for $R_{\mathbb{L}}^{\mathbb{G}}$ in the classical groups. Let $\Lambda=(S,T)$ be a symbol. Define $\Lambda^-=(S^-,T^-)$ to be the symbol obtained from Λ by moving all odd entries from S to T and vice versa. In fact, as $(S,T)=(T,S)$ by definition, we might as well exchange the even entries of S and T to obtain Λ^-. We claim $\sigma^{\mathbb{G}}$ is the mapping

$$\boldsymbol{\gamma}_\Lambda\mapsto\epsilon_\Lambda\boldsymbol{\gamma}_{\Lambda^-},$$

where ϵ_Λ is the parity of the degree in x of $\mathrm{Deg}(\boldsymbol{\gamma}_\Lambda)$ (note that ϵ_Λ is the sign by which one has to adjust $\mathrm{Deg}(\boldsymbol{\gamma}_\Lambda)(-q)$ to obtain a positive value). To prove 3.3, observe that any Levi subgroup of a classical group contains at most one factor of classical type, *i.e.,* not of type A. Since all unipotent characters of type-A groups are uniform, the formulae of Asai 3.5, 3.9, enable us to compute the decomposition of $R_{\mathbb{L}}^{\mathbb{G}}$ for any Levi subgroup $\mathbb{L}$ of any classical group. For type A the desired commutativity has already been proved, and the definition of the sign ϵ_Λ agrees with the one given there. So by induction it therefor suffices to prove the commutativity of 3.3 in the case that $\mathbb{L}$ has the particular form assumed in Asai's formula.

The operation $\Lambda\mapsto\Lambda^-$ interchanges odd hooks and odd cohooks, and leaves even hooks and even cohooks alone. It follows that the symbols for constituents of $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\gamma}_{\Lambda^-})$

are just the images under $^-$ of the constituents of $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\gamma}_\Lambda)$. Hence we are reduced to checking the correctness of the signs. So assume that $\mathbb{L} = \mathbb{T} \times \mathbb{H}$ as in 3.5, and Ψ is obtained by adding an f-hook at position x of Λ. Shifting the symbol if necessary, we may assume that x is even. The equality to be proved after evaluating 3.5 and 3.9 is

$$(-1)^{M_{S,x}} = \epsilon_\Lambda \epsilon_\Psi (-1)^{N_{S^-,x}+N_{T^-,x+f}}.$$

By the definition of S^-, T^-, M and N, this may be written as

$$\epsilon_\Lambda \epsilon_\Psi = (-1)^{1+N_{S,x}+N_{T,x}+N_1},$$

where $N_1 := |\{s \in S \mid x \leq s < x+f, s \text{ even}\}| + |\{t \in T \mid x \leq t < x+f, t \text{ even}\}|$. But by the degree formula for $\boldsymbol{\gamma}_\Lambda$ given in [Ca], 13.8, the right hand side is the change in parity of the degree of $\mathrm{Deg}(\boldsymbol{\gamma}_\Lambda)$ when Λ is replaced by Ψ, whence the desired result follows for odd length hooks. (Note that if $\mathbb{L}$ is of type D, exactly three of the four groups $\mathbb{L}$, $\mathbb{L}^-$, $\mathbb{G}$ and $\mathbb{G}^-$ are of the same orthogonal type. This accounts for a factor -1 in the above equation.) If f is even, one obtains similarly

$$\epsilon_\Lambda \epsilon_\Psi = (-1)^{1+N_2},$$

where $N_2 := |\{s \in S \mid x \leq s < x+f, s \text{ odd}\}| + |\{t \in T \mid x \leq t < x+f, t \text{ odd}\}|$. Again this identity is verified using the degree formulae. This completes the proof of 3.3 for classical groups.

The case of exceptional groups.

We now prove theorem 3.2 for exceptional groups of Lie type, and "Ennola"–duality 3.3 in the case of d–split Levi subgroups of such groups.

We assume that $\mathbb{G}$ has a connected Dynkin diagram of exceptional type, so that the corresponding groups of Lie type are simple of exceptional type. In this case we do not have such a nice result on the decomposition of $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda})$ as Asai's formula for the classical groups (*cf.* 3.5, 3.9). But since we are dealing with groups of bounded rank, it can be replaced by the Mackey formula for twisted induction 1.34 (which had been already proved by Deligne if q is large enough, and which follows in general from Shoji's results, *cf.* §1.B). Using the Mackey formula the decomposition may uniquely be determined except for some ambiguities stemming from algebraic conjugate eigenvalues of Frobenius. These we can not resolve, but the attainable information is sufficient to prove the fundamental theorem 3.2.

Our approach will consist in a case by case analysis, but some frequently used arguments are collected at the beginning. Clearly if we know the decomposition of $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda})$ for each d–split generic Levi subgroup $\mathbb{L}$ of $\mathbb{G}$ and each unipotent character $\boldsymbol{\lambda} \in \mathrm{Uch}(\mathbb{L})$, then the assertion of the theorem may be checked. Namely the disjointness and the positivity will follow from the lists, and the existence of the collection of maps $I_{(\mathbb{L},\boldsymbol{\lambda})}^{\mathbb{M}}$ just by comparing these finitely many decompositions with tables showing the decompositions of induced characters in the corresponding Weyl groups. This trivial step of verification will not be made explicit in the proof. We shall just sketch how to find the decomposition of $R_{\mathbb{L}}^{\mathbb{G}}$, and list the relevant centralizers $W_{\mathbb{G}}(\mathbb{L}, \boldsymbol{\lambda})$ in the Weyl group.

If our d–split Levi subgroup $\mathbb{L}$ happens to be a maximal torus $\mathbb{T}$, then $R_{\mathbb{L}}^{\mathbb{G}}$ is just the usual Deligne–Lusztig induction $R_{\mathbb{T}}^{\mathbb{G}}$. The decomposition of $R_{\mathbb{T}}^{\mathbb{G}}$ is known in all cases by the work of Lusztig. This already accounts for about half the cases in the exceptional groups. When $\mathbb{L}$ is contained in a Harish–Chandra generic subgroup $\mathbb{M}$, *i.e.,* centralizes a 1–subgroup (see [BoTi], Sect. 4), then $R_{\mathbb{L}}^{\mathbb{M}}$ is known by induction and $R_{\mathbb{M}}^{\mathbb{G}}$ coincides with ordinary Harish–Chandra induction, so the decomposition follows from the result of Howlett and Lehrer [HoLe] (the case $d = 1$). Finally, if $\boldsymbol{\lambda}$ lies in the space of uniform functions of $\mathbb{L}$ then it may be written as a linear combination of Deligne–Lusztig characters $R_{\mathbb{T}}^{\mathbb{L}}(\mathbf{1})$. By transitivity of Lusztig–induction we can thus also express $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda})$ as a linear combination of $R_{\mathbb{T}}^{\mathbb{G}}(\mathbf{1})$'s. Hence the determination of the decomposition is reduced to computations with Weyl group characters. So far we don't even have to assume the Mackey formula to obtain the decompositions.

Unfortunately, not all cases $(\mathbb{G}, d, \mathbb{L}, \boldsymbol{\lambda})$ are of one of the above types. We now first collect a list of all quadruples $(\mathbb{G}, d, \mathbb{L}, \boldsymbol{\lambda})$, where $\mathbb{G}$ is simple exceptional, d an integer with $\Phi_d(x)$ dividing $|\mathbb{G}|$, $\mathbb{L}$ a d–split generic Levi subgroup and $\boldsymbol{\lambda} \in \mathrm{Uch}(\mathbb{L})$ is d–cuspidal (this last information is either already known if $\mathbb{L}$ is classical, or by induction for exceptional $\mathbb{L}$). This list appears as Table 1. The notation for the unipotent characters of the exceptional groups follows [Ca], 13.9. For groups of type A_l, the unipotent characters are indexed by partitions, for types D_l and ${}^2D_l$ by symbols (see the proof of 3.2).

However we have omitted the cases with $d = 1$, as these are known by the classical Howlett–Lehrer theory, and the cases where $\mathbb{L} = \mathbb{T}$, (hence) $\boldsymbol{\lambda} = \mathbf{1}$, (then d is regular for $\mathbb{G}$ in the sense of Springer [Sp1]), because here the decomposition is also clear due to $\mathbb{L} = \mathbb{T}$. The cases are numbered, and a * indicates that in this case the decomposition of $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda})$ follows from the above considerations, *i.e.,* without having to apply to the results of Deligne and Shoji.

Note that the groups $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ occurring in case $d = 2$ are all well known Coxeter groups, as they should be by the "Ennola"–duality, since we have $G(2,1,2) \cong W(B_2)$, $G(1,1,3) \cong W(A_2)$, $G(2,1,3) \cong W(B_3)$, $G_{28} \cong W(F_4)$ and $G(6,6,2) \cong W(G_2)$.

The table already shows that for the types ${}^2B_2$, ${}^2G_2$, G_2, ${}^3D_4$ and ${}^2F_4$ the decomposition of $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda})$ for all cuspidal pairs $(\mathbb{L}, \boldsymbol{\lambda})$ can be determined from the knowledge of $R_{\mathbb{T}}^{\mathbb{G}}$. But this is not the case for the decomposition of all $R_{\mathbb{M}}^{\mathbb{G}}(\boldsymbol{\lambda})$, where $\mathbb{M} < \mathbb{G}$ is maximal d–split but $\boldsymbol{\lambda}$ not cuspidal. In particular we will have to return to ${}^2F_4$ later on.

3.10 Theorem. *The decomposition of the* $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda})$ *for the* $(\mathbb{G}, d, \mathbb{L}, \boldsymbol{\lambda})$ *listed in Table 1 is given by Table 2.*

From this, the fundamental theorem 3.2 follows for exceptional groups. At the same time, the "Ennola"–duality 3.3 follows for all Levi subgroups $\mathbb{L}$ *which are* d*–split for some* d*.*

Remark.

In the cases marked by a * in the last column, the decomposition can be proved without using the general Mackey formula.

Remark.

The decompositions for $\mathbb{G}$ not of type E_8, *i.e.,* cases 1–38, were already obtained by Schewe [Sch] assuming a conjecture on Shintani descent, but note that some of his tables contain inaccuracies.

Outline of the proof of 3.10. In all cases marked with a * in the next to last column of Table 1, the d–cuspidal character $\boldsymbol{\lambda}$ of $\mathbb{L}$ lies in the space of uniform functions of $\mathbb{L}$. So by our above remarks on transitivity of Deligne–Lusztig induction, $R_{\mathbb{L}}^{\mathbb{G}}$ can be computed purely mechanical from information inside the Weyl group. All these computations with characters of Weyl groups were done with the computer algebra system CAS [NPP].

This leaves 28 cases $(\mathbb{G}, d, \mathbb{L}, \boldsymbol{\lambda})$ to be treated. These can not be solved with the methods discussed so far. We shall now assume in addition the validity of the Mackey formula for $R_{\mathbb{L}}^{\mathbb{G}}$ in full generality, as allowed by 1.34 and 1.32.

It enables us to calculate the norm of $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda})$ from the knowledge of $W_{\mathbb{G}}(\mathbb{L})$. Now for $\boldsymbol{\lambda} \in \mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{L})$, denote by $\pi_u^{\mathbb{L}}(\boldsymbol{\lambda})$ its projection to the space of uniform class functions on $\mathbb{L}$. By definition, this projection may be written as a linear combination of Deligne–Lusztig characters $R_{\mathbb{T}}^{\mathbb{G}}(1)$, so as above we may calculate $R_{\mathbb{L}}^{\mathbb{G}}(\pi_u^{\mathbb{L}}(\boldsymbol{\lambda}))$ from the character tables of $W_{\mathbb{L}}$ and $W_{\mathbb{G}}$. Since we have $R_{\mathbb{L}}^{\mathbb{G}}(\pi_u^{\mathbb{L}}(\boldsymbol{\lambda})) = \pi_u^{\mathbb{G}}(R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda}))$ by 1.32 (2), the uniform part of $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda})$ is known in all cases, as is the norm by the Mackey formula. These two informations suffice to determine the decomposition except for some small ambiguities resulting from irrationalities. Namely, it will turn out that there exists an essentially unique element $\boldsymbol{\gamma} \in \mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$ with $\pi_u^{\mathbb{G}}(\boldsymbol{\gamma}) = \pi_u^{\mathbb{G}}(R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda}))$ of minimal norm, and that this norm coincides with the norm of $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda})$ calculated from the Mackey formula.

By transitivity of Deligne–Lusztig induction we may recover the decomposition of $R_{\mathbb{L}}^{\mathbb{G}}$ if for some maximal Levi subgroup $\mathbb{M} < \mathbb{G}$ containing $\mathbb{L}$ we know the decomposition of $R_{\mathbb{M}}^{\mathbb{G}}$ (and the one of $R_{\mathbb{L}}^{\mathbb{M}}$ by induction). We will therefore from now on assume that $\mathbb{L}$ is already maximal in $\mathbb{G}$.

If $\mathbb{M}$ is a direct product of two groups $\mathbb{M} = \mathbb{M}_1 \times \mathbb{M}_2$, and all unipotent characters of $\mathbb{M}_1$ happen to be uniform, then $R_{\mathbb{M}}^{\mathbb{G}}$ may be written as a sum of $R_{\mathbb{T}\times\mathbb{M}_2}^{\mathbb{G}}$, for some maximal subtori $\mathbb{T}$ of $\mathbb{M}_1$. If moreover $\mathbb{T} \times \mathbb{M}_2$ is contained in another maximal Levi subgroup $\mathbb{M}'$ for which we know $R_{\mathbb{M}'}^{\mathbb{G}}$, then by induction we can also compute $R_{\mathbb{M}}^{\mathbb{G}}$. This is for example useful if $R_{\mathbb{M}'}^{\mathbb{G}}$ tends to contain only few constituents (and hence is easily determined), while $R_{\mathbb{M}}^{\mathbb{G}}$ has large norm. As an example we cite the case $d = 2$ in $\mathbb{G}$ of type E_8, where $\mathbb{M}$ has type $D_5 + A_2$ and $\mathbb{M}'$ has type E_7.

We now consider the restriction of the uniform part $\pi_u^{\mathbb{G}}(R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda}))$ to a family $\mathcal{F}$ of unipotent characters as defined by Lusztig (see [Ca], chap. 13). The characters in a family, as well as their uniform projections, span pairwise orthogonal spaces of class functions. If a family consists of just one unipotent character, then this character coincides with its uniform projection and hence its multiplicity in $R_{\mathbb{L}}^{\mathbb{G}}$ may be recovered from the uniform projection.

Next assume that the family $\mathcal{F}$ contains four unipotent characters $\rho_1, \ldots, \rho_4$, *i.e.*, belongs to the group $\mathfrak{S}_2$. Then except for one case in E_7 and two cases in E_8 the orthogonal complement of uniform functions has a one dimensional intersection with $\mathcal{F}$ and is spanned (after a suitable renumbering of the ρ_i) by $\rho_1 - \rho_2 - \rho_3 + \rho_4$. It turns out that in all cases of interest to us the uniform projections of the $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda})$, restricted to a four element family, coincide with the uniform projection of exactly one ρ_i. It is now clear that the minimal norm is achieved if indeed $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda})|_{\mathcal{F}} = \rho_i$. In case that adding up the minimal norms one just arrives at the norm of $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda})$, the decomposition is uniquely determined. This accounts for the cases (17), (27), (62), (63), (64), (65). For cases (42) and (43) the decomposition can not be deduced unambiguously, since there two unipotent characters in the four–element family have the same uniform projection, but at least the sum of the two decompositions follows from the norm condition. Since the pairs of characters concerned do only occur with equal multiplicity in all other $R_{\mathbb{L}}^{\mathbb{G}}$, the ambiguity here does not harm the proof of the theorem. Whichever of the possibilities holds, the collection of maps $I_{(\mathbb{L},\boldsymbol{\lambda})}^{\mathbb{M}}$ exists.

Next assume we consider the eight member family associated to the group $\mathfrak{S}_3$. Here the orthogonal complement to the space of uniform functions has dimension three, except for type G_2, but there all unipotent characters of every proper Levi subgroup are uniform. In the other cases, the vector of shortest length in the sublattice spanned by generalized characters is $\rho_7 - \rho_8$ of length two; all other vectors having length at least four. Checking the cases one sees that (5), (9), (11) and (14) follow from this observation.

In the case of the 21–member family in F_4 associated to $\mathfrak{S}_4$, the shortest vectors in the lattice of generalized characters orthogonal to all uniform functions are $F_4[\theta] - F_4[\theta^2]$ and $F_4[i] - F_4[-i]$ of norm two, and all other vectors have length at least four. This determines the cases (1), (2) and (3).

Finally in the 39–member family $\mathcal{F}$ in E_8 associated to $\mathfrak{S}_5$ one again finds that all vectors of length two come from pairs of unipotent characters with algebraically conjugate eigenvalues of Frobenius. Moreover, all other vectors have length at least four. This information is not yet sufficient do obtain all missing decompositions. So for example in case (47) we have $\mathbb{G}(q) = E_8(q)$, $\mathbb{M}(q) = \Phi_3 E_6(q)$, and it turns out that the norm of $R_{\mathbb{M}}^{\mathbb{G}}(\boldsymbol{\lambda})$ restricted to the family $\mathcal{F}$ may be as big as seven. But at least we know how to decompose $R_{\mathbb{M}}^{\mathbb{G}}(\boldsymbol{\gamma})$ for those $\boldsymbol{\gamma}$ lying in the principal series of $E_6(q)$. Also, by the Mackey formula, the scalar products of these with the still unknown $R_{\mathbb{M}}^{\mathbb{G}}(\boldsymbol{\lambda})$ are known. This, combined with the information on nonuniform characters in $\mathcal{F}$, allows to compute the decomposition completely.

The same type of reasoning takes care of the 13–member family of characters in groups of type ${}^2F_4$ for Φ_8'–split subgroups $\mathbb{M}$ with structure $\mathbb{M}(q) = \Phi_8'\,{}^2B_2(q)$. Although some ambiguities with respect to characters with algebraically conjugate eigenvalues of Frobenius remain, these are not relevant to the proof of the theorem, since it holds whichever of the different possible cases occurs.

After the determination of all $R_{\mathbb{M}}^{\mathbb{G}}$ it remains to compare the decompositions for $\mathbb{M}$ maximal with those of the corresponding characters induced from $W_{\mathbb{M}}(\mathbb{L},\boldsymbol{\lambda})$ to $W_{\mathbb{G}}(\mathbb{L},\boldsymbol{\lambda})$. The groups $W_{\mathbb{G}}(\mathbb{L},\boldsymbol{\lambda})$ turn out to be irreducible complex reflection groups in all cases. We have indicated these groups, using the standard notation given for example in [Be] or [Co], in the last column of Table 1 for the cases listed there, and collected them in all other cases where $d \neq 1, 2$ and $W_{\mathbb{G}}(\mathbb{L},\boldsymbol{\lambda})$ is noncyclic in Table 3. (For $d = 1, 2$ they are the usual Weyl groups of $\mathbb{G}$ or $\mathbb{G}^-$.)

In the two cases (22) and (23), $W_{\mathbb{G}}(\mathbb{L})$ is the reflection group G_8 but the stabilizer of the d–cuspidal character $\boldsymbol{\lambda}$ of $\mathbb{L}$ is a Sylow 2–subgroup of G_8 of index three, isomorphic to the imprimitive reflection group $G(4, 1, 2)$.

With the help of the reflection representations for the relevant reflection groups given in [Be] it is now possible to find the fusion of $W_{\mathbb{M}}(\mathbb{L},\boldsymbol{\lambda})$ into $W_{\mathbb{G}}(\mathbb{L},\boldsymbol{\lambda})$ and then compute the decomposition of all induced characters, using the computer algebra system CAS [NPP]. Then a straightforward verification proves the existence of the maps $I_{(\mathbb{L},\boldsymbol{\lambda})}^{\mathbb{M}}$. The details are omitted. This completes the proof of 3.10. □

It might be noted that the bijections $I^{\mathbb{M}}_{(\mathbb{L},\boldsymbol{\lambda})}$ are not at all unique if $W_{\mathbb{G}}(\mathbb{L},\boldsymbol{\lambda})$ is cyclic; indeed on this level any bijective map is admissible. But if we have once chosen the bijections on the level of cyclic groups, then it is "almost" determined for all bigger groups $W_{\mathbb{G}}(\mathbb{L})$. To be more precise, the characters in the $(\mathbb{L},\boldsymbol{\lambda})$–series are determined by their multiplicities in the $R^{\mathbb{G}}_{\mathbb{M}}(\boldsymbol{\gamma})$ for $(\mathbb{L},\boldsymbol{\lambda}) \preccurlyeq (\mathbb{M},\boldsymbol{\gamma})$ (provided that some numbering is chosen on the 1–dimensional level), up to a small number of cases which basically result from algebraically conjugate eigenvalues of Frobenius, making characters indistinguishable by induction. This is again a remarkable analogy to the case $d = 1$, where Benson and Curtis have proved a similar result (here the "cyclic" case is just the trivial group), also up to one ambiguity in E_7 and two in E_8. □

B. *d*–Harish–Chandra theory for unipotent characters..

In §3.B, we draw some straightforward consequences of the preceding fundamental theorem about the structure of the poset of d–split pairs.

The following statements generalize Harish–Chandra theory, which corresponds here to the case $d = 1$. They were recently conjectured or partially proved by many authors (see for example [Sch] or [FoSr]), and are immediate consequences of theorem 3.2.

3.11. Theorem. *(transitivity) Let $(\mathbb{L},\boldsymbol{\lambda})$ be a d–cuspidal pair, and let $(\mathbb{M},\boldsymbol{\mu})$ and $(\mathbb{G},\boldsymbol{\gamma})$ be d–split pairs such that $(\mathbb{L},\boldsymbol{\lambda}) \preccurlyeq (\mathbb{M},\boldsymbol{\mu})$ and $(\mathbb{M},\boldsymbol{\mu}) \preccurlyeq (\mathbb{G},\boldsymbol{\gamma})$. Then $(\mathbb{L},\boldsymbol{\lambda}) \preccurlyeq (\mathbb{G},\boldsymbol{\gamma})$.*

This theorem, together with the disjointness asserted in 3.2(1), shows the existence of a "*d*–Harish–Chandra theory". We draw some straightforward consequences of the existence of such a theory.

3.12. Proposition. *Let $(\mathbb{L},\boldsymbol{\lambda})$ be a d–cuspidal pair, and let $(\mathbb{M},\boldsymbol{\mu})$ and $(\mathbb{G},\boldsymbol{\gamma})$ be d–split pairs such that $(\mathbb{L},\boldsymbol{\lambda}) \preccurlyeq (\mathbb{G},\boldsymbol{\gamma})$ and $(\mathbb{M},\boldsymbol{\mu}) \preccurlyeq (\mathbb{G},\boldsymbol{\gamma})$. Then $(\mathbb{L},\boldsymbol{\lambda}) \preccurlyeq_{W_{\mathbb{G}}} (\mathbb{M},\boldsymbol{\mu})$.*

Proof of 3.12. By 3.2, (1), there exists a *d*–cuspidal pair $(\mathbb{L}',\boldsymbol{\lambda}')$ such that $(\mathbb{L}',\boldsymbol{\lambda}') \preccurlyeq (\mathbb{M},\boldsymbol{\mu})$. Then by 3.11 we have $(\mathbb{L}',\boldsymbol{\lambda}') \preccurlyeq (\mathbb{G},\boldsymbol{\gamma})$. By 3.2, (1) again, we then see that $(\mathbb{L}',\boldsymbol{\lambda}')$ is $W_{\mathbb{G}}$–conjugate to $(\mathbb{L},\boldsymbol{\lambda})$. □

3.13. Corollary. *The d–cuspidal pairs are the minimal d–split pairs for the relation $\preccurlyeq$.*

Proof of 3.13. Apply the preceding proposition with $(\mathbb{L},\boldsymbol{\lambda}) = (\mathbb{G},\boldsymbol{\gamma})$. □

The following consequence of *d*-Harish-Chandra theory is a result of "control of fusion".

3.14. Proposition. *Let $(\mathbb{L},\boldsymbol{\lambda})$ and $(\mathbb{M},\boldsymbol{\mu})$ be two d–split pairs. We assume that $(\mathbb{L},\boldsymbol{\lambda})$ is d–cuspidal, and that $(\mathbb{L},\boldsymbol{\lambda}) \preccurlyeq (\mathbb{M},\boldsymbol{\mu})$.*

(1) *Assume that $w \in W_{\mathbb{G}}$ is such that also $(\mathbb{L},\boldsymbol{\lambda}) \preccurlyeq (\mathbb{M},\boldsymbol{\mu})^w$. Then there exists $v \in W_{\mathbb{M}}$ and $n \in N_{W_{\mathbb{G}}}(\mathbb{L},\boldsymbol{\lambda})$ such that $w = vn$.*
(2) *The natural map $N_{W_{\mathbb{G}}}((\mathbb{L},\boldsymbol{\lambda}),(\mathbb{M},\boldsymbol{\mu}))/N_{W_{\mathbb{M}}}(\mathbb{L},\boldsymbol{\lambda}) \to N_{W_{\mathbb{G}}}(\mathbb{M},\boldsymbol{\mu})/W_{\mathbb{M}}$ is an isomorphism. In particular (with obvious notation) we have*

$$N_{W_{\mathbb{G}}(\mathbb{L},\boldsymbol{\lambda})}(\mathbb{M},\boldsymbol{\mu})/W_{\mathbb{M}}(\mathbb{L},\boldsymbol{\lambda}) = W_{\mathbb{G}}(\mathbb{M},\boldsymbol{\mu}) \ .$$

Proof of 3.14. If $(\mathbb{L},\boldsymbol{\lambda})$ is contained in both $(\mathbb{M},\boldsymbol{\mu})$ and $(\mathbb{M},\boldsymbol{\mu})^w$, then $(\mathbb{L},\boldsymbol{\lambda})$ and ${}^w(\mathbb{L},\boldsymbol{\lambda})$ are d–cuspidal pairs contained in $(\mathbb{M},\boldsymbol{\mu})$, and hence (see 3.13) are conjugate under $W_{\mathbb{M}}$. This proves (1). We get (2) by applying (1) to the case where $w \in N_{W_{\mathbb{G}}}(\mathbb{M},\boldsymbol{\mu})$. □

The next result is also an immediate consequence of 3.12.

3.15. Proposition. *Suppose that $(\mathbb{L},\boldsymbol{\lambda})$ is a d–cuspidal pair such that $(\mathbb{L},\boldsymbol{\lambda}) \preccurlyeq (\mathbb{G},\boldsymbol{\gamma})$. Then*

$$ {}^*R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\gamma}) = (\boldsymbol{\gamma}, R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda}))_{\mathbb{G}} \sum_{w\in W_{\mathbb{G}}(\mathbb{L})/W_{\mathbb{G}}(\mathbb{L},\boldsymbol{\lambda})} {}^w\boldsymbol{\lambda}\,. $$

In particular, we have

$$ \mathrm{Deg}({}^*R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\gamma})) = (\boldsymbol{\gamma}, R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda}))_{\mathbb{G}} |W_{\mathbb{G}}(\mathbb{L}) : W_{\mathbb{G}}(\mathbb{L},\boldsymbol{\lambda})| \mathrm{Deg}(\boldsymbol{\lambda})\,, $$

and $\mathrm{Deg}({}^*R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\gamma})) \neq 0$.

4. Generic Φ_d–blocks.

A. More on regular unipotent characters and Mackey formula.

In §4.A, using §3, we split unipotent regular characters into a sum of characters, for which we refine results of §2.B. We also refine the Mackey formula.

Let $(\mathbb{L},\boldsymbol{\lambda})$ be a d–cuspidal pair. We denote by $\mathrm{pr}_{\mathbb{L},\boldsymbol{\lambda}}^{\mathbb{G}}$ the orthogonal projection from $\mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$ onto the submodule with basis $\mathrm{Uch}(\mathbb{G},(\mathbb{L},\boldsymbol{\lambda}))$.

We define

$$ \text{(4.1)} \qquad \mathrm{UReg}_{\mathbb{L},\boldsymbol{\lambda}}^{\mathbb{G}} := \mathrm{pr}_{\mathbb{L},\boldsymbol{\lambda}}^{\mathbb{G}}(\mathrm{UReg}^{\mathbb{G}}) = \sum_{\{\boldsymbol{\gamma}\,;\,(\mathbb{L},\boldsymbol{\lambda})\preccurlyeq(\mathbb{G},\boldsymbol{\gamma})\}} \mathrm{Deg}(\boldsymbol{\gamma}).\boldsymbol{\gamma}\,, $$

and we have (see 2.12)

$$ \text{(4.2)} \qquad \mathrm{c}_d(\mathrm{UReg}_{\mathbb{L},\boldsymbol{\lambda}}^{\mathbb{G}}) = \mathrm{pr}_{\mathbb{L},\boldsymbol{\lambda}}^{\mathbb{G}}(\mathrm{c}_d(\mathrm{UReg}^{\mathbb{G}})) = \sum_{\{\boldsymbol{\gamma}\,;\,(\mathbb{L},\boldsymbol{\lambda})\preccurlyeq(\mathbb{G},\boldsymbol{\gamma})\}} \mathrm{Deg}(c_d(\boldsymbol{\gamma})).\boldsymbol{\gamma}\,. $$

We have the following version of 2.13:

4.3. Proposition. *Let $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G},(\mathbb{L},\boldsymbol{\lambda}))$, and let $\psi \in \mathbb{Q}[x]\,\mathrm{Uch}(\mathbb{G})$ be a uniform function. Then*

$$ (\boldsymbol{\gamma},\psi)_{\mathbb{G}} = \sum_{[(\mathbb{M},\boldsymbol{\mu})\,;\,(\mathbb{L},\boldsymbol{\lambda})\preccurlyeq_{W_{\mathbb{G}}}(\mathbb{M},\boldsymbol{\mu})]_{W_{\mathbb{G}}}} \frac{(\boldsymbol{\gamma}, R_{\mathbb{M}}^{\mathbb{G}}(\boldsymbol{\mu}))_{\mathbb{G}}}{|W_{\mathbb{G}}(\mathbb{M},\boldsymbol{\mu})|} (\boldsymbol{\mu}, c_d({}^*R_{\mathbb{M}}^{\mathbb{G}}(\psi))_{\mathbb{M}} $$

Proof. From 2.13, we have

$$ \begin{aligned} (\boldsymbol{\gamma},\psi)_{\mathbb{G}} &= \sum_{[\mathbb{M}\ d\text{–split}]_{W_{\mathbb{G}}}} \frac{1}{|W_{\mathbb{G}}(\mathbb{M})|} ({}^*R_{\mathbb{M}}^{\mathbb{G}}\boldsymbol{\gamma}, c_d({}^*R_{\mathbb{M}}^{\mathbb{G}}(\psi)))_{\mathbb{M}} \\ &= \sum_{[\mathbb{M}\ d\text{–split}]_{W_{\mathbb{G}}}} \frac{1}{|W_{\mathbb{G}}(\mathbb{M})|} \sum_{\boldsymbol{\mu}\in\mathrm{Uch}(\mathbb{M})} (\boldsymbol{\gamma}, R_{\mathbb{M}}^{\mathbb{G}}(\boldsymbol{\mu}))_{\mathbb{G}} (\boldsymbol{\mu}, c_d({}^*R_{\mathbb{M}}^{\mathbb{G}}(\psi))_{\mathbb{M}} \end{aligned} $$

whence the result, since the pairs $(\mathbb{M},\boldsymbol{\mu})$ in the summation have to be above $(\mathbb{L},\boldsymbol{\lambda})$. □

From 4.3 we deduce the following refinement of 2.16.

4.4. Proposition. *We have*

$$\mathrm{UReg}^{\mathbb{G}}_{\mathbb{L},\boldsymbol{\lambda}} = \sum_{[(\mathbb{M},\boldsymbol{\mu})\,;\,(\mathbb{L},\boldsymbol{\lambda})\preccurlyeq_{W_{\mathbb{G}}}(\mathbb{M},\boldsymbol{\mu})]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R^{\mathbb{G}}_{\mathbb{M}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{M},\boldsymbol{\mu})|} R^{\mathbb{G}}_{\mathbb{M}}(\mathrm{Deg}(c_d(\boldsymbol{\mu})).\boldsymbol{\mu})\,.$$

Finally, the preceding machinery allows us to prove a refined version of the Mackey formula (see 1.34). The proof of the following statements follows from 1.34 and from 3.14 and is left to the reader.

4.5. Proposition. *If* $(\mathbb{M}_1,\boldsymbol{\mu}_1)$ *and* $(\mathbb{M}_2,\boldsymbol{\mu}_2)$ *are two d–split pairs containing the d–cuspidal pair* $(\mathbb{L},\boldsymbol{\lambda})$*, we have*

$$(R^{\mathbb{G}}_{\mathbb{M}_1}(\boldsymbol{\mu}_1), R^{\mathbb{G}}_{\mathbb{M}_2}(\boldsymbol{\mu}_2))_{\mathbb{G}} = \sum_{w\in W_{\mathbb{M}_2}(\mathbb{L},\boldsymbol{\lambda})\backslash W_{\mathbb{G}}(\mathbb{L},\boldsymbol{\lambda})/W_{\mathbb{M}_1}(\mathbb{L},\boldsymbol{\lambda})} ({}^*R^{{}^w\mathbb{M}_1}_{\mathbb{M}_2\cap{}^w\mathbb{M}_1}({}^w\boldsymbol{\mu}_1), {}^*R^{\mathbb{M}_2}_{\mathbb{M}_2\cap{}^w\mathbb{M}_1}(\boldsymbol{\mu}_2))_{\mathbb{M}_2\cap{}^w\mathbb{M}_1}\,.$$

The preceding statement can be reformulated in terms entirely analogous to the previous "Mackey formulae", provided we consider maps "over $(\mathbb{L},\boldsymbol{\lambda})$".

For a d–split generic Levi subgroup $\mathbb{M}$ containing $\mathbb{L}$, let us denote by

$$R^{\mathbb{G}}_{\mathbb{M}}(\mathbb{L},\boldsymbol{\lambda})\colon \mathbb{Z}\,\mathrm{Uch}(\mathbb{M},(\mathbb{L},\boldsymbol{\lambda})) \to \mathbb{Z}\,\mathrm{Uch}(\mathbb{G},(\mathbb{L},\boldsymbol{\lambda}))$$

the restriction of $R^{\mathbb{G}}_{\mathbb{M}}$, and by

$${}^*R^{\mathbb{G}}_{\mathbb{M}}(\mathbb{L},\boldsymbol{\lambda})\colon \mathbb{Z}\,\mathrm{Uch}(\mathbb{G},(\mathbb{L},\boldsymbol{\lambda})) \to \mathbb{Z}\,\mathrm{Uch}(\mathbb{M},(\mathbb{L},\boldsymbol{\lambda}))$$

its adjoint, *i.e.*, the restriction to $\mathbb{Z}\,\mathrm{Uch}(\mathbb{G},(\mathbb{L},\boldsymbol{\lambda}))$ of $\mathrm{pr}^{\mathbb{M}}_{\mathbb{L},\boldsymbol{\lambda}}\cdot{}^*R^{\mathbb{G}}_{\mathbb{M}}$.

4.6. Proposition. *Let* $(\mathbb{L},\boldsymbol{\lambda})$ *be a d–cuspidal pair, and let* $\mathbb{M}_1$ *and* $\mathbb{M}_2$ *be two d–split Levi subgroups of* $\mathbb{G}$ *containing* $\mathbb{L}$*. Then*

$${}^*R^{\mathbb{G}}_{\mathbb{M}_2}(\mathbb{L},\boldsymbol{\lambda})\cdot R^{\mathbb{G}}_{\mathbb{M}_1}(\mathbb{L},\boldsymbol{\lambda}) = \sum_{w\in W_{\mathbb{M}_2}(\mathbb{L},\boldsymbol{\lambda})\backslash W_{\mathbb{G}}(\mathbb{L},\boldsymbol{\lambda})/W_{\mathbb{M}_1}(\mathbb{L},\boldsymbol{\lambda})} R^{\mathbb{M}_2}_{\mathbb{M}_2\cap{}^w\mathbb{M}_1}(\mathbb{L},\boldsymbol{\lambda})\cdot{}^*R^{{}^w\mathbb{M}_1}_{\mathbb{M}_2\cap{}^w\mathbb{M}_1}(\mathbb{L},\boldsymbol{\lambda})\cdot\mathrm{ad}(w)\,.$$

B. Φ_d–defect groups.

In §4.B, we reformulate some of the previous results in order to define a suitable notion of generic Φ_d–defect group of a unipotent character.

The d–Harish–Chandra theory allows us to define the notion of "Φ_d–defect groups" or "defect (generic) tori" of an element $\boldsymbol{\gamma}\in\mathrm{Uch}(\mathbb{G})$, as follows.

4.7. Definition. *Let* $\boldsymbol{\gamma}\in\mathrm{Uch}(\mathbb{G})$.

(1) *For each d let $\mathcal{S}_d(\boldsymbol{\gamma})$ be the set of all Φ_d–tori $\mathbb{S}$ of $\mathbb{G}$ for which there exists a maximal torus $\mathbb{T}$ containing $\mathbb{S}$ such that* ${}^*R^{\mathbb{G}}_{\mathbb{T}}(\boldsymbol{\gamma})\neq 0$.

(2) *The maximal elements of $\mathcal{S}_d(\boldsymbol{\gamma})$ are called the Φ_d–defect tori of $\boldsymbol{\gamma}$.*

4.8. Theorem. *Let $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})$, and let $(\mathbb{L}, \boldsymbol{\lambda})$ be a d–cuspidal pair (unique up to $W_{\mathbb{G}}$–conjugation) such that $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \boldsymbol{\lambda}))$. Then $\mathbb{S} \in \mathcal{S}_d(\boldsymbol{\gamma})$ if and only if $\mathbb{S}$ is $W_{\mathbb{G}}$–conjugate to the maximal Φ_d–subgroup of $\mathrm{Rad}(\mathbb{L})$. In particular the defect tori of $\boldsymbol{\gamma}$ are the $W_{\mathbb{G}}$–conjugates of $\mathrm{Rad}(\mathbb{L})_d$.*

Remark. An element $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})$ is d–cuspidal if and only if the Sylow Φ_d–subtorus of $\mathrm{Rad}(\mathbb{G})$ is its defect torus.

Proof of 4.8. We set $\mathbb{M} := C_{\mathbb{G}}(\mathbb{S})$.

Assume first that $\mathbb{S} \in \mathcal{S}_d(\boldsymbol{\gamma})$. Let $\mathbb{T}$ be a maximal torus of $\mathbb{G}$ such that $\mathbb{T} \supset \mathbb{S}$ and ${}^*R_{\mathbb{T}}^{\mathbb{G}}(\boldsymbol{\gamma}) \neq 0$. Since $\mathbb{T} \subseteq \mathbb{M}$, we have ${}^*R_{\mathbb{M}}^{\mathbb{G}}(\boldsymbol{\gamma}) \neq 0$ and there exists $\boldsymbol{\mu} \in \mathrm{Uch}(\mathbb{M})$ such that $(\mathbb{M}, \boldsymbol{\mu}) \preccurlyeq (\mathbb{G}, \boldsymbol{\gamma})$. By 3.2(i) there exists $w \in W_{\mathbb{G}}$ such that $\mathbb{L}$ is a subgroup of $\mathbb{M}^w$, from which it follows that $\mathbb{S}^w$ is contained in the maximal Φ_d–subtorus of $\mathrm{Rad}(\mathbb{L})$.

Assume now that $\mathbb{S}$ is contained in the maximal Φ_d–subgroup of $\mathrm{Rad}(\mathbb{L})$. Then $\mathbb{L} \subseteq \mathbb{M}$. Since (*cf.* 3.15) $\mathrm{Deg}({}^*R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\gamma})) \neq 0$, we see that there exists a maximal torus $\mathbb{T}$ of $\mathbb{L}$, whence of $\mathbb{M}$, such that ${}^*R_{\mathbb{T}}^{\mathbb{G}}(\boldsymbol{\gamma}) \neq 0$, which shows that $\mathbb{S} \in \mathcal{S}_d(\boldsymbol{\gamma})$. $\square$

For the notation used below, the reader may refer to §2.A.

4.9. Theorem. *Let $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})$ with Φ_d–defect group $\mathbb{S}$. Let $\mathbb{L} := C_{\mathbb{G}}(\mathbb{S})$. Then*

(1) $\mathbb{S} = \mathrm{Rad}(\mathbb{L})_d$ *and* $m_{\boldsymbol{\gamma}} \in \mathcal{CF}_d(\mathbb{G}, \mathbb{L})$,
(2) $\mathrm{Deg}(\boldsymbol{\gamma})_d = |\mathbb{G}|_d / |\mathbb{S}|$,
(3) ${}^*R_{\mathbb{M}}^{\mathbb{G}}(\boldsymbol{\gamma}) = 0$ *whenever $\mathbb{M}$ is a d–split Levi subgroup of $\mathbb{G}$ such that $Z_d(\mathbb{M})$ is not $W_{\mathbb{G}}$–conjugate to a subgroup of $\mathbb{S}$.*

Proof of 4.9.

(1) results from the definition of a defect group.

(2) Since $|\mathbb{G}|_d/|\mathbb{S}| = |\mathbb{L}_{\mathrm{ss}}|_d$, we see by 2.17 that

$$\frac{\mathrm{Deg}(\boldsymbol{\gamma})}{|\mathbb{L}_{\mathrm{ss}}|_d} \equiv \frac{\mathrm{Deg}(R_{\mathbb{L}}^{\mathbb{G}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{L})|} \frac{\mathrm{Deg}({}^*R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\gamma}))}{|\mathbb{L}_{\mathrm{ss}}|_d} \mod \Phi_d \,,$$

and it suffices to prove that $\dfrac{\mathrm{Deg}({}^*R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\gamma}))}{|\mathbb{L}_{\mathrm{ss}}|_d}$ is not divisible by Φ_d. But this results from 3.15 and from 2.9.

(3) $Z_d(\mathbb{M})$ is not $W_{\mathbb{G}}$–conjugate to a subgroup of $\mathbb{S}$ if and only if $\mathbb{L}$ is not conjugate to a subgroup of $\mathbb{M}$. Thus (3) is an easy consequence of 3.2(i). $\square$

Remark. We shall see later on that a generic defect group does behave, in many respects, as the right “generization” of an actual defect group. Moreover, for suitable choices of q and ℓ (see §4), the Sylow ℓ– subgroups of the rational points of a generic defect group are the defect groups of a corresponding actual ℓ–block.

Assertions (2) and (3) of 4.9 may be stated in the following way, which generalizes the properties of d–cuspidal elements of $\mathrm{Uch}(\mathbb{G})$.

4.10. Proposition. *Let $(\mathbb{L}, \boldsymbol{\lambda})$ be a d-cuspidal pair, and let $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G}, (\mathbb{L}, \boldsymbol{\lambda}))$. Then*

(1) $\mathrm{Deg}(\boldsymbol{\gamma})_d = |\mathbb{L}_{\mathrm{ss}}|_d$,
(2) *for a d–split generic Levi subgroup $\mathbb{M}$ of $\mathbb{G}$, we have ${}^*R_{\mathbb{M}}^{\mathbb{G}}(\boldsymbol{\gamma}) = 0$ unless $\mathbb{L} \subseteq_{W_{\mathbb{G}}} \mathbb{M}$.*

Finally, we have the following version of 2.13

4.11. Proposition. *Let* $\boldsymbol{\gamma} \in \mathrm{Uch}(\mathbb{G})$ *and let* $\mathbb{S} \in \mathcal{S}_d(\boldsymbol{\gamma})$*; let* ψ *be a uniform function in* $\mathbb{Q}[X]\,\mathrm{Uch}(\mathbb{G})$*. Then*

$$(\boldsymbol{\gamma}, \psi)_{\mathbb{G}} = \sum_{[\mathbb{M}\ d\text{–split};\mathrm{Rad}(\mathbb{M})_d \leq \mathbb{S}]_{W_{\mathbb{G}}}} \frac{1}{|W_{\mathbb{G}}(\mathbb{M})|} (\boldsymbol{\gamma}, R_{\mathbb{M}}^{\mathbb{G}}(c_d({}^*R_{\mathbb{M}}^{\mathbb{G}}\psi)))_{\mathbb{G}}$$

Proof. This is immediate from 2.13 if we notice that $c_d({}^*R_{\mathbb{M}}^{\mathbb{G}}\psi)$ is a sum $R_{\mathbb{T}}^{\mathbb{M}}$'s where $\mathbb{T} \in \mathcal{T}_d(\mathbb{M})$: this shows that, for the scalar product to be non zero, those $\mathbb{T}$'s must have $\mathbb{T}_d = \mathrm{Rad}(\mathbb{M})_d \leq \mathbb{S}$.

□

C. Regular characters of generic d–blocks.

In §4.C, results of §2.C are refined the same way results of §2.B were refined in §4.A. In particular we define what must be considered as the regular character of the generic Φ_d*–blocks.*

Following the notation introduced in §2.C, we have a decomposition

$$\mathcal{R}_d(\mathbb{G}) = \bigoplus_{[\mathbb{L},\boldsymbol{\lambda}]_{W_{\mathbb{G}}}} \mathcal{R}_{\mathbb{L},\boldsymbol{\lambda}}(\mathbb{G})$$

where $(\mathbb{L}, \boldsymbol{\lambda})$ runs over the set of d–cuspidal pairs of $\mathbb{G}$, and where we denote by $\mathcal{R}_{\mathbb{L},\boldsymbol{\lambda}}(\mathbb{G})$ the free $\mathbb{Q}[x]$–submodule of $\mathcal{R}_d(\mathbb{G})$ generated by $\{\chi_{(\mathbb{M},\boldsymbol{\mu})}^{\mathbb{G}}\}_{(\mathbb{L},\boldsymbol{\lambda}) \preccurlyeq_{W_{\mathbb{G}}} (\mathbb{M},\boldsymbol{\mu})}$.

We "extend" the definition of the orthogonal projection $\mathrm{pr}_{\mathbb{L},\boldsymbol{\lambda}}^{\mathbb{G}}$ (see §3.A above) by defining

$$\mathrm{pr}_{\mathbb{L},\boldsymbol{\lambda}}^{\mathbb{G}}(\chi_{(\mathbb{M},\boldsymbol{\mu})}^{\mathbb{G}}) := \begin{cases} \chi_{(\mathbb{M},\boldsymbol{\mu})}^{\mathbb{G}} & \text{if } (\mathbb{L}, \boldsymbol{\lambda}) \preccurlyeq_{W_{\mathbb{G}}} (\mathbb{M}, \boldsymbol{\mu}) \\ 0 & \text{if not}\,. \end{cases}$$

The following property of $\mathrm{pr}_{\mathbb{L},\boldsymbol{\lambda}}^{\mathbb{G}}$ is a translation of the consequences of 3.2 listed in §3.A above.

4.12. Lemma. *Whenever* $\mathbb{N}$ *and* $\mathbb{M}$ *are two* d*–split generic Levi subgroups of* $\mathbb{G}$ *such that* $\mathbb{N} \subseteq \mathbb{M}$*, and* $\boldsymbol{\nu} \in \mathrm{Uch}(\mathbb{N})$*, we have*

$$\mathrm{pr}_{\mathbb{L},\boldsymbol{\lambda}}^{\mathbb{G}}(R_{\mathbb{N}}^{\mathbb{G}}(\boldsymbol{\Theta}_{\mathbb{G},\mathbb{M}}^{\mathbb{N}} \otimes \boldsymbol{\nu})) = \begin{cases} R_{\mathbb{N}}^{\mathbb{G}}(\boldsymbol{\Theta}_{\mathbb{G},\mathbb{M}}^{\mathbb{N}} \otimes \boldsymbol{\nu}) & \textit{if } (\mathbb{L}, \boldsymbol{\lambda}) \preccurlyeq_{W_{\mathbb{G}}} (\mathbb{N}, \boldsymbol{\nu}) \\ 0 & \textit{if not}. \end{cases}$$

Proof of 4.12. By 2.30, we know that

$$R_{\mathbb{N}}^{\mathbb{G}}(\boldsymbol{\Theta}_{\mathbb{G},\mathbb{M}}^{\mathbb{N}} \otimes \boldsymbol{\nu}) = R_{\mathbb{M}}^{\mathbb{G}}(\boldsymbol{\Theta}_{\mathbb{G}}^{\mathbb{M}} \otimes R_{\mathbb{N}}^{\mathbb{M}}(\boldsymbol{\nu}))\,,$$

or, in other words,

$$R_{\mathbb{N}}^{\mathbb{G}}(\boldsymbol{\Theta}_{\mathbb{G},\mathbb{M}}^{\mathbb{N}} \otimes \boldsymbol{\nu}) = \sum_{\{\boldsymbol{\mu}\,;\,(\mathbb{N},\boldsymbol{\nu}) \preccurlyeq (\mathbb{M},\boldsymbol{\mu})\}} (R_{\mathbb{N}}^{\mathbb{M}}(\boldsymbol{\nu}), \boldsymbol{\mu})_{\mathbb{M}} R_{\mathbb{M}}^{\mathbb{G}}(\boldsymbol{\Theta}_{\mathbb{G}}^{\mathbb{M}} \otimes R_{\mathbb{N}}^{\mathbb{M}}(\boldsymbol{\nu}))\,.$$

The result then follows from the following property:

Assume $(\mathbb{N},\boldsymbol{\nu}) \preccurlyeq (\mathbb{M},\boldsymbol{\mu})$. *Then*

$$(\mathbb{L},\boldsymbol{\lambda}) \preccurlyeq_{W_{\mathbb{G}}} (\mathbb{M},\boldsymbol{\mu}) \textit{ if and only if } (\mathbb{L},\boldsymbol{\lambda}) \preccurlyeq_{W_{\mathbb{G}}} (\mathbb{N},\boldsymbol{\nu})\,,$$

which is immediate by 3.13. □

We define (*cf.* 2.28)

$$\mathrm{Reg}^{\mathbb{G}}_{(\mathbb{L},\boldsymbol{\lambda})} := \mathrm{pr}^{\mathbb{G}}_{\mathbb{L},\boldsymbol{\lambda}}(\mathrm{Reg}^{\mathbb{G}}_d)\,.$$

Thus by definition,

$$\mathrm{Reg}^{\mathbb{G}}_{(\mathbb{L},\boldsymbol{\lambda})} = \sum_{[(\mathbb{M},\boldsymbol{\mu})\,;\,(\mathbb{L},\boldsymbol{\lambda})\preccurlyeq_{\mathbb{G}}(\mathbb{M},\boldsymbol{\mu})]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R^{\mathbb{G}}_{\mathbb{M}}(\boldsymbol{\mu}))}{|W_{\mathbb{G}}(\mathbb{M},\boldsymbol{\mu})|}.\chi^{\mathbb{G}}_{(\mathbb{M},\boldsymbol{\mu})}$$

$$(4.13)\qquad = \sum_{[(\mathbb{M},\boldsymbol{\mu})\,;\,(\mathbb{L},\boldsymbol{\lambda})\preccurlyeq_{\mathbb{G}}(\mathbb{M},\boldsymbol{\mu})]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R^{\mathbb{G}}_{\mathbb{M}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{M},\boldsymbol{\mu})|} R^{\mathbb{G}}_{\mathbb{M}}(\boldsymbol{\Theta}^{\mathbb{M}}_{\mathbb{G}} \otimes \mathrm{Deg}(\boldsymbol{\mu}).\boldsymbol{\mu})\ ,$$

and also

$$(4.14)\qquad \mathrm{Reg}^{\mathbb{G}}_d = \sum_{[(\mathbb{L},\boldsymbol{\lambda})]_{W_{\mathbb{G}}}} \mathrm{Reg}^{\mathbb{G}}_{(\mathbb{L},\boldsymbol{\lambda})}\ ,$$

where $(\mathbb{L},\boldsymbol{\lambda})$ runs over the set of d–cuspidal pairs of $\mathbb{G}$.

4.15. Theorem. *We have*

$$\mathrm{Reg}^{\mathbb{G}}_{(\mathbb{L},\boldsymbol{\lambda})} = \sum_{[(\mathbb{M},\boldsymbol{\mu})\,;\,(\mathbb{L},\boldsymbol{\lambda})\preccurlyeq_{\mathbb{G}}(\mathbb{M},\boldsymbol{\mu})]_{W_{\mathbb{G}}}} \frac{\mathrm{Deg}(R^{\mathbb{G}}_{\mathbb{M}}(\mathbf{1}))}{|W_{\mathbb{G}}(\mathbb{M},\boldsymbol{\mu})|} R^{\mathbb{G}}_{\mathbb{M}}(\mathrm{Ab}_d\,\mathrm{Reg}^{\mathbb{M}} \otimes \mathrm{Deg}(c_d(\boldsymbol{\mu})).\boldsymbol{\mu})\,.$$

Proof of 4.15. It suffices to apply $\mathrm{pr}^{\mathbb{G}}_{\mathbb{L},\boldsymbol{\lambda}}$ to both sides of the corresponding formula for $\mathrm{Reg}^{\mathbb{G}}_d$ (see 2.33), and to use lemma 4.12 above. □

The preceeding results allows us to improve 2.39, by decomposing the character $\mathrm{Reg}^{\mathbf{G}^F}_d$ (see §2.D) into a sum of characters with similar divisibility properties.

The reader may refer to §2.D above for the notation used here.

Let $(\mathbf{G},\mathbf{T},F)$ be a (q,ϕ)–triple associated with $\mathbb{G}$, chosen in such a way that $\mathbf{T}$ contains a Sylow Φ_d–subgroup of $\mathbf{G}$.

Let $(\mathbf{L},\boldsymbol{\lambda})$ be a d–cuspidal pair of $\mathbf{G}$.

We set

$$\mathrm{Reg}^{\mathbf{G}^F}_{\mathbf{L},\boldsymbol{\lambda}} := \sum_{[(\mathbf{M},\boldsymbol{\mu})\,;\,(\mathbf{L},\boldsymbol{\lambda})\preccurlyeq_{\mathbf{G}^F}(\mathbf{M},\boldsymbol{\mu})]_{\mathbf{G}^F}} \frac{\mathrm{Deg} R^{\mathbf{G}}_{\mathbf{M}}(\boldsymbol{\mu})}{|W_{\mathbf{G}^F}(\mathbf{M},\boldsymbol{\mu})|} R^{\mathbf{G}}_{\mathbf{M}}(\Theta^{\mathbf{M}^F}_{\mathbf{G}}.\boldsymbol{\mu})\,,$$

4.16. Theorem.

(1) *We have* $\mathrm{Reg}^{\mathbf{G}^F}_d = \sum_{[(\mathbf{L},\boldsymbol{\lambda})]_{\mathbf{G}^F}} \mathrm{Reg}^{\mathbf{G}^F}_{\mathbf{L},\boldsymbol{\lambda}}$.

(2) *All the values of* $|W_{\mathbf{G}}|\,\mathrm{Reg}^{\mathbf{G}^F}_{\mathbf{L},\boldsymbol{\lambda}}$ *are divisible by* $|\mathbb{G}|_d(q)$.

Proof of 4.16. The first assertion is immediate. As for the proof of 2.39, the second assertion is a consequence of 2.42. □

ℓ–Blocks.

The following hypothesis are in force in what follows.

(H1) Let $\mathbf{G}$ be a connected reductive algebraic group over an algebraic closure of $\mathbb{F}_q$, endowed with a Frobenius endomorphism $F\colon \mathbf{G} \to \mathbf{G}$ defining a rational structure over $\mathbb{F}_q$.

(H2) Let ℓ be a prime number which does not divide q, which divides $|\mathbf{G}^F|$, and which is excellent for $(\mathbf{G}, F)$.

We denote by $\mathbb{G}$ a generic finite reductive group corresponding to $(\mathbf{G}, F)$, and we set $\mathbb{G} = (\Gamma, W\phi)$.

Following [BrMi], we call unipotent ℓ–blocks of $\mathbf{G}^F$ the primitive central ℓ–idempotents which are constituents of the "unipotent idempotent" $e_\ell^{\mathbf{G}^F}$. We denote by $\mathcal{O}$ the extension of the ring of ℓ–adic integers by a root of unity whose order is the l.c.m. of the orders of the elements of $\mathbf{G}^F$. Thus $e_\ell^{\mathbf{G}^F} \in Z\mathcal{O}\mathbf{G}^F$, and the unipotent ℓ–blocks are the primitive idempotents of the algebra $Z\mathcal{O}\mathbf{G}^F e_\ell^{\mathbf{G}^F}$.

The following omnibus theorem is an immediate consequence of the preceding sections (specially of theorem 5.15), as well as a consequence of general results about isotypies (see [Br1]). We make free use of the notation which has been introduced previously.

4.17. Theorem.

(I). Let $(\mathbf{L}, \boldsymbol{\lambda})$ be a d–cuspidal pair of $\mathbf{G}$. We assume that ℓ is $(d, (\mathbf{L}, \boldsymbol{\lambda}))$–adapted.

(1) *The set of all irreducible constituents of the virtual characters $R_{\mathbf{L}}^{\mathbf{G}}(\theta\boldsymbol{\lambda})$, where $\theta \in \mathrm{Ab}_\ell\mathrm{Irr}(\mathbf{L}^F)$, is the set of all irreducible characters of an ℓ–block of $\mathbf{G}^F$, which we denote by $e_{\ell,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F}$.*

(2) *$(Z^o(\mathbf{L})_\ell^F, e_{\ell,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{L}^F})$ is a maximal $e_{\ell,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F}$–subpair of $\mathbf{G}^F$. In particular, $Z^o(\mathbf{L})_\ell^F$ is a defect group of $e_{\ell,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F}$.*

(3) *Whenever S is a subgroup of $Z^o(\mathbf{L})_\ell^F$, its centralizer $\mathbf{M}$ in $\mathbf{G}$ is a d–split Levi subgroup, and the Brauer correspondent $\mathrm{Br}_S(e_{\ell,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F})$ of $e_{\ell,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F}$ is given by the following formula:*

$$\mathrm{Br}_S(e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F}) = \sum_{(\mathbf{L}',\boldsymbol{\lambda}')} e_{\pi,(\mathbf{L}',\boldsymbol{\lambda}')}^{\mathbf{M}^F}$$

where $(\mathbf{L}', \boldsymbol{\lambda}')$ runs over a complete set of representatives for the $\mathbf{M}^F$–conjugacy classes of d–cuspidal pairs of $\mathbf{M}$ which are $\mathbf{G}^F$–conjugate to $(\mathbf{L}, \boldsymbol{\lambda})$.

(4) *$(\mathbf{G}^F, e_{\ell,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F})$ and $(Z^o(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F}(\mathbf{L}, \boldsymbol{\lambda}), 1)$ are isotypic, and so in particular (see* [Br1]*, thm. 1.5 and §4.D)*

(a) the block $e_{\ell,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F}$ of $\mathbf{G}^F$ has respectively the same numbers of ordinary and modular characters as the group $Z^o(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F}(\mathbf{L}, \boldsymbol{\lambda})$,

(b) all irreducible characters in $e_{\ell,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F}$ have height zero,

(c) the Cartan matrix of $e_{\ell,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F}$ defines the same integral quadratic form as the Cartan matrix of the group $Z^o(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F}(\mathbf{L}, \boldsymbol{\lambda})$,

(d) the generalized decomposition matrix of $(\mathbf{G}^F, e_{\ell,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F})$ *defines the same quadratic form (over* $\mathbb{Z}[e^{2i\pi/\ell^a}]$*, where* a *is the valuation in* ℓ *of* $\Phi_d(q)$*) as the generalized decomposition matrix of* $(Z^o(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda}), 1)$*,*

(e) the algebras $Z\mathcal{O}\mathbf{G}^F e_{\ell,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F}$ *and* $Z\mathcal{O}[Z^o(\mathbf{L})_\ell^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})]$ *are isomorphic.*

(II). Assume that ℓ *does not divide* $|W{<}\phi{>}|$*. Let* d *be the unique integer such that* $\Phi_d(x)$ *divides* $|\mathbb{G}|$ *and* ℓ *divides* $\Phi_d(q)$*. Then*

$$e_\ell^{\mathbf{G}^F} = \sum_{[(\mathbf{L},\boldsymbol{\lambda})]_{\mathbf{G}^F}} e_{\ell,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F}\,,$$

where $(\mathbf{L},\boldsymbol{\lambda})$ *runs over the set of all* d*–cuspidal pairs of* $\mathbf{G}^F$ *(module* $\mathbf{G}^F$*–conjugation). In other words,* $(\mathbf{L},\boldsymbol{\lambda}) \mapsto e_{\ell,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F}$ *is a bijection between the set of all unipotent* ℓ*-blocks of* $\mathbf{G}^F$ *and the* $\mathbf{G}^F$*–conjugacy classes of* d*–cuspidal pairs*

Notice that, still assuming ℓ is large (*i.e.,* $\ell \nmid |W{<}\phi{>}|$), the preceding theorem implies that several of the current conjectures in modular representation theory are true for groups of type $\mathbf{G}^F$ and unipotent blocks : Alperin conjecture, Alperin–McKay conjecture, all implied in these cases (blocks with abelian defect groups) by conjecture 6.1 in [Br1].

Notice also that the preceding remark can be strengthened by extending it to non–unipotent blocks of finite reductive groups, using the fact (see [Br1], theorems 2.3 and 5.6) that in many cases the “Jordan decomposition of blocks” is an isotypie.

5. π–BLOCKS AND ISOTYPIES

In §5, we apply what preceeds to the study of ordinary blocks of actual finite reductive groups.

The preceding “generic” approach shows that the group $\mathbf{G}^F$ (at least as far as unipotent characters are concerned) hardly differentiates between two prime numbers dividing the same factor $\Phi_d(q)$. Thus, many usual objects and properties of the classical ℓ–modular representation theory can be replaced, without harm, by corresponding objects or properties when the single prime ℓ is replaced by a set of primes dividing the same factor $\Phi_d(q)$. In this section, we illustrate this with an application of the generic theory, assuming that we are dealing with “large” primes (which in practice means primes not dividing $|W_{\mathbb{G}}\phi|$).

Since many notions (Brauer morphisms, generalized decomposition maps, isotypies, etc.) are often known only for the case of a single prime, we give (or recall) briefly, when needed, their definitions for a set of primes. The reader may refer to [Rob] and [RoSt] for more details — or read what follows assuming that π consists of a single prime.

A. π–blocks, Brauer morphisms.

In §5.A, for suitable sets of primes π*, we describe* π*–idempotents which turn out to be the* π*–blocks of* $\mathbf{G}^F$*. We compute their images under the usual Brauer morphisms.*

Large primes.

Let $\mathbb{G}$ be a generic finite reductive group, let p be a prime number, and let $(\mathbf{G}, \mathbf{T}, F)$ be a (q, ϕ)–triple associated to $\mathbb{G}$ such that q is a power of p. Let ℓ be a prime number such that $\ell \neq p$, ℓ divides $|\mathbf{G}^F|$.

5.1. Definition. *We say that ℓ is large (for $\mathbb{G}$) if $\ell \nmid |W_{\mathbb{G}}\phi|$.*

5.2. Proposition. *If $\ell \neq p$, $\ell \nmid |W\phi|$, then*

(1) *there exists a unique integer d such that Φ_d divides $|\mathbb{G}|$ and ℓ divides $\Phi_d(q)$ (In other words, we have $|\mathbf{G}^F|_\ell = |\mathbb{G}|_d(q)_\ell$),*

(2) *ℓ is good for $\mathbf{G}$,*

(3) *ℓ divides neither $|(Z(\mathbf{G})/Z^o(\mathbf{G}))^F|$ nor $|(Z(\mathbf{G}^*)/Z^o(\mathbf{G}^*))^{F^*}|$.*

Proof. (1) : See [Ge].

(2) follows from the fact that bad primes divide $|W|$.

(3). The group of characters of $Z(\mathbf{G})/Z^o(\mathbf{G})$ is isomorphic to the p'-part of the torsion subgroup of $X/Q(R)$. Since the torsion subgroup of $X/Q(R)$, namely the group $Q(R)^{\perp\perp}/Q(R)$, is a subgroup of $P(R)/Q(R)$, it is enough to check that $|P(R)/Q(R)|$ and $|P(R^\vee)/Q(R^\vee)|$ (this last to get the result for $\mathbf{G}^*$) divides $|W|$. But by definition $|P(R)/Q(R)|$ is the connection index f_R of R, and $f_R = f_{R^\vee}$ divides $|W|$ by [Bou], ch. VI, §2, Prop.7. □

5.3. Definition. *Let π be a set of prime numbers such that $p \notin \pi$. We say that π is $(\mathbf{G}, F)$–adapted if there exists an integer d such that $|\mathbf{G}^F|_\pi = |\mathbb{G}|_d(q)_\pi$. We then say that π is $(\mathbf{G}, F, d)$–adapted.*

Remarks.

1. If π is $(\mathbf{G}, F, d)$–adapted and $\mathbf{L}$ is an F–stable Levi subgroup of $\mathbf{G}$, then π is $(\mathbf{L}, F, d)$–adapted.
2. If $\pi = \{\ell\}$ where ℓ is a large prime, then π is $(\mathbf{G}, F)$–adapted by 5.2.

5.4. Proposition. *Let π be a set of large prime numbers which is $(\mathbf{G}, F, d)$–adapted. Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$.*

(1) *Let $\mathbf{S} = Z^o\mathbf{L}$. Let S be a π-subgroup of $\mathbf{S}^F$. Then $C^o_{\mathbf{G}}(S)$ is a d–split Levi subgroup of $\mathbf{G}$.*

(2) *Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{L}$, and let θ be a π-character of $(\mathbf{L}/[\mathbf{L}, \mathbf{L}])^F$. Then $\mathbf{G}(\mathbf{T}, \theta)$ is a d–split Levi subgroup of $\mathbf{G}$.*

Proof. (1) Since every element of π is good for $\mathbf{G}$, the group $C^o_{\mathbf{G}}(S)$ is a Levi subgroup of $\mathbf{G}$. Moreover, by 5.2, (3), we have $S \subset Z^o(C^o_{\mathbf{G}}(S))$. Now $C^o_{\mathbf{G}}(S)$ contains $C_{\mathbf{G}}(\mathbf{S})$, a Levi subgroup whose center by assumption is $\mathbf{S}$, so we have $Z^o(C^o_{\mathbf{G}}(S)) \subset \mathbf{S}$, and thus π is d–adapted for $Z^o(C^o_{\mathbf{G}}(S))$. It follows that S is contained in the Sylow Φ_d-subgroup $Z^o(C^o_{\mathbf{G}}(S))_d$ of $Z^o(C^o_{\mathbf{G}}(S))$, since the quotient of the corresponding groups of rational points is a π'-group. Then we have $C^o_{\mathbf{G}}(S) = C^o_{\mathbf{G}}(Z^o(C^o_{\mathbf{G}}(S))_d)$, which shows that $C^o_{\mathbf{G}}(S)$ is a d–split Levi subgroup of $\mathbf{G}$.

(2) The proof of the second assertion follows from the first one applied in the group $\mathbf{G}^*$. □

π–blocks.

Let π be a $(\mathbf{G},F,d)$–adapted set of large primes, and let $\mathrm{Reg}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}$ be the projection of $\mathrm{Reg}^{\mathbf{G}^F}_{\mathbf{L},\boldsymbol{\lambda}}$ (see §3.C above) onto the subspace generated by characters in

$$\mathcal{E}_\pi(\mathbf{G}^F,1) := \bigcup_{s\in(\mathbf{G}^{*F^*})_\pi} \mathcal{E}(\mathbf{G}^F,(s))\,,$$

(*cf.* [BrMi]). We will prove $\mathrm{Reg}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}$ is the regular character of a central "π–idempotent" of $\mathbf{G}^F$ (which will turn out to be a block if $\pi=\{\ell\}$).

From now on, we assume that $(\mathbf{G},\mathbf{T},F)$ is chosen so that $\mathbf{T}$ contains a Sylow Φ_d–subgroup of $\mathbf{G}$. So $\mathbf{T}$ is contained up to $\mathbf{G}^F$–conjugation in all d–split Levi subgroups of $\mathbf{G}$.

5.5. Theorem. *Let π be a set of large primes which is $(\mathbf{G},F,d)$–adapted. Then*

(1) *The projection of* $\mathrm{Reg}^{\mathbf{G}^F}_d$ *onto* $\mathcal{E}_\pi(\mathbf{G}^F,1)$ *is the character* $\mathrm{Reg}^{\mathbf{G}^F}_\pi$.

(2) *Let $(\mathbf{L},\boldsymbol{\lambda})$ be a d–cuspidal pair of $\mathbf{G}$ such that $\mathbf{T}\subseteq\mathbf{L}$. Then*

$$\mathrm{Reg}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} = \sum_{[(\mathbf{M},\theta,\boldsymbol{\mu})]_{\mathbf{G}^F}} \chi^{\mathbf{G}^F}_{(\mathbf{M},\theta,\boldsymbol{\mu})}(1).\chi^{\mathbf{G}^F}_{(\mathbf{M},\theta,\boldsymbol{\mu})}$$

where

- *θ runs over* $\mathrm{Ab}_\pi\,\mathrm{Irr}(\mathbf{L}^F)$,
- $\mathbf{M}=\mathbf{G}(\mathbf{T},\theta)$ *(cf. 2.24).*
- *$(\mathbf{M},\boldsymbol{\mu})$ is a d–split pair such that $(\mathbf{L},\boldsymbol{\lambda})\preccurlyeq_{\mathbf{G}^F}(\mathbf{M},\boldsymbol{\mu})$,*

The character $\mathrm{Reg}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}$ is the character associated to a central π–idempotent $e^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}$. In particular, its values are divisible by $|\mathbf{G}^F|_\pi$.

(3) *We have*

$$\mathrm{Reg}^{\mathbf{G}^F}_\pi = \sum_{[(\mathbf{L},\boldsymbol{\lambda})]_{\mathbf{G}^F}} \mathrm{Reg}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}\ ,$$

(in other words, $e^{\mathbf{G}^F}_\pi = \sum_{[(\mathbf{L},\boldsymbol{\lambda})]_{\mathbf{G}^F}} e^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}$), where $(\mathbf{L},\boldsymbol{\lambda})$ runs over the set of all d–cuspidal pairs of $\mathbf{G}$.

Proof of 5.5.

(1) For a d–split Levi subgroup $\mathbf{M}$ of $\mathbf{G}$, let us denote by $\mathrm{Irr}_\pi(\Theta^{\mathbf{M}^F}_{\mathbf{G}})$ the set of π–elements of $\mathrm{Irr}(\Theta^{\mathbf{M}^F}_{\mathbf{G}})$. Then, if $\mathrm{pr}^{\mathbf{G}^F}_\pi$ denotes the projection onto the subspace generated by characters in $\mathcal{E}_\pi(\mathbf{G}^F,1)$, we have

$$\mathrm{pr}^{\mathbf{G}^F}_\pi(R^{\mathbf{G}}_{\mathbf{M}}(\Theta^{\mathbf{M}^F}_{\mathbf{G}}.\boldsymbol{\mu})) = \sum_{\theta\in\mathrm{Irr}_\pi(\Theta^{\mathbf{M}^F}_{\mathbf{G}})} R^{\mathbf{G}}_{\mathbf{M}}(\theta.\boldsymbol{\mu})\ .$$

We need the following lemma, which follows from 5.2 applied to a dual group $\mathbf{L}^*$ of $\mathbf{L}$). The reader may refer to 2.38 for the notation.

5.6. Lemma. *Let* $\mathbf{L}$ *be an* F*–stable Levi subgroup of* $\mathbf{G}$ *containing* $\mathbf{T}$*. Let* π *be a* $(Z^o(\mathbf{L}), d)$*–adapted set of prime numbers none of which divides* $|W\phi|$*. Let* $\mathbf{M}$ *be a* d*–split Levi subgroup of* $\mathbf{G}$ *containing* $\mathbf{L}$*. Let* θ *be a* π*–element of* $\mathrm{Irr}(\Theta_{\mathbf{G}}^{\mathbf{M}^F})$*. Then* $\mathbf{M} = \mathbf{G}(\mathbf{T}, \theta)$*.*

It then results from 5.6 that

$$\mathrm{pr}_\pi^{\mathbf{G}^F}(R_{\mathbf{M}}^{\mathbf{G}}(\Theta_{\mathbf{G}}^{\mathbf{M}^F}.\boldsymbol{\mu})) = \sum_{\{\theta\,;\,(\theta\;\pi\text{–element})(\mathbf{G}(\mathbf{T},\theta)=\mathbf{M})\}} R_{\mathbf{M}}^{\mathbf{G}}(\theta.\boldsymbol{\mu})\ .$$

(1), as well as the first part of (2), are now immediate. Let us now prove that the values of $\mathrm{Reg}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F}$ are divisible by $|\mathbf{G}^F|_\pi$. Let $e_\pi^{\mathbf{G}^F}$ denote the central idempotent corresponding to $\mathcal{E}_\pi(\mathbf{G}^F, 1)$. We know (see [BrMi], th. 2.2) that $e_\pi^{\mathbf{G}^F}$ is a π–idempotent, *i.e.*, $e_\pi^{\mathbf{G}^F} \in \bar{\mathbb{Z}}[1/r]_{r\in\pi'}$. Since π is d–adapted for $\mathbf{G}$ and no element of π divides $|W_{\mathbb{G}}|$, we know by 4.16 that all values of $\mathrm{Reg}_{\mathbf{L},\boldsymbol{\lambda}}^{\mathbf{G}^F}$ are divisible by $|\mathbf{G}^F|_\pi$. For all $g \in \mathbf{G}^F$, we have $\mathrm{Reg}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F}(g) = \mathrm{Reg}_{\mathbf{L},\boldsymbol{\lambda}}^{\mathbf{G}^F}(g.e_\pi^{\mathbf{G}^F})$ and this proves now the second assertion of 5.5.

Finally the third assertion of 5.5 is an immediate consequence of 4.14. □

Brauer morphisms.

Let π be a set of prime numbers, and let s be a π–element of $\mathbf{G}^F$. Whenever e is a central π–idempotent of $\bar{\mathbb{Q}}\mathbf{G}^F$, we denote by $\mathrm{Br}_s(e)$ the image of e through the corresponding Brauer morphism (*cf.* [Rob] or [RoSt], and also [BrMi], §3). For the convenience of the reader, we recall the definition of $\mathrm{Br}_s(e)$.

• Let $\varpi_s^{\mathbf{G}^F}$ be the class function on $\mathbf{G}^F$ with value $|C_{\mathbf{G}^F}(s)|$ on the $\mathbf{G}^F$–conjugacy class of s and 0 elsewhere.

• For a class function ψ on $\mathbf{G}^F$, let $\mathrm{dec}_\pi^{s,\mathbf{G}^F}(\psi)$ denote the class function on $C_{\mathbf{G}^F}(s)$ such that

$$\mathrm{dec}_\pi^{s,\mathbf{G}^F}(\psi)(t) = \begin{cases} \psi(st) & \text{if } t \text{ is a } \pi'\text{–element} \\ 0 & \text{if not.} \end{cases}$$

• For a class function ψ on $\mathbf{G}^F$, let $e.\psi$ be the class function on $\mathbf{G}^F$ defined by $e.\psi(g) := \psi(eg)$.

Then the following formula defines the idempotent $\mathrm{Br}_s(e)$:

$$\mathrm{Br}_s(e).\,\mathrm{Reg}^{C_{\mathbf{G}^F}(s)} = \mathrm{dec}_\pi^{s,\mathbf{G}^F}(e.\varpi_s^{\mathbf{G}^F})\ . \tag{5.7}$$

Thus $\mathrm{Br}_s(e)$ is a central π–idempotent of $\bar{\mathbb{Q}}C_{\mathbf{G}^F}(s)$.

Iteration of 5.7 allows us to define (see for example [Rob]) an idempotent $\mathrm{Br}_S(e)$ for all abelian π–subgroups S of $\mathbf{G}^F$. In the case where $\pi = \{\ell\}$, this definition coincides with the original definition of Brauer.

5.8. Theorem. *Let* $(\mathbf{L}, \boldsymbol{\lambda})$ *be a* d*–cuspidal pair of* $\mathbf{G}$*. Let* π *be a set of large prime numbers which is* $(\mathbf{G}, F, d)$*–adapted for* $\mathbf{L}$*. Let* S *be a subgroup of* $Z^o(\mathbf{L})_\pi^F$ *and set* $\mathbf{M} := C_{\mathbf{G}}^o(S)$*. Then*

$$\mathrm{Br}_S(e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F}) = \sum_{(\mathbf{L}',\boldsymbol{\lambda}')} e_{\pi,(\mathbf{L}',\boldsymbol{\lambda}')}^{\mathbf{M}^F}$$

where $(\mathbf{L}', \boldsymbol{\lambda}')$ *runs over a complete set of representatives for the* $\mathbf{M}^F$*–conjugacy classes of* d*–cuspidal pairs of* $\mathbf{M}$ *which are* $\mathbf{G}^F$*–conjugate to* $(\mathbf{L}, \boldsymbol{\lambda})$*.*

Proof of 5.8. We may assume by induction that S is cyclic. Let s be a generator of S.
By "Curtis–type formula" (see [Br3], 4.3), we know that

$$\operatorname{dec}_\pi^{s,\mathbf{G}^F} = \operatorname{dec}_\pi^{s,\mathbf{M}^F} \cdot {}^*R_\mathbf{M}^\mathbf{G} \,.$$

So by 5.7 we must prove that

$$(\operatorname{dec}_\pi^{s,\mathbf{M}^F} \cdot {}^*R_\mathbf{M}^\mathbf{G})(e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F} . \varpi_s^{\mathbf{G}^F}) = \left(\sum_{(\mathbf{L}',\boldsymbol{\lambda}')} e_{\pi,(\mathbf{L}',\boldsymbol{\lambda}')}^{\mathbf{M}^F} \right) . \operatorname{Reg}^{\mathbf{M}^F} \,.$$

Let us denote by $\mathcal{E}_\pi(\mathbf{G}^F, (\mathbf{L}, \boldsymbol{\lambda}))$ the subspace of the space of all class functions ψ on $\mathbf{G}^F$ such that $e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F} . \psi = \psi$. Since $R_\mathbf{M}^\mathbf{G}$ and ${}^*R_\mathbf{M}^\mathbf{G}$ interchange $\mathcal{E}_\pi(\mathbf{G}^F, (\mathbf{L}, \boldsymbol{\lambda}))$ and $\bigcup_{(\mathbf{L}',\boldsymbol{\lambda}')} \mathcal{E}_\pi(\mathbf{M}^F, (\mathbf{L}', \boldsymbol{\lambda}'))$, we see that

$${}^*R_\mathbf{M}^\mathbf{G}(e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F} . \varpi_s^{\mathbf{G}^F}) = \left(\sum_{(\mathbf{L}',\boldsymbol{\lambda}')} e_{\pi,(\mathbf{L}',\boldsymbol{\lambda}')}^{\mathbf{M}^F} \right) . {}^*R_\mathbf{M}^\mathbf{G}(\varpi_s^{\mathbf{G}^F}) \,,$$

and so

$$\begin{aligned}
(\operatorname{dec}_\pi^{s,\mathbf{M}^F} \cdot {}^*R_\mathbf{M}^\mathbf{G})(e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F} . \varpi_s^{\mathbf{G}^F}) &= \left(\sum_{(\mathbf{L}',\boldsymbol{\lambda}')} e_{\pi,(\mathbf{L}',\boldsymbol{\lambda}')}^{\mathbf{M}^F} \right) . (\operatorname{dec}_\pi^{s,\mathbf{M}^F} \cdot {}^*R_\mathbf{M}^\mathbf{G})(\varpi_s^{\mathbf{G}^F}) \\
&= \left(\sum_{(\mathbf{L}',\boldsymbol{\lambda}')} e_{\pi,(\mathbf{L}',\boldsymbol{\lambda}')}^{\mathbf{M}^F} \right) . \operatorname{dec}_\pi^{s,\mathbf{G}^F}(\varpi_s^{\mathbf{G}^F}) \\
&= \left(\sum_{(\mathbf{L}',\boldsymbol{\lambda}')} e_{\pi,(\mathbf{L}',\boldsymbol{\lambda}')}^{\mathbf{M}^F} \right) . \operatorname{Reg}^{\mathbf{M}^F} \,.
\end{aligned}$$

□

B. Isotypies.

In §5.B, we check the the isotypie conjectures of [Br1] *hold for our* π*–idempotents.*

Preliminaries.

The following notation and hypothesis are in force throughout this section.
Let π be a set of large prime numbers for $\mathbb{G}$, which is $((\mathbf{G}, F, d)$–adapted.
Let $(\mathbf{L}, \boldsymbol{\lambda})$ be a d–cuspidal pair of $\mathbf{G}$. Let $\mathbf{T}$ be an F–stable maximal torus of $\mathbf{G}$ which contains $Z^o(\mathbf{L})$.

5.9. Proposition.

(1) *The restriction map from* $\mathbf{L}^F$ *down to* $Z^o(\mathbf{L})^F$ *induces an isomorphism*

$$\mathrm{Ab}_\pi \operatorname{Irr}(\mathbf{L}^F) \xrightarrow{\sim} \operatorname{Irr}(Z^o(\mathbf{L})^F_\pi) \ .$$

(2) *Let* θ *be a character of* $Z^o(\mathbf{L})^F_\pi$*, and let* θ *denote as well the linear character of* $\mathbf{T}^F$ *corresponding to it by (1). Then the inertial subgroup of* θ *in* $W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})$ *is* $W_{\mathbf{G}^F(\mathbf{T},\theta)}(\mathbf{L},\boldsymbol{\lambda})$*.*

Proof of 5.9. We know that $\mathbf{L} = [\mathbf{L},\mathbf{L}]Z^o(\mathbf{L})$. Moreover, $Z^o(\mathbf{L}) \cap [\mathbf{L},\mathbf{L}]$ is a subgroup of the finite group $Z([\mathbf{L},\mathbf{L}])$. Since π consists of large prime numbers for $\mathbb{G}$, we see by 5.2 that $Z^o(\mathbf{L}) \cap [\mathbf{L},\mathbf{L}]$ is a π'–group. Thus the projection $Z^o(\mathbf{L})^F_\pi \to \mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F$ is injective. Since $|Z^o(\mathbf{L})^F_\pi| = |\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F|_\pi$, the image of $Z^o(\mathbf{L})^F_\pi$ in $\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F$ is a supplement of $(\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F)_{\pi'}$, which proves 5.9. $\square$

In order to simplify the notation, we set

$$\mathbf{G}(\theta) := \mathbf{G}(\mathbf{T},\theta) \ .$$

Let $\tilde\theta$ be the character of the group $Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F(\theta)}(\mathbf{L},\boldsymbol{\lambda})$ defined by $\tilde\theta(zw) := \theta(z)$ where $z \in Z^o(\mathbf{L})^F_\pi$, $w \in W_{\mathbf{G}^F(\theta)}(\mathbf{L},\boldsymbol{\lambda})$. Then, whenever τ is an irreducible character of $W_{\mathbf{G}^F(\theta)}(\mathbf{L},\boldsymbol{\lambda})$, the character $\operatorname{Ind}_{Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F(\theta)}(\mathbf{L},\boldsymbol{\lambda})}^{Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}(\tilde\theta.\tau)$ is an irreducible character of the group $Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})$. We have (with the above notation)

$$\operatorname{Irr}(Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})) = \{\operatorname{Ind}_{Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F(\theta)}(\mathbf{L},\boldsymbol{\lambda})}^{Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}(\tilde\theta.\tau)\}_{[(\theta,\tau)]_{W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}} \ .$$

We denote by $e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F}$ the central π–idempotent of $\mathbf{G}^F$ associated with π and $(\mathbf{L},\boldsymbol{\lambda})$ (*cf.* 5.5). For $x \in \mathbf{G}^F$ we set

$$\mathbf{G}(x) := C^o_{\mathbf{G}}(x) \text{ and } \mathbf{G}^F(x) := C^o_{\mathbf{G}}(x)^F \ .$$

The centralizer in $Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})$ of an element $x \in Z^o(\mathbf{L})^F_\pi$ is $Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F(x)}(\mathbf{L},\boldsymbol{\lambda})$.

The next proposition is a result about control of fusion.

5.10. Proposition. *The map* $x \mapsto (x, e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F(x)})$ *induces a bijection between the conjugacy classes of* π*–elements of the group* $Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})$ *and the* $\mathbf{G}^F$*–conjugacy classes of pairs* $(x, e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F(x)})$ *(for a* π*–element* x *of* $Z^o(\mathbf{L})^F_\pi$*).*

Proof of 5.10.

Since $e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F(x)} = e_{\pi,(\mathbf{L}',\boldsymbol{\lambda}')}^{\mathbf{G}^F(x)}$ if and only if $(\mathbf{L},\boldsymbol{\lambda})$ and $(\mathbf{L}',\boldsymbol{\lambda}')$ are $\mathbf{G}^F(x)$–conjugate, we must check the following property: the elements x and x' of $Z^o(\mathbf{L})^F_\pi$ are conjugate under $Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})$ if and only if there exists $g \in \mathbf{G}^F$ such that

(a) $x' = x^g$,

(b) $(\mathbf{L},\boldsymbol{\lambda})^g$ is $\mathbf{G}^F(x')$–conjugate to $(\mathbf{L},\boldsymbol{\lambda})$.

This is obvious since $W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda}) = N_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})/\mathbf{L}^F$. $\square$

π–perfect isometries.

We shall use the notion of "perfect isometry" as defined in [Br1]. Since this was only defined for ℓ–idempotents and not for π–idempotents, we give here briefly a whole account of the necessary definitions and elementary properties of π–perfect isometries.

Let H be a finite group and let π be a set of prime numbers. Let K be a field of characteristic zero which is a splitting field for the group H. Let $\mathcal{O}$ denote the integral closure in K of the ring $\mathbb{Z}[\{1/\ell\}_{\ell\notin\pi}]$.

Let e be an idempotent of the center $Z\mathcal{O}H$ (a "central π–idempotent" for H).

We denote by $\mathrm{CF}(H,e;K)$ the K–vector space consisting of class functions α on H with values in K and such that $\alpha(eh) = \alpha(h)$ for all $h \in H$. Let $\mathrm{CF}_{\pi'}(H,e;K)$ be the subspace of functions vanishing outside of the set $H_{\pi'}$ of π'–elements of H. We define the $\mathcal{O}$–modules $\mathrm{CF}(H,e;\mathcal{O})$ and $\mathrm{CF}_{\pi'}(H,e;\mathcal{O})$ in an analogous way. We denote by $\mathrm{Irr}(H,e)$ the set of irreducible characters of H on K which belong to $\mathrm{CF}(H,e;K)$, and so to $\mathrm{CF}(H,e;\mathcal{O})$.

5.11. Definition. *(see* [Br1]*, 4.1) Let H and H' be finite groups, and let e and e' be two central π–idempotents for H and H' respectively. Let*

$$I\colon \mathbb{Z}\mathrm{Irr}(H,e) \xrightarrow{\sim} \mathbb{Z}\mathrm{Irr}(H',e')$$

be an isometry. I is π–perfect if the following two conditions are fulfilled:

(P1) *I extends linearly to an isometric isomorphism*

$$\mathrm{CF}(H,e;\mathcal{O}) \xrightarrow{\sim} \mathrm{CF}(H',e';\mathcal{O}) \ .$$

(P2) *The restriction of I to* $\mathrm{CF}_{\pi'}(H,e;\mathcal{O})$ *induces an isometric isomorphism*

$$\mathrm{CF}_{\pi'}(H,e;\mathcal{O}) \xrightarrow{\sim} \mathrm{CF}_{\pi'}(H',e';\mathcal{O}) \ .$$

The proof of the following property of π–perfect isometries is the same as the proof of the corresponding result given in [Br1], 1.5, for the case $\pi = \{\ell\}$.

5.12. Proposition. *Assume that*

$$I\colon \mathbb{Z}\mathrm{Irr}(H,e) \xrightarrow{\sim} \mathbb{Z}\mathrm{Irr}(H',e')$$

is a π–perfect isometry. Then the algebra isomorphism from $ZKHe$ onto $ZKH'e'$ defined by I induces an algebra isomorphism from $Z\mathcal{O}He$ onto $Z\mathcal{O}H'e'$. In particular e is primitive if and only if e' is primitive.

Generalized decomposition maps.

For the convenience of the reader, we also recall the definition of generalized decomposition maps in the particular settings we are working in (*cf.* for example [Br1], 4A).

5.13. Definition.

(1) Given a class function ψ *of* $\mathbf{G}^F$, *let* $\mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{x,\mathbf{G}^F}(\psi)$ *be the class function on* $\mathbf{G}^F(x)$ *defined as follows (cf. for example* [Br1] *4.3):*

- $\mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{x,\mathbf{G}^F}(\psi)$ *vanishes outside of the set of* π'*–elements of* $\mathbf{G}^F(x)$,
- *for* $x' \in \mathbf{G}^F(x)_{\pi'}$,

$$\mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{x,\mathbf{G}^F}(\psi)(x') := \psi(xx'e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F(x)}) \ .$$

(2) Similarly, given a class function φ *of* $Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})$, *let* $\mathrm{dec}_\pi^{x,Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}(\varphi)$ *be the class function on the group* $Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F(x)}(\mathbf{L},\boldsymbol{\lambda})$ *defined as follows:*

- $\mathrm{dec}_\pi^{x,Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}(\varphi)$ *vanishes outside of the set of* π'*–elements of the group* $Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F(x)}(\mathbf{L},\boldsymbol{\lambda})$,
- *for a* π'*–element* x' *of* $Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F(x)}(\mathbf{L},\boldsymbol{\lambda})$,

$$\mathrm{dec}_\pi^{x,Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}(\varphi)(x') := \varphi(xx') \ .$$

Let ${}^t\mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{x,\mathbf{G}^F}$ denote the adjoint of $\mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{x,\mathbf{G}^F}$. For a class function φ of $\mathbf{G}^F(x)$ with support on $\mathbf{G}^F(x)_{\pi'}$, ${}^t\mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{x,\mathbf{G}^F}(\varphi)$ is the class function of $\mathbf{G}^F$ with support on the π–section of x in $\mathbf{G}^F$ given by

$${}^t\mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{x,\mathbf{G}^F}(\varphi)(xx') = \varphi(x')$$

for all $x' \in \mathbf{G}^F(x)_{\pi'}$.

Similarly, ${}^t\mathrm{dec}_\pi^{x,Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}$ denotes the adjoint of $\mathrm{dec}_\pi^{x,Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}$.

The following result is a classical consequence of Brauer's second main theorem (see for example [Br1], 4A, last part, for the case where π consists of a single prime).

(5.14)

(1) *If* ψ *is a linear combination of elements of* $\mathcal{E}_\pi(\mathbf{G}^F,(\mathbf{L},\boldsymbol{\lambda}))$, *then*

$$\psi = \sum_{[x\in Z^o(\mathbf{L})_\pi^F]_{W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}} ({}^t\mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{x,\mathbf{G}^F} \cdot \mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{x,\mathbf{G}^F})(\psi) \ .$$

(2) *If* ξ *is a class function on* $Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})$, *then*

$$\xi = \sum_{[x\in Z^o(\mathbf{L})_\pi^F]_{W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}} ({}^t\mathrm{dec}_\pi^{x,Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{dec}_\pi^{x,Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})})(\xi) \ .$$

The main theorem.

Notation.

Let $(\mathbf{L},\boldsymbol{\lambda})$ be a d–cuspidal pair of $\mathbf{G}$. We denote by $\mathrm{Uch}(\mathbf{M},(\mathbf{L},\boldsymbol{\lambda}))$ the set of unipotent characters $\boldsymbol{\gamma}$ of $\mathbf{G}^F$ such that $<R_{\mathbf{L}}^{\mathbf{G}}(\boldsymbol{\lambda}),\boldsymbol{\gamma}> \neq 0$. It follows from 3.2 that there exists a collection of isometries

$$I_{(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{M}} \colon \mathbb{Z}\mathrm{Irr}(W_{\mathbf{M}^F}(\mathbf{L},\boldsymbol{\lambda})) \to \mathbb{Z}\,\mathrm{Uch}(\mathbf{M},(\mathbf{L},\boldsymbol{\lambda}))\,,$$

where $\mathbf{M}$ runs over the set of all d–split Levi subgroups of $\mathbf{G}$, such that

$$R_{\mathbf{M}}^{\mathbf{G}} \cdot I_{(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{M}} = I_{(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}} \cdot \mathrm{Ind}_{W_{\mathbf{M}^F}(\mathbf{L},\boldsymbol{\lambda})}^{W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}\,.$$

5.15. Theorem.

(1) Let $\mathbf{N}$ run over the set of d–split Levi subgroups of $\mathbf{G}$ containing $\mathbf{L}$. Then the maps

$$\mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{N}^F} \colon \mathbb{Z}\mathrm{Irr}\left(Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{N}^F}(\mathbf{L},\boldsymbol{\lambda})\right) \longrightarrow \mathbb{Z}\mathrm{Irr}(\mathbf{N}^F, e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{N}^F})$$

such that

$$\mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{N}^F} \colon \mathrm{Ind}_{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{N}^F(\theta)}(\mathbf{L},\boldsymbol{\lambda})}^{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{N}^F}(\mathbf{L},\boldsymbol{\lambda})}(\tilde{\theta}.\tau) \mapsto R_{\mathbf{N}(\theta)}^{\mathbf{N}}(\theta.I_{(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{N}(\theta)}(\tau))$$

commute with the generalized decomposition maps in the following sense: whenever $x \in Z^o(\mathbf{L})_\pi^F$, then

$$\text{(a)}\qquad \mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{x,\mathbf{G}^F} \cdot \mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F} = \mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F(x)} \cdot \mathrm{dec}_\pi^{x,Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}\,.$$

(2) Each $\mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{N}^F}$ is a π–perfect isometry between $(\mathbf{N}^F, e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{N}^F})$ and $(Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{N}^F}(\mathbf{L},\boldsymbol{\lambda}),1)$.

Remark. Had we given the general definition of a "π-isotypy" (see [Br1], 4.6, for the definition of an ℓ–isotypy), assertions (1) and (2) above would mean that $(\mathbf{N}^F, e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{N}^F})$ and $(Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{N}^F}(\mathbf{L},\boldsymbol{\lambda}),1)$ are π–isotypic.

Proof of 5.15, (1).

For $x \in Z^o(\mathbf{L})_\pi^F$, we have the "Curtis type formula" (*cf.* [Br3], 4.3)

$$\text{(5.16)}\qquad \mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{x,\mathbf{G}^F} = \mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{x,\mathbf{G}^F(x)} \cdot {}^*R_{\mathbf{G}(x)}^{\mathbf{G}}(\mathbf{L},\boldsymbol{\lambda})$$

and similarly it is easy to see that

$$\mathrm{dec}_\pi^{x,Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} = \mathrm{dec}_\pi^{x,Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F(x)}(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{Res}_{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F(x)}(\mathbf{L},\boldsymbol{\lambda})}^{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}\,.$$

Thus we see that (a) in the statement is equivalent to

$$\begin{aligned} &\mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{x,\mathbf{G}^F(x)} \cdot {}^*R_{\mathbf{G}(x)}^{\mathbf{G}}(\mathbf{L},\boldsymbol{\lambda}) \cdot \mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F} = \\ \text{(b)}\qquad &\mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F(x)} \cdot \mathrm{dec}_\pi^{x,Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F(x)}(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{Res}_{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F(x)}(\mathbf{L},\boldsymbol{\lambda})}^{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}\,. \end{aligned}$$

The proof of 5.15 is in several steps.

Step 1. The first step allows us to reduce the proof of (b) to the case where $x \in Z^o(\mathbf{G})_\pi^F$.

5.17. Proposition. *Whenever* $\mathbf{N}$ *is a* d*–split Levi subgroup of* $\mathbf{G}$*,*

$$ {}^*R_{\mathbf{N}}^{\mathbf{G}}(\mathbf{L},\boldsymbol{\lambda}) \cdot \mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F} = \mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{N}^F} \cdot \mathrm{Res}_{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{N}^F}(\mathbf{L},\boldsymbol{\lambda})}^{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} \,. $$

Proof of 5.17. It suffices to prove the equivalent adjoint statement:

$$ R_{\mathbf{N}}^{\mathbf{G}}(\mathbf{L},\boldsymbol{\lambda}) \cdot \mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{N}^F} = \mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F} \cdot \mathrm{Ind}_{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{N}^F}(\mathbf{L},\boldsymbol{\lambda})}^{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} \,. $$

Thus we must check that for all d–split Levi subgroups $\mathbf{M}$ of $\mathbf{N}$ such that $\mathbf{L} \subseteq \mathbf{M}$, for $\theta \in \mathrm{Ab}_\pi \mathrm{Irr}(Z^o(\mathbf{L})^F)$ such that $\mathbf{N}(\theta) = \mathbf{M}$, and for all irreducible characters τ of $W_{\mathbf{M}}(\mathbf{L},\boldsymbol{\lambda})$,

$$ R_{\mathbf{N}}^{\mathbf{G}}(\mathbf{L},\boldsymbol{\lambda}) \cdot \mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{N}^F} \left(\mathrm{Ind}_{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{M}^F}(\mathbf{L},\boldsymbol{\lambda})}^{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{N}^F}(\mathbf{L},\boldsymbol{\lambda})} (\tilde{\theta}\tau) \right) = \mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F} \cdot \mathrm{Ind}_{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{N}^F}(\mathbf{L},\boldsymbol{\lambda})}^{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} \left(\mathrm{Ind}_{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{M}^F}(\mathbf{L},\boldsymbol{\lambda})}^{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{N}^F}(\mathbf{L},\boldsymbol{\lambda})} (\tilde{\theta}\tau) \right) \,. $$

This last equality is equivalent to

$$ R_{\mathbf{G}(\theta)}^{\mathbf{G}} \left(\theta R_{\mathbf{M}}^{\mathbf{G}(\theta)} (I_{(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{M}}(\tau)) \right) = R_{\mathbf{G}(\theta)}^{\mathbf{G}} \left(\theta I_{(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}(\theta)} \left(\mathrm{Ind}_{W_{\mathbf{M}^F}(\mathbf{L},\boldsymbol{\lambda})}^{W_{\mathbf{G}^F(\theta)}(\mathbf{L},\boldsymbol{\lambda})} (\tau) \right) \right) \,. $$

This follows now from the fundamental theorem 3.2 applied in $\mathbf{G}(\theta)$. $\square$

By 5.17, we see that (b) is equivalent to

$$ \mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{x,\mathbf{G}^F(x)} \cdot \mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F(x)} \cdot \mathrm{Res}_{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F(x)}(\mathbf{L},\boldsymbol{\lambda})}^{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} = \mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F(x)} \cdot \mathrm{dec}_{\pi}^{x, Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F(x)}(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{Res}_{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F(x)}(\mathbf{L},\boldsymbol{\lambda})}^{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} \,. $$

Thus we see that it is sufficient to prove that, for $x \in Z^o(\mathbf{G})_\pi^F$,

(c)
$$ \mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{x,\mathbf{G}^F} \cdot \mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F} = \mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F} \cdot \mathrm{dec}_{\pi}^{x, Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} \,. $$

Step 2. The usual decomposition maps are defined by the formulae

$$ \mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F} := \mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{1,\mathbf{G}^F} \quad \text{and} \quad \mathrm{dec}_{\pi}^{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} := \mathrm{dec}_{\pi}^{1, Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} \,. $$

For $x \in Z^o(\mathbf{G})_\pi^F$, and for a class function ψ on $\mathbf{G}^F$, we denote by $\mathrm{t}^{x,\mathbf{G}^F}(\psi)$ the class function on $\mathbf{G}^F$ defined by $\mathrm{t}^{x,\mathbf{G}^F}(\psi)(g) := \psi(xg)$.

Similarly, for $x \in Z^o(\mathbf{G})_\pi^F$, and for a class function φ on $Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})$, we denote by $\mathrm{t}^{x,Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}(\varphi)$ the class function on $Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})$ defined by $\mathrm{t}^{x,\mathbf{G}^F}(\varphi)(h) := \varphi(xh)$.

Thus for $x \in Z^o(\mathbf{G})_\pi^F$ we have

$$ \begin{cases} \mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{x,\mathbf{G}^F} = \mathrm{dec}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{G}^F} \cdot \mathrm{t}^{x,\mathbf{G}^F} \\ \mathrm{dec}_{\pi}^{x, Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} = \mathrm{dec}_{\pi}^{Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{t}^{x, Z^o(\mathbf{L})_\pi^F \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} \end{cases} \,. $$

Equation (c) becomes then

$$\mathrm{dec}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{t}^{x,\mathbf{G}^F} \cdot \mathbf{I}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} = \mathbf{I}^{\mathbf{G}^F}_{(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{dec}_\pi^{Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{t}^{x,Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} .$$

We see that it suffices to prove

$$\text{(d)}\qquad \mathrm{t}^{x,\mathbf{G}^F} \cdot \mathbf{I}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} = \mathbf{I}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{t}^{x,Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}$$

and

$$\text{(e)}\qquad \mathrm{dec}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathbf{I}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} = \mathbf{I}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{dec}_\pi^{Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} .$$

Equation (d) is an immediate consequence of the values of Deligne–Lusztig and ordinary induction. Indeed, we see that

$$\begin{aligned}\left(\mathrm{t}^{x,\mathbf{G}^F} \cdot \mathbf{I}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}\right)&(\mathrm{Ind}^{Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}_{Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F(\theta)}(\mathbf{L},\boldsymbol{\lambda})}(\tilde{\theta}.\tau)) = \\ &\left(\mathbf{I}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{t}^{x,Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}\right)(\mathrm{Ind}^{Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}_{Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F(\theta)}(\mathbf{L},\boldsymbol{\lambda})}(\tilde{\theta}.\tau)) = \\ &\theta(x) R^{\mathbf{G}}_{\mathbf{G}(\theta)}(\theta. I^{\mathbf{G}(\theta)}_{(\mathbf{L},\boldsymbol{\lambda})}(\tau)) .\end{aligned}$$

Step 3 : proof of (e).

The proof is by induction on $\mathrm{rk}_{ss}(\mathbf{G}) - \mathrm{rk}_{ss}(\mathbf{L})$, where we denote by $\mathrm{rk}_{ss}(\mathbf{G})$ the semi–simple rank of $\mathbf{G}$.

1. The case $\mathbf{G} = \mathbf{L}$.

This is the case where $\mathrm{rk}_{ss}(\mathbf{G}) = \mathrm{rk}_{ss}(\mathbf{L})$. The irreducible characters of $\mathbf{L}^F$ to be considered are the $\theta.\boldsymbol{\lambda}$ where $\theta \in \mathrm{Ab}_\pi \mathrm{Irr}(\mathbf{L}^F)$. The equality (e) is then proved by the following lemma.

5.18. Lemma.

(1) *For a d–cuspidal irreducible character $\boldsymbol{\lambda}$ of $\mathbf{L}^F$ and $\theta \in \mathrm{Ab}_\pi \mathrm{Irr}(\mathbf{L}^F)$, we have*

$$\mathrm{dec}^{\mathbf{L}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}(\theta.\boldsymbol{\lambda}) = \frac{1}{|Z^o(\mathbf{L})^F_\pi|} \mathrm{Ab}_\pi \mathrm{Reg}^{\mathbf{L}^F} \cdot \boldsymbol{\lambda} = \frac{1}{|Z^o(\mathbf{L})^F_\pi|} \sum_{\eta \in \mathrm{Irr}(Z^o(\mathbf{L})^F_\pi)} \eta.\boldsymbol{\lambda} .$$

(2) *For $\theta \in \mathrm{Irr}(Z^o(\mathbf{L})^F_\pi)$, we have*

$$\mathrm{dec}_\pi^{Z^o(\mathbf{L})^F_\pi}(\theta) = \frac{1}{|Z^o(\mathbf{L})^F_\pi|} \mathrm{Reg}^{Z^o(\mathbf{L})^F_\pi} = \frac{1}{|Z^o(\mathbf{L})^F_\pi|} \sum_{\eta \in \mathrm{Irr}(Z^o(\mathbf{L})^F_\pi)} \eta .$$

Proof of 5.18. The second assertion is obvious. To prove the first one, we notice the following property of d–cuspidal characters.

5.19. Let $\boldsymbol{\gamma}$ be a d–cuspidal irreducible unipotent character of $\mathbf{G}^F$. Let π be a set of large primes which is $(\mathbf{G}, F, d)$–adapted. Let $g \in \mathbf{G}^F$. Then $\boldsymbol{\gamma}(g) = 0$ unless the π–component g_π of g belongs to $Z^o(\mathbf{G})^F$.

Proof of 5.19. It results from the Curtis type formula (see 5.16) that, for $\mathbf{M} := C^o_{\mathbf{G}}(g_\pi)$, we have

$$\boldsymbol{\gamma}(g) = {}^*R^{\mathbf{G}}_{\mathbf{M}}(\boldsymbol{\gamma})(g) \ .$$

Since $\mathbf{M}$ is a d–split Levi subgroup of $\mathbf{G}$ (by 5.4), and since $\boldsymbol{\gamma}$ is d–cuspidal, we see by 4.10, (2), that ${}^*R^{\mathbf{G}}_{\mathbf{M}}(\boldsymbol{\gamma}) = 0$ unless $\mathbf{M} = \mathbf{G}$, which proves 5.19. □

□

Remark. The preceding result is a particular case of the following property, which also follows from 4.10, (2).

Let $\boldsymbol{\gamma}$ be a unipotent character of $\mathbf{G}^F$ such that $\left(\boldsymbol{\gamma}, R^{\mathbf{G}}_{\mathbf{L}}(\boldsymbol{\lambda})\right)_{\mathbf{G}^F} \neq 0$, where $\mathbf{L}$ is a d–split Levi subgroup of $\mathbf{G}$, and where $\boldsymbol{\lambda}$ is a d–cuspidal unipotent character of $\mathbf{L}^F$. Let π be a set of large primes which is $(\mathbf{G}, F, d)$–adapted. Let $g \in \mathbf{G}^F$. Then $\boldsymbol{\gamma}(g) = 0$ unless the π–component g_π of g belongs to $Z^o(\mathbf{L})^F$.

This property can also be viewed as a consequence of Brauer's second main theorem and theorem 5.8 (see for example formula 5.14, (1)).

2. Applying the induction hypothesis.

By the induction hypothesis, we assume now that (e) holds whenever $\mathbf{G}$ is replaced by any of its proper d–split Levi subgroups. It follows that (c) (whence (b) and (a) as well) holds for all $x \in Z^o(\mathbf{L})^F_\pi - Z^o(\mathbf{G})^F_\pi$. The formulae 5.14 applied with $\psi = \mathbf{I}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}(\xi)$ then show that

$$(5.20)\quad \mathbf{I}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} \cdot \left(\sum_{z \in Z^o(\mathbf{G})^F_\pi} {}^t\mathrm{dec}_\pi^{z, Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{dec}_\pi^{z, Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} \right) = \left(\sum_{z \in Z^o(\mathbf{G})^F_\pi} {}^t\mathrm{dec}^{x,\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{dec}^{z,\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} \right) \cdot \mathbf{I}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} \ .$$

To prove that (e) holds, again by the induction hypothesis and by 5.17, it suffices to prove that, for $\theta \in \mathrm{Irr}_\pi(Z^o(\mathbf{G})^F)$ and $\tau \in \mathrm{Irr}(W_{\mathbf{G}^F}(\mathbf{L}, \boldsymbol{\lambda}))$, we have

$$(\mathrm{dec}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathbf{I}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})})(\theta.\tau) = (\mathbf{I}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{dec}_\pi^{Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})})(\theta.\tau) \ ,$$

i.e.,

$$(\mathrm{dec}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}(\theta.I^{\mathbf{G}}_{(\mathbf{L},\boldsymbol{\lambda})}(\tau)) = (\mathbf{I}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{dec}_\pi^{Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})})(\theta.\tau) \ .$$

It is easy to check that

$$\sum_{\theta \in \mathrm{Irr}_\pi(Z^o(\mathbf{G})^F)} \sum_{z \in Z^o(\mathbf{G})^F_\pi} {}^t\mathrm{dec}_\pi^{z, Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{dec}_\pi^{z, Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}(\theta\tau) = |Z^o(\mathbf{G})^F_\pi| \mathrm{dec}_\pi^{Z^o(\mathbf{L})^F_\pi}(\tau) \ ,$$

and also that

$$\sum_{\theta\in\mathrm{Irr}_\pi(Z^o(\mathbf{G})^F)}\ \sum_{z\in Z^o(\mathbf{G})^F_\pi} {}^t\mathrm{dec}^{z,\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}\cdot \mathrm{dec}^{z,\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}(I^{\mathbf{G}^F}_{(\mathbf{L},\boldsymbol{\lambda})}(\theta\tau)) = $$
$$|Z^o(\mathbf{G})^F_\pi|\mathrm{dec}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}(I^{\mathbf{G}^F}_{(\mathbf{L},\boldsymbol{\lambda})}(\tau)) \ .$$

Since

$$\mathrm{dec}_\pi^{Z^o(\mathbf{L})^F_\pi\rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}(\theta.\tau) = \mathrm{dec}_\pi^{Z^o(\mathbf{L})^F_\pi\rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}(\tau)$$

and

$$\mathrm{dec}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}(\theta.I^{\mathbf{G}^F}_{(\mathbf{L},\boldsymbol{\lambda})}(\tau)) = \mathrm{dec}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}(I^{\mathbf{G}^F}_{(\mathbf{L},\boldsymbol{\lambda})}(\tau)) \ ,$$

we see that (e) follows then from what precedes. This completes step (3) and thus the proof of 5.15, (1). □

Proof of 5.15, (2). .

As in the proof of step 3 above, the proof is by induction on $\mathrm{rk}_{ss}(\mathbf{G}) - \mathrm{rk}_{ss}(\mathbf{L})$, where we denote by $\mathrm{rk}_{ss}(\mathbf{G})$ the semi–simple rank of $\mathbf{G}$.

1. The case where $\mathbf{G} = \mathbf{L}$.

We use notation previously introduced (see 5.11). We shall need the following property of d–cuspidal characters.

5.21. Lemma. *Let $\boldsymbol{\gamma}$ be a d–cuspidal unipotent character of $\mathbf{G}^F$. Let π be a set of large prime numbers which is $(\mathbf{G}, F, d)$–adapted for $\mathbf{G}$. Then for all $g \in \mathbf{G}^F$, $|C_{\mathbf{G}^F}(g) : Z^o(\mathbf{G})^F|_\pi$ divides $\boldsymbol{\gamma}(g)$ (in other words, $\dfrac{\boldsymbol{\gamma}(g)}{|C_{\mathbf{G}^F}(g) : Z^o(\mathbf{G})^F|} \in \mathcal{O}$).*

Proof of 5.21.

We see by 2.9 (*cf.* proof of 2.42) that for each $\ell \in \pi$, we have $\boldsymbol{\gamma}(1)_\ell = |\mathbf{G}^F : Z^o(\mathbf{G})^F|_\ell$, and so $\boldsymbol{\gamma}$ is a character with ℓ–defect zero of the group $\mathbf{G}^F/Z^o(\mathbf{G})^F$. Henceforth, we know that, for all $g \in \mathbf{G}^F$, $|C_{\mathbf{G}^F}(g) : Z^o(\mathbf{G})^F|_\ell$ divides $\boldsymbol{\gamma}(g)$, from which it follows that $|C_{\mathbf{G}^F}(g) : Z^o(\mathbf{G})^F|_\pi$ divides $\boldsymbol{\gamma}(g)$. □

We must check that the map

$$\mathbf{I}^{\mathbf{L}^F}_{\pi(\mathbf{L},\boldsymbol{\lambda})} : \mathbb{Z}\mathrm{Irr}(Z^o(\mathbf{L})^F_\pi) \xrightarrow{\sim} \mathbb{Z}\mathrm{Irr}(\mathbf{L}^F, e^{\mathbf{L}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})})$$

such that $\theta \mapsto \theta\boldsymbol{\lambda}$ (here we identify $\mathrm{Ab}_\pi\,\mathrm{Irr}(\mathbf{L}^F)$ and $\mathrm{Irr}(Z^o(\mathbf{L})^F_\pi)$ after proposition 5.9, (1)), is a π–perfect isometry.

• The $\mathcal{O}$–module $\mathrm{CF}(Z^o(\mathbf{L})^F_\pi;\mathcal{O})$ is generated by the set of characteristic functions $\delta_x^{Z^o(\mathbf{L})^F_\pi}$ of the elements x of $Z^o(\mathbf{L})^F_\pi$, while the submodule $\mathrm{CF}_{\pi'}(Z^o(\mathbf{L})^F_\pi;\mathcal{O})$ is generated by the characteristic function $\delta_1^{Z^o(\mathbf{L})^F_\pi}$. We have

(a)
$$\delta_x^{Z^o(\mathbf{L})^F_\pi} = \frac{1}{|Z^o(\mathbf{L})^F_\pi|}\sum_{\theta\in\mathrm{Irr}(Z^o(\mathbf{L})^F_\pi)} \theta(x^{-1})\theta \ .$$

• The $\mathcal{O}$–module $\mathrm{CF}(\mathbf{L}^F, e^{\mathbf{L}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})};\mathcal{O})$ is generated by the set of "projections" of the characteristic functions of the conjugacy classes of $\mathbf{L}^F$ "onto the idempotent $e^{\mathbf{L}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}$".

By 5.19, we see that all these characteristic functions project onto zero, except the characteristic functions $\delta_{(xx')}^{\mathbf{L}^F}$ of elements $xx' \in \mathbf{L}^F$ whose π–component x belongs to $Z^o(\mathbf{L})^F_\pi$. The projection of such a function is then

$$\text{(b)} \qquad e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{L}^F} \cdot \delta_{(xx')}^{\mathbf{L}^F} := \frac{\boldsymbol{\lambda}(x'^{-1})}{|C_{\mathbf{L}^F}(x')|} \sum_{\theta \in \mathrm{Ab}_\pi \operatorname{Irr}(\mathbf{L}^F)} \theta(x^{-1})\theta\boldsymbol{\lambda} \ .$$

Since

$$e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{L}^F} \cdot \delta_{(xx')}^{\mathbf{L}^F} := \frac{\boldsymbol{\lambda}(x'^{-1})}{|C_{\mathbf{L}^F}(x')|} e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{L}^F} \cdot \delta_{(x)}^{\mathbf{L}^F} \ ,$$

we see by 5.21 that the module $\mathrm{CF}(\mathbf{L}^F, e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{L}^F}; \mathcal{O})$ is actually generated by the functions $e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{L}^F} \cdot \delta_{(x)}^{\mathbf{L}^F}$ for $x \in Z^o(\mathbf{L})^F_\pi$.

The submodule $\mathrm{CF}_{\pi'}(\mathbf{L}^F, e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{L}^F}; \mathcal{O})$ is generated by the functions $e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{L}^F} \cdot \delta_{(x')}^{\mathbf{L}^F}$. For the same reasons as above, it is actually generated by $e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{L}^F} \cdot \delta_{(1)}^{\mathbf{L}^F}$.

By (a) and (b), we see that

$$e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{L}^F} \cdot \delta_{(x)}^{\mathbf{L}^F} = \frac{\boldsymbol{\lambda}(1)}{|\mathbf{L}^F : Z^o(\mathbf{L})^F_\pi|} \mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{L}^F}(\delta_x^{Z^o(\mathbf{L})^F_\pi}) \ ,$$

and this proves indeed that $\mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{L}^F}$ induces a bijection between $\mathrm{CF}(\mathbf{L}^F, e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{L}^F}; \mathcal{O})$ and $\mathrm{CF}(Z^o(\mathbf{L})^F_\pi; \mathcal{O})$ as well as between $\mathrm{CF}_{\pi'}(\mathbf{L}^F, e_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{L}^F}; \mathcal{O})$ and $\mathrm{CF}_{\pi'}(Z^o(\mathbf{L})^F_\pi; \mathcal{O})$.

2. Applying the induction hypothesis

The following proof is the analogue of the proof of lemma 4.5 in [Br1].

We first need to prove an easy general result about finite groups. Let H be a finite group, and let e be a central π–idempotent for H. Let us denote by

$$\mathrm{dec}_\pi^H : \mathrm{CF}(H, e; \mathcal{O}) \to \mathrm{CF}_{\pi'}(H, e; \mathcal{O})$$

the "π–decomposition map", defined as usual (for $f \in \mathrm{CF}(H, e; \mathcal{O})$, and for $h \in H_{\pi'}$, we have $\mathrm{dec}_\pi^H(f)(h) = f(h)$).

5.22. The $\mathcal{O}$–module $\mathrm{CF}_{\pi'}(H, e; \mathcal{O})$ is generated by $\mathrm{dec}_\pi^H(\mathrm{Irr}(H, e))$.

Proof of 5.22. Since dec_π^H commutes with the "projection onto e", we may assume $e = 1$. Let h be a π'–element of H, and let $\delta_{\pi,h}^H$ be the characteristic function of the π'–section of h (namely, we have $\delta_{\pi,h}^H(g) = 1$ if the π'–component of g is conjugate to h, and $\delta_{\pi,h}^H(g) = 0$ otherwise). It suffices to prove that $\delta_{\pi,h}^H$ is a linear combination with coefficients in $\mathcal{O}$ of elements of $\mathrm{Irr}(H)$. This is an easy consequence of Brauer's characterization of characters.

This completes the induction and thus the proof of 5.15, (2). □

We now assume that for every proper d–split Levi subgroup $\mathbf{N}$ of $\mathbf{G}$ containing $\mathbf{L}$, the map $\mathbf{I}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}^{\mathbf{N}^F}$ is a π–perfect isometry.

Then by the definition of a perfect isometry (5.11) and by assertion (1) of theorem 5.15, we see that it suffices to prove that $\mathbf{I}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}$ induces an isomorphism between $\mathrm{CF}_{\pi'}(\mathbf{G}^F, e^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}; \mathcal{O})$ and $\mathrm{CF}_{\pi'}(Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda}); \mathcal{O})$.

By 5.22, this follows from the fact (5.15, (1)) that

$$\mathrm{dec}^{\mathbf{G}^F}_{\pi(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathbf{I}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} = \mathbf{I}^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} \cdot \mathrm{dec}^{Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})}_{\pi,(\mathbf{L},\boldsymbol{\lambda})} .$$

5.23. Corollary. *The idempotent $e^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}$ is a π–block of $\mathbf{G}^F$ (i.e., it is a primitive central π–idempotent).*

Proof of 5.23. By 5.15, (2), and by 5.12, it suffices to prove that 1 is a π–block of $Z^o(\mathbf{L})^F_\pi \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})$. By [Rob], thm. 9 (a non trivial generalization to any π of a result which is well known in the case where π consists of a single prime), it suffices to check that $W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})$ acts faithfully on $Z^o(\mathbf{L})^F_\pi$. This follows from the fact that $\mathbf{L} = C_{\mathbf{G}}(Z^o(\mathbf{L})^F_\pi)$. □

C. ℓ–Blocks.

The following hypothesis are in force in what follows.

(H1) Let $\mathbf{G}$ be a connected reductive algebraic group over an algebraic closure of $\mathbb{F}_q$, endowed with a Frobenius endomorphism $F\colon \mathbf{G} \to \mathbf{G}$ defining a rational structure over $\mathbb{F}_q$.

(H2) Let ℓ be a prime number which does not divide q, which divides $|\mathbf{G}^F|$, and which does not divide $|W_{\mathbf{G}^F}\phi|$.

We denote by $\mathbb{G}$ a generic finite reductive group corresponding to $(\mathbf{G}, F)$, and we set $\mathbb{G} = (\Gamma, W\phi)$.

Following [BrMi], we call unipotent ℓ–blocks of $\mathbf{G}^F$ the primitive central ℓ–idempotents which are constituents of the "unipotent idempotent" $e^{\mathbf{G}^F}_\ell$. We denote by $\mathcal{O}$ the extension of the ring of ℓ–adic integers by a root of unity whose order is the l.c.m. of the orders of the elements of $\mathbf{G}^F$. Thus $e^{\mathbf{G}^F}_\ell \in Z\mathcal{O}\mathbf{G}^F$, and the unipotent ℓ–blocks are the primitive idempotents of the algebra $Z\mathcal{O}\mathbf{G}^F e^{\mathbf{G}^F}_\ell$.

The following omnibus theorem is an immediate consequence of the preceding sections (specially of theorem 5.15), as well as a consequence of general results about isotypies (see [Br1]). We make free use of the notation which has been introduced previously.

5.24. Theorem. *Let d be the unique integer such that $\Phi_d(x)$ divides $|\mathbb{G}|$ and ℓ divides $\Phi_d(q)$.*

Let $(\mathbf{L},\boldsymbol{\lambda})$ be a d–cuspidal pair of $\mathbf{G}$.

(1) *The set of all irreducible constituents of the virtual characters $R^{\mathbf{G}}_{\mathbf{L}}(\theta\boldsymbol{\lambda})$, where $\theta \in \mathrm{Ab}_\ell\mathrm{Irr}(\mathbf{L}^F)$, is the set of all irreducible characters of an ℓ–block of $\mathbf{G}^F$, which we denote by $e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\boldsymbol{\lambda})}$.*

(2) *$(Z^o(\mathbf{L})^F_\ell, e^{\mathbf{L}^F}_{\ell,(\mathbf{L},\boldsymbol{\lambda})})$ is a maximal $e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\boldsymbol{\lambda})}$–subpair of $\mathbf{G}^F$. In particular, $Z^o(\mathbf{L})^F_\ell$ is a defect group of $e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\boldsymbol{\lambda})}$.*

(3) *Whenever S is a subgroup of $Z^o(\mathbf{L})^F_\ell$, its centralizer $\mathbf{M}$ in $\mathbf{G}$ is a d–split Levi subgroup, and the Brauer correspondent* $\mathrm{Br}_S(e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\boldsymbol{\lambda})})$ *of* $e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\boldsymbol{\lambda})}$ *is given by the following formula:*

$$\mathrm{Br}_S(e^{\mathbf{G}^F}_{\pi,(\mathbf{L},\boldsymbol{\lambda})}) = \sum_{(\mathbf{L}',\boldsymbol{\lambda}')} e^{\mathbf{M}^F}_{\pi,(\mathbf{L}',\boldsymbol{\lambda}')}$$

where $(\mathbf{L}',\boldsymbol{\lambda}')$ runs over a complete set of representatives for the $\mathbf{M}^F$–conjugacy classes of d–cuspidal pairs of $\mathbf{M}$ which are $\mathbf{G}^F$–conjugate to $(\mathbf{L},\boldsymbol{\lambda})$.

(4) $(\mathbf{G}^F, e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\boldsymbol{\lambda})})$ *and* $(Z^o(\mathbf{L})^F_\ell \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda}), 1)$ *are isotypic, and so in particular (see* [Br1]*, thm. 1.5 and §4.D)*

(a) the block $e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\boldsymbol{\lambda})}$ of $\mathbf{G}^F$ has respectively the same numbers of ordinary and modular characters as the group $Z^o(\mathbf{L})^F_\ell \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})$,

(b) all irreducible characters in $e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\boldsymbol{\lambda})}$ have height zero,

(c) the Cartan matrix of $e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\boldsymbol{\lambda})}$ defines the same integral quadratic form as the Cartan matrix of the group $Z^o(\mathbf{L})^F_\ell \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})$,

(d) the generalized decomposition matrix of $(\mathbf{G}^F, e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\boldsymbol{\lambda})})$ defines the same quadratic form (over $\mathbb{Z}[e^{2i\pi/\ell^a}]$, where a is the valuation in ℓ of $\Phi_d(q)$) as the generalized decomposition matrix of $(Z^o(\mathbf{L})^F_\ell \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda}), 1)$,

(e) the algebras $Z\mathcal{O}\mathbf{G}^F e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\boldsymbol{\lambda})}$ and $Z\mathcal{O}[Z^o(\mathbf{L})^F_\ell \rtimes W_{\mathbf{G}^F}(\mathbf{L},\boldsymbol{\lambda})]$ are isomorphic.

We have

$$e^{\mathbf{G}^F}_\ell = \sum_{[(\mathbf{L},\boldsymbol{\lambda})]_{\mathbf{G}^F}} e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\boldsymbol{\lambda})},$$

where $(\mathbf{L},\boldsymbol{\lambda})$ runs over the set of all d–cuspidal pairs of $\mathbf{G}^F$ (module $\mathbf{G}^F$–conjugation). In other words, $(\mathbf{L},\boldsymbol{\lambda}) \mapsto e^{\mathbf{G}^F}_{\ell,(\mathbf{L},\boldsymbol{\lambda})}$ is a bijection between the set of all unipotent ℓ-blocks of $\mathbf{G}^F$ and the $\mathbf{G}^F$–conjugacy classes of d–cuspidal pairs

Notice that, still assuming ℓ is large (*i.e.*, $\ell \nmid |W\phi|$), the preceding theorem implies that several of the current conjectures in modular representation theory are true for groups of type $\mathbf{G}^F$ and unipotent blocks : Alperin conjecture, Alperin–McKay conjecture, all implied in these cases (blocks with abelian defect groups) by conjecture 6.1 in [Br1].

Notice also that the preceding remark can be strengthened by extending it to non–unipotent blocks of finite reductive groups, using the fact (see [Br1], theorems 2.3 and 5.6) that in many cases the "Jordan decomposition of blocks" is an isotypie.

6. Appendix : Tables for exceptional groups

Table 1. Some d-series for exceptional groups.

$\mathbb{G}$	d	$\mathbb{L}(q)$	$\boldsymbol{\lambda}$	case #	$W_{\mathbb{G}}(\mathbb{L},\boldsymbol{\lambda})$
F_4	2	$(q+1)^2.B_2(q)$	$\phi_{1,1}$	1	$G(2,1,2)$
	4	$(q^2+1).B_2(q)$	$\phi_{11,-}, \phi_{-,2}$	2, 3	Z_4
E_6	2	$(q+1).A_5(q)$	ϕ_{321}	4^*	Z_2
	3	$(q^2+q+1).{}^3D_4(q)$	${}^3D_4[-1]$	5	Z_3
	4	$(q^2+1)(q-1).{}^2A_3(q)$	ϕ_{22}	6^*	Z_4
	5	$(q^5-1).A_1(q)$	ϕ_2, ϕ_{11}	$7^*, 8^*$	Z_5
${}^2E_6$	2	$(q+1)^2.D_4(q)$	$\phi_{13,02}$	9	$G(1,1,3)$
	4	$(q^2+1)(q+1).A_3(q)$	ϕ_{22}	10^*	Z_4
	6	$(q^2-q+1).{}^3D_4(q)$	$\phi_{2,1}$	11	Z_3
	10	$(q^5+1).A_1(q)$	ϕ_2, ϕ_{11}	$12^*, 13^*$	Z_5
E_7	2	$(q+1)^3.D_4(q)$	$\phi_{13,02}$	14	$G(2,1,3)$
	2	$(q+1).{}^2E_6(q)$	${}^2E_6[\theta],\ {}^2E_6[\theta^2]$	15, 16	Z_2
	3	$(q^3-1).{}^3D_4(q)$	${}^3D_4[-1]$	17	Z_6
	3	$(q^2+q+1).A_5(q)$	ϕ_{42}, ϕ_{2211}	$18^*, 19^*$	Z_6
	4	$(q^2+1)^2.A_1(q)^3$	ϕ_2^3, ϕ_{11}^3	$20^*, 21^*$	G_8
			$\phi_2^2\phi_{11}, \phi_2\phi_{11}^2$	$22^*, 23^*$	$G(4,1,2)$
	5	$(q^5-1).A_2(q)$	$\phi_3, \phi_{21}, \phi_{111}$	$24^*, 25^*, 26^*$	Z_{10}
	6	$(q^3+1).{}^3D_4(q)$	$\phi_{2,1}$	27	Z_6
	6	$(q^2-q+1).{}^2A_5(q)$	ϕ_{42}, ϕ_{2211}	$28^*, 29^*$	Z_6
	8	$(q^4+1).A_1(q^2).A_1(q)$	$\phi_2^2, \phi_2\phi_{11}, \phi_{11}\phi_2, \phi_{11}^2$	$30^*, 31^*, 32^*, 33^*$	Z_8
	10	$(q^5+1).{}^2A_2(q)$	$\phi_3, \phi_{21}, \phi_{111}$	$34^*, 35^*, 36^*$	Z_{10}
	12	$(q^4-q^2+1).A_1(q^3)$	ϕ_2, ϕ_{11}	$37^*, 38^*$	Z_{12}
E_8	2	$(q+1)^4.D_4(q)$	$\phi_{13,02}$	39	G_{28}
	2	$(q+1)^2.{}^2E_6(q)$	${}^2E_6[\theta],\ {}^2E_6[\theta^2]$	40, 41	$G(6,6,2)$
	2	$(q+1).E_7(q)$	$\phi_{512,11}, \phi_{512,12}$	42, 43	Z_2
	3	$(q^2+q+1)^2.{}^3D_4(q)$	${}^3D_4[-1]$	44	G_5
	3	$(q^2+q+1).E_6(q)$	$\phi_{81,6}, \phi_{81,10}, \phi_{90,8}$	$45^*, 46^*, 47$	Z_6
	4	$(q^2+1)^2.D_4(q)$	$\phi_{3,1}, \phi_{123,013}, \phi_{23,01}, \phi_{12,03}$	$48^*, 49^*, 50, 51$	G_8
	5	$(q^4+q^3+q^2+q+1).A_4(q)$	ϕ_{32}, ϕ_{221}	$52^*, 53^*$	Z_{10}
	6	$(q^2-q+1)^2.{}^3D_4(q)$	$\phi_{2,1}$	54	G_5
	6	$(q^2-q+1).{}^2E_6(q)$	$\phi'_{9,6}, \phi''_{9,6}, \phi''_{6,6}$	$55^*, 56^*, 57$	Z_6
	7	$(q^7-1).A_1(q)$	ϕ_2, ϕ_{11}	$58^*, 59^*$	Z_{14}
	8	$(q^4+1).{}^2D_4(q)$	$\phi_{13,-}, \phi_{0123,13}$	$60^*, 61^*$	Z_8
			$\phi_{023,1}, \phi_{123,0}, \phi_{013,2}, \phi_{012,3}$	62, 63, 64, 65	Z_8
	9	$(q^6+q^3+1).A_2(q)$	$\phi_3, \phi_{21}, \phi_{111}$	$66^*, 67^*, 68^*$	Z_{18}
	10	$(q^4-q^3+q^2-q+1).{}^2A_4(q)$	ϕ_{32}, ϕ_{221}	$69^*, 70^*$	Z_{10}
	12	$(q^4-q^2+1).{}^3D_4(q)$	$\phi'_{1,3}, \phi''_{1,3}, \phi_{2,2},\ {}^3D_4[1]$	$71^*, 72^*, 73, 74$	Z_{12}
	14	$(q^7+1).A_1(q)$	ϕ_2, ϕ_{11}	$75^*, 76^*$	Z_{14}
	18	$(q^6-q^3+1).{}^2A_2(q)$	$\phi_3, \phi_{21}, \phi_{111}$	$77^*, 78^*, 79^*$	Z_{18}

The notation for the unipotent characters in column 3 and in the following table is as in [Ca], the one for the reflection groups in the last column is standard, see [Be].

Table 2. Decomposition of some $R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda})$.

case	$R_{\mathbb{L}}^{\mathbb{G}}(\boldsymbol{\lambda})$
1	$\phi_{4,1}+\phi_{4,13}-\phi''_{4,7}-\phi'_{4,7}-2B_2,r$
2	$\phi'_{2,4}+\phi'_{2,16}-\phi'_{4,7}+B_2,\epsilon'$
3	$\phi''_{2,4}+\phi''_{2,16}-\phi''_{4,7}+B_2,\epsilon''$
4	$\phi_{64,4}-\phi_{64,13}$
5	$D_4,1+D_4,\epsilon-D_4,r$
6	$\phi_{20,2}-\phi_{60,5}+\phi_{60,11}-\phi_{20,20}$
7	$\phi_{1,0}-\phi_{24,6}+\phi_{81,10}-\phi_{64,13}+\phi_{6,25}$
8	$\phi_{6,1}-\phi_{64,4}+\phi_{81,6}-\phi_{24,12}+\phi_{1,36}$
9	$\phi'_{8,3}+\phi''_{8,9}+2\phi_{16,5}$
10	$\phi_{4,1}+\phi'_{4,7}-\phi''_{4,7}-\phi_{4,13}$
11	$\phi'_{8,3}+\phi''_{8,9}-\phi_{16,5}$
12	$\phi_{1,0}-\phi''_{8,3}+\phi''_{9,6}-{}^2A_5,\epsilon-\phi''_{2,16}$
13	$\phi'_{2,4}-{}^2A_5,1-\phi'_{9,6}+\phi'_{8,9}-\phi_{1,24}$
14	$\phi_{56,3}-\phi_{120,4}-3\phi_{216,9}-2\phi_{336,11}+2\phi_{336,14}+3\phi_{216,16}+\phi_{120,25}-\phi_{56,30}$ $+3\phi_{280,8}-3\phi_{280,17}$
15 + 16	$E_6[\theta],1-E_6[\theta],\epsilon+E_6[\theta^2],1-E_6[\theta^2],\epsilon$
17	$D_4,1+D_4,\epsilon_1-D_4,\sigma_2-D_4,\sigma'_2+D_4,\epsilon_2+D_4,\epsilon$
18	$\phi_{27,2}+\phi_{189,5}-\phi_{378,9}+\phi_{189,20}-\phi_{216,16}+\phi_{189,17}$
19	$\phi_{189,7}-\phi_{378,14}+\phi_{189,22}+\phi_{27,37}-\phi_{216,9}+\phi_{189,10}$
20	$\phi_{1,0}-2\phi_{210,6}-3\phi_{105,6}-3\phi_{189,22}+\phi_{21,36}+2\phi_{56,3}-2D_4,1+3\phi_{405,8}+3\phi_{189,10}$ $-4\phi_{336,11}-4D_4,\sigma_2+2\phi_{120,25}-2D_4,\epsilon_2+\phi_{315,16}+2\phi_{70,18}+\phi_{35,22}$
21	$\phi_{21,3}-3\phi_{189,5}-3\phi_{105,21}-2\phi_{210,21}+\phi_{1,63}+2\phi_{120,4}-2D_4,\epsilon_1-4\phi_{336,14}$ $-4D_4,\sigma'_2+3\phi_{405,15}+3\phi_{189,17}+2\phi_{56,30}-2D_4,\epsilon+\phi_{315,7}+2\phi_{70,9}+\phi_{35,13}$
22	$\phi_{7,1}-\phi_{189,7}+2\phi_{378,9}-\phi_{105,15}+2\phi_{210,13}+\phi_{27,37}-\phi_{15,7}+\phi_{105,5}-2\phi_{216,16}$ $-2D_4,r\epsilon_2-\phi_{35,31}+\phi_{21,33}-2\phi_{280,8}-2D_4,r$
23	$\phi_{27,2}+2\phi_{210,10}-\phi_{105,12}+2\phi_{378,14}-\phi_{189,20}+\phi_{7,46}-\phi_{35,4}+\phi_{21,6}-2\phi_{216,9}$ $-2D_4,r\epsilon_1-\phi_{15,28}+\phi_{105,26}-2\phi_{280,17}-2D_4,r\epsilon$
24	$\phi_{1,0}+\phi_{21,3}-\phi_{189,7}-\phi_{189,22}-\phi_{84,12}+\phi_{336,11}+\phi_{216,16}-\phi_{189,17}+\phi_{56,30}+\phi_{21,33}$
25	$\phi_{7,1}+\phi_{27,2}-\phi_{168,6}-\phi_{378,9}-\phi_{378,14}-\phi_{168,21}+\phi_{27,37}+\phi_{7,46}+\phi_{512,11}+\phi_{512,12}$
26	$-\phi_{189,5}-\phi_{189,20}+\phi_{21,36}+\phi_{1,63}+\phi_{56,3}+\phi_{21,6}+\phi_{216,9}-\phi_{189,10}-\phi_{84,15}+\phi_{336,14}$
27	$\phi_{56,3}-\phi_{120,4}+\phi_{336,11}-\phi_{336,14}+\phi_{120,25}-\phi_{56,30}$
28	$\phi_{27,2}-\phi_{189,5}+\phi_{378,9}+\phi_{189,20}-\phi_{405,15}+D_4,r\epsilon_2$
29	$\phi_{189,7}+\phi_{378,14}-\phi_{189,22}+\phi_{27,37}-\phi_{405,8}+D_4,r\epsilon_1$
30	$\phi_{1,0}-\phi_{105,6}+\phi_{189,22}-\phi_{21,36}+\phi_{216,9}+D_4,r\epsilon_1-\phi_{280,17}-D_4,r\epsilon$
31	$\phi_{7,1}+\phi_{189,7}-\phi_{105,15}-\phi_{27,37}-\phi_{120,4}+D_4,\epsilon_1+\phi_{56,30}-D_4,\epsilon$

32	$\phi_{27,2}+\phi_{105,12}-\phi_{189,20}-\phi_{7,46}-\phi_{56,3}+D_4,1+\phi_{120,25}-D_4,\epsilon_2$
33	$\phi_{21,3}-\phi_{189,5}+\phi_{105,21}-\phi_{1,63}-\phi_{216,16}-D_4,r\epsilon_2+\phi_{280,8}+D_4,r$
34	$\phi_{1,0}-\phi_{21,3}+\phi_{189,7}-\phi_{189,22}-\phi_{420,10}+D_4,\sigma_2+\phi_{405,15}-D_4,r\epsilon_2+\phi_{35,31}+D_4,\epsilon$
35	$-\phi_{7,1}+\phi_{27,2}-\phi_{168,6}+\phi_{378,9}-\phi_{378,14}+\phi_{168,21}-\phi_{27,37}+\phi_{7,46}+E_7[\xi]+E_7[-\xi]$
36	$-\phi_{189,5}+\phi_{189,20}-\phi_{21,36}+\phi_{1,63}+\phi_{35,4}+D_4,1+\phi_{405,8}-D_4,r\epsilon_1-\phi_{420,13}+D_4,\sigma_2'$
37	$\phi_{1,0}+\phi_{210,6}+\phi_{21,36}-\phi_{56,3}+D_4,1-\phi_{336,11}-D_4,\sigma_2-\phi_{120,25}+D_4,\epsilon_2+\phi_{280,18}$ $+E_6[\theta],\epsilon+E_6[\theta^2],\epsilon$
38	$\phi_{21,3}+\phi_{210,21}+\phi_{1,63}-\phi_{120,4}+D_4,\epsilon_1-\phi_{336,14}-D_4,\sigma_2'-\phi_{56,30}+D_4,\epsilon+\phi_{280,9}$ $+E_6[\theta],1+E_6[\theta^2],1$
39	$\phi_{112,3}-2\phi_{160,7}-\phi_{400,7}-9\phi_{1296,13}-4\phi_{2240,10}-2\phi_{3360,13}+9\phi_{2800,13}+8\phi_{3200,16}$ $+8\phi_{3200,22}-2\phi_{3360,25}+9\phi_{2800,25}-4\phi_{2240,28}-9\phi_{1296,33}-\phi_{400,43}-2\phi_{160,55}+\phi_{112,63}$ $+4\phi_{1344,8}-8D_4,\phi_{8,3}'-8D_4,\phi_{8,9}''+4\phi_{1344,38}-4\phi_{1344,19}-6\phi_{2016,19}-6\phi_{5600,19}$ $-12\phi_{448,25}-16D_4,\phi_{16,5}$
40 + 41	$E_6[\theta],\phi_{1,0}-E_6[\theta],\phi_{1,3}'-E_6[\theta],\phi_{1,3}''+E_6[\theta],\phi_{1,6}-2E_8[\theta]-2E_8[-\theta]+$ $E_6[\theta^2],\phi_{1,0}-E_6[\theta^2],\phi_{1,3}'-E_6[\theta^2],\phi_{1,3}''+E_6[\theta^2],\phi_{1,6}-2E_8[\theta^2]-2E_8[-\theta^2]$
42 + 43	$\phi_{4096,11}-\phi_{4096,26}+\phi_{4096,12}-\phi_{4096,27}$
44	$D_4,\phi_{1,0}-D_4,\phi_{2,4}'+D_4,\phi_{1,12}'-2D_4,\phi_{4,7}'-D_4,\phi_{2,4}''+2D_4,\phi_{8,3}''+2D_4,\phi_{8,9}'-D_4,\phi_{2,16}'$ $-2D_4,\phi_{4,7}''+D_4,\phi_{1,12}''-D_4,\phi_{2,16}''+D_4,\phi_{1,24}-2D_4,\phi_{4,1}+2D_4,\phi_{8,3}'+2D_4,\phi_{8,9}''$ $-2D_4,\phi_{4,13}+D_4,\phi_{4,8}-2D_4,\phi_{16,5}-3E_8[-\theta]-3E_8[-\theta^2]+3E_8[-1]$
45	$\phi_{567,6}+\phi_{3240,9}-\phi_{4536,13}-\phi_{2835,22}+\phi_{2268,30}+\phi_{1296,33}$
46	$-\phi_{2835,14}-\phi_{4536,23}+\phi_{3240,31}+\phi_{567,46}+\phi_{2268,10}+\phi_{1296,13}$
47	$\phi_{1008,9}+\phi_{1575,10}+\phi_{1575,35}+\phi_{1008,39}-\phi_{3150,18}-\phi_{2016,19}$
48	$\phi_{8,1}+2\phi_{560,5}+3\phi_{4536,13}+\phi_{4200,21}-3\phi_{3240,31}-3\phi_{1400,11}+3\phi_{840,13}-4\phi_{3200,16}$ $-4D_4,\phi_{8,3}''+2\phi_{2240,28}-2D_4,\phi_{4,7}''-2\phi_{1344,8}-2D_4,\phi_{4,1}+\phi_{1400,37}+2\phi_{1008,39}+\phi_{56,49}$
49	$-3\phi_{3240,9}+\phi_{4200,15}+3\phi_{4536,23}+2\phi_{560,47}+\phi_{8,91}+2\phi_{2240,10}-2D_4,\phi_{4,7}'-4\phi_{3200,22}$ $-4D_4,\phi_{8,9}'-3\phi_{1400,29}+3\phi_{840,31}+\phi_{1400,7}+2\phi_{1008,9}+\phi_{56,19}-2\phi_{1344,38}-2D_4,\phi_{4,13}$
50	$\phi_{28,8}+2\phi_{160,7}+\phi_{300,8}-3\phi_{972,12}+2\phi_{840,14}-3\phi_{700,16}+2\phi_{840,26}-3\phi_{700,28}-3\phi_{972,32}$ $+\phi_{300,44}+2\phi_{160,55}+\phi_{28,68}+4\phi_{1344,19}-2D_4,\phi_{6,6}'-2D_4,\phi_{6,6}''-4D_4,\phi_{12,4}$
51	$\phi_{84,4}-2D_4,\phi_{2,4}'-\phi_{700,6}+3\phi_{2268,10}-2\phi_{4200,12}-3\phi_{2100,16}-2\phi_{4200,24}-3\phi_{2100,28}$ $+3\phi_{2268,30}-\phi_{700,42}-2D_4,\phi_{2,16}''+\phi_{84,64}+2\phi_{2016,19}+2\phi_{5600,19}+4\phi_{448,25}-4D_4,\phi_{4,8}$
52	$\phi_{35,2}+\phi_{560,5}-\phi_{3240,9}+\phi_{2835,22}-\phi_{840,14}+\phi_{3360,13}-\phi_{2240,28}-\phi_{840,31}+\phi_{210,52}+\phi_{160,55}$
53	$\phi_{2835,14}-\phi_{3240,31}+\phi_{560,47}+\phi_{35,74}+\phi_{210,4}+\phi_{160,7}-\phi_{2240,10}-\phi_{840,13}-\phi_{840,26}+\phi_{3360,25}$
54	$\phi_{112,3}+\phi_{160,7}-\phi_{400,7}+2\phi_{2240,10}+\phi_{3360,13}+2\phi_{3200,16}+2\phi_{3200,22}+\phi_{3360,25}$ $+2\phi_{2240,28}-\phi_{400,43}+\phi_{160,55}+\phi_{112,63}-2\phi_{1344,8}-2D_4,\phi_{8,3}'-2D_4,\phi_{8,9}''-2\phi_{1344,38}$ $-3\phi_{7168,17}-\phi_{1344,19}+2D_4,\phi_{16,5}+3E_6[\theta],\phi_{2,2}+3E_6[\theta^2],\phi_{2,2}$
55	$\phi_{567,6}-\phi_{3240,9}+\phi_{4536,13}-\phi_{2835,22}+\phi_{972,32}+D_4,\phi_{9,10}$
56	$-\phi_{2835,14}+\phi_{4536,23}-\phi_{3240,31}+\phi_{567,46}+\phi_{972,12}+D_4,\phi_{9,2}$

57	$\phi_{1008,9}-\phi_{1575,10}-\phi_{1575,34}+\phi_{1008,39}+\phi_{1134,20}+D_4,\phi'_{6,6}$
58	$\phi_{1,0}-\phi_{3240,9}-\phi_{6075,14}+\phi_{8,91}+\phi_{400,7}-\phi_{300,8}+\phi_{4096,11}+\phi_{4096,12}+\phi_{3200,22}$ $-\phi_{2400,23}-\phi_{972,32}+\phi_{1296,33}+\phi_{50,56}-\phi_{160,55}$
59	$\phi_{8,1}-\phi_{6075,22}-\phi_{3240,31}+\phi_{1,120}+\phi_{50,8}-\phi_{160,7}-\phi_{972,12}+\phi_{1296,13}+\phi_{3200,16}$ $-\phi_{2400,17}+\phi_{4096,26}+\phi_{4096,27}+\phi_{400,43}-\phi_{300,44}$
60	$\phi_{8,1}+\phi_{4536,13}-\phi_{4200,21}+\phi_{3240,31}-\phi_{2240,10}+D_4,\phi'_{4,7}-\phi_{1344,38}-D_4,\phi_{4,13}$
61	$-\phi_{3240,9}+\phi_{4200,15}-\phi_{4536,23}-\phi_{8,91}+\phi_{2240,28}-D_4,\phi''_{4,7}+\phi_{1344,8}+D_4,\phi_{4,1}$
62	$\phi_{84,4}-\phi_{400,7}+D_4,\phi_{9,2}+\phi_{700,16}-D_4,\phi''_{9,6}-\phi_{972,32}+\phi_{700,42}-\phi_{112,63}$
63	$D_4,\phi_{1,0}+\phi_{300,8}-\phi_{2268,10}+\phi_{2800,13}-\phi_{2100,28}+\phi_{1296,33}-D_4,\phi''_{1,12}-\phi_{28,68}$
64	$\phi_{112,3}-\phi_{700,6}+\phi_{972,12}+D_4,\phi'_{9,6}-\phi_{700,28}-D_4,\phi_{9,10}+\phi_{400,43}-\phi_{84,64}$
65	$\phi_{28,8}+D_4,\phi'_{1,12}-\phi_{1296,13}+\phi_{2100,16}-\phi_{2800,25}+\phi_{2268,30}-\phi_{300,44}-D_4,\phi_{1,24}$
66	$\phi_{1,0}-\phi_{560,47}-\phi_{50,8}+\phi_{160,7}+\phi_{2800,13}+\phi_{700,16}-\phi_{5600,21}-\phi_{3200,22}+\phi_{4096,26}+\phi_{4096,27}$ $+\phi_{112,63}+\phi_{28,68}-\phi_{1008,9}-E_6[\theta],\phi_{1,0}-E_6[\theta^2],\phi_{1,0}-\phi_{1575,34}-E_6[\theta],\phi''_{1,3}-E_6[\theta^2],\phi''_{1,3}$
67	$\phi_{8,1}+\phi_{35,2}+\phi_{35,74}+\phi_{8,91}-\phi_{700,6}-\phi_{400,7}+\phi_{2240,10}+\phi_{1400,11}+\phi_{2240,28}+\phi_{1400,29}$ $-\phi_{700,42}-\phi_{400,43}-\phi_{3150,18}-\phi_{2016,19}-E_6[\theta],\phi_{2,1}-E_6[\theta^2],\phi_{2,1}-E_6[\theta],\phi_{2,2}-E_6[\theta^2],\phi_{2,2}$
68	$-\phi_{560,5}+\phi_{1,120}+\phi_{112,3}+\phi_{28,8}+\phi_{4096,11}+\phi_{4096,12}-\phi_{5600,15}-\phi_{3200,16}+\phi_{2800,25}+\phi_{700,28}$ $-\phi_{50,56}+\phi_{160,55}-\phi_{1575,10}-E_6[\theta],\phi'_{1,3}-E_6[\theta^2],\phi'_{1,3}-\phi_{1008,39}-E_6[\theta],\phi_{1,6}-E_6[\theta^2],\phi_{1,6}$
69	$\phi_{35,2}-\phi_{560,5}+\phi_{3240,9}+\phi_{2835,22}-\phi_{4200,12}+D_4,\phi''_{2,4}-\phi_{1400,29}-D_4,\phi''_{4,7}+\phi_{50,56}+D_4,\phi''_{2,16}$
70	$\phi_{2835,14}+\phi_{3240,31}-\phi_{560,47}+\phi_{35,74}+\phi_{50,8}+D_4,\phi'_{2,4}-\phi_{1400,11}-D_4,\phi'_{4,7}-\phi_{4200,24}+D_4,\phi'_{2,16}$
71	$\phi_{8,1}-\phi_{560,5}+\phi_{4200,21}-\phi_{3200,16}-D_4,\phi''_{8,3}-\phi_{2240,28}+D_4,\phi''_{4,7}+\phi_{1344,8}+D_4,\phi_{4,1}$ $+\phi_{448,39}+E_6[\theta],\phi_{1,6}+E_6[\theta^2],\phi_{1,6}$
72	$\phi_{4200,15}-\phi_{560,47}+\phi_{8,91}-\phi_{2240,10}+D_4,\phi'_{4,7}-\phi_{3200,22}-D_4,\phi'_{8,9}+\phi_{1344,38}+D_4,\phi_{4,13}$ $+\phi_{448,9}+E_6[\theta],\phi_{1,0}+E_6[\theta^2],\phi_{1,0}$
73	$\phi_{84,4}+D_4,\phi'_{2,4}-\phi_{700,6}+\phi_{4200,12}+\phi_{4200,24}-\phi_{700,42}+D_4,\phi''_{2,16}+\phi_{84,64}-\phi_{7168,17}$ $-D_4,\phi_{4,8}+E_6[\theta],\phi_{2,2}+E_6[\theta^2],\phi_{2,2}$
74	$\phi_{28,8}-\phi_{160,7}+\phi_{300,8}-\phi_{840,14}-\phi_{840,26}+\phi_{300,44}-\phi_{160,55}+\phi_{28,68}+\phi_{1344,19}$ $-E_8[-\theta]-E_8[-\theta^2]+E_8[-1]$
75	$\phi_{1,0}+\phi_{3240,9}-\phi_{6075,14}-\phi_{8,91}-\phi_{700,6}+D_4,\phi'_{1,12}+E_7[\xi],1+E_7[-\xi],1+\phi_{5600,21}$ $-D_4,\phi'_{8,9}-\phi_{2268,30}+D_4,\phi_{9,10}+\phi_{210,52}-D_4,\phi''_{2,16}$
76	$\phi_{8,1}+\phi_{6075,22}-\phi_{3240,31}-\phi_{1,120}-\phi_{210,4}+D_4,\phi'_{2,4}+\phi_{2268,10}-D_4,\phi_{9,2}-\phi_{5600,15}$ $+D_4,\phi''_{8,3}+E_7[\xi],\epsilon+E_7[-\xi],\epsilon+\phi_{700,42}-D_4,\phi''_{1,12}$
77	$\phi_{1,0}+\phi_{560,47}-\phi_{210,4}+D_4,\phi'_{2,4}-\phi_{2100,16}-D_4,\phi'_{9,6}+\phi_{2400,23}+D_4,\phi'_{8,9}-E_7[\xi],\epsilon-E_7[-\xi],\epsilon$ $-\phi_{84,64}-D_4,\phi_{1,24}+\phi_{1008,9}+E_6[\theta],\phi_{1,0}+E_6[\theta^2],\phi_{1,0}-\phi_{1575,34}-E_6[\theta],\phi''_{1,3}-E_6[\theta^2],\phi''_{1,3}$
78	$-\phi_{8,1}+\phi_{35,2}+\phi_{35,74}-\phi_{8,91}-\phi_{300,8}-D_4,\phi'_{1,12}+\phi_{840,13}+D_4,\phi'_{4,7}+\phi_{840,31}+D_4,\phi''_{4,7}$ $-\phi_{300,44}-D_4,\phi''_{1,12}-\phi_{1134,20}-D_4,\phi'_{6,6}-E_8[-\theta]-E_8[-\theta^2]-E_8[\theta]-E_8[\theta^2]$
79	$\phi_{560,5}+\phi_{1,120}-\phi_{84,4}-D_4,\phi_{1,0}+E_7[\xi],1+E_7[-\xi],1+\phi_{2400,17}+D_4,\phi''_{8,3}-\phi_{2100,28}-D_4,\phi''_{9,6}$ $-\phi_{210,52}+D_4,\phi''_{2,16}-\phi_{1575,10}-E_6[\theta],\phi'_{1,3}-E_6[\theta^2],\phi'_{1,3}+\phi_{1008,39}+E_6[\theta],\phi_{1,6}+E_6[\theta^2],\phi_{1,6}$

Table 3. $W_{\mathbb{G}}(\mathbb{T})$ for d regular and $\mathbb{T}$ noncyclic.

$\mathbb{G}$	d	$\lvert\mathbb{T}\rvert$	$W_{\mathbb{G}}(\mathbb{T})$	$\mathbb{G}$	d	$\lvert\mathbb{T}\rvert$	$W_{\mathbb{G}}(\mathbb{T})$
${}^3D_4$	3	Φ_3^2	G_4	${}^2E_6$	3	$\Phi_3^2\Phi_6$	G_5
	6	Φ_6^2	G_4		4	$\Phi_4^2\Phi_2^2$	G_8
${}^2F_4$	4	Φ_4^2	G_{12}		6	Φ_6^3	G_{25}
	$8'$	${\Phi_8'}^2$	G_8	E_7	3	$\Phi_3^3\Phi_1$	G_{26}
	$8''$	${\Phi_8''}^2$	G_8		6	$\Phi_6^3\Phi_2$	G_{26}
F_4	3	Φ_3^2	G_5	E_8	3	Φ_3^4	G_{32}
	4	Φ_4^2	G_8		4	Φ_4^4	G_{31}
	6	Φ_6^2	G_5		5	Φ_5^2	G_{16}
E_6	3	Φ_3^3	G_{25}		6	Φ_6^4	G_{32}
	4	$\Phi_4^2\Phi_1^2$	G_8		8	Φ_8^2	G_9
	6	$\Phi_6^2\Phi_3$	G_5		10	Φ_{10}^2	G_{16}
					12	Φ_{12}^2	G_{10}

The notation for the reflection groups $W_{\mathbb{G}}(\mathbb{T})$ is as in [Be]. Note that the $\mathbb{T}$ are uniquely determined by their orders, being Φ_d-Sylow tori.

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