DEFECT ZERO CHARACTERS PREDICTED BY LOCAL STRUCTURE

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Abstract. Let G be a finite group and let p be a prime. Assume that there exists a prime q dividing $|G|$ which does not divide the order of any p-local subgroup of G. If G is p-solvable or q divides $p-1$, then G has a p-block of defect zero. The case $q = 2$ is a well-known result by Brauer and Fowler.

1. Introduction

Let G be a finite group and let p be a prime. A p-defect zero character of G is an irreducible complex character $\chi \in \mathrm{Irr}(G)$ whose degree has p-part $\chi(1)_p = |G|_p$. For several well-known reasons, p-defect zero characters play an important role in Representation Theory, and they are the subject of key questions in this field by Richard Brauer (as Problem 19 of [1], solved by the third author in [11]) or Walter Feit (Problem VI, Chapter IV of [3]).

This note is a small contribution to Feit's Problem VI, in which he asks about necessary and sufficient conditions for the existence of characters of defect zero. There are too many results in this area, by R. Brauer and K. A. Fowler, N. Ito, G. R. Robinson, Y. Tsushima, T. Wada and others, to be listed here. It is elementary to show, however, that a necessary condition for the existence of a p-defect zero character in a finite group G is that the group has no non-trivial normal p-subgroups. In this note, we consider local subgroups (normalisers of non-trivial p-subgroups), and some special primes to give a sufficient condition.

Theorem A. *Let G be a finite group and let p and q be different primes dividing $|G|$. If G is not p-solvable, assume that $q|(p-1)$. If q does not divide $|\mathbf{N}_G(R)|$ for every p-subgroup $R > 1$ of G, then G has a p-defect zero character.*

We point out that the proof of Theorem A, if G is not p-solvable, uses the Classification of Finite Simple Groups. In the case where $q = 2$, however, Theorem A follows from a classical result by Brauer and Fowler [2, (5F)]. In fact, as in the Brauer–Fowler theorem, we only need to consider subgroups R of G of order p, and therefore our main result can be regarded as a version of Brauer-Fowler for odd primes.

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As a by-product of our proof in Theorem 3.7 we also classify for $p \geq 5$ the almost simple groups G with G/G' cyclic of prime power order p^a having no p-defect zero characters. The analogous classification in the case when $p \leq 3$ seems quite a bit more involved.

2. Main Results

As we have pointed out in the introduction, for primes q dividing $p-1$, the hypothesis that q does not divide $|\mathbf{N}_G(R)|$ for every p-subgroup $R > 1$ of G is equivalent to the condition that q does not divide $|\mathbf{N}_G(R)|$ for every subgroup R of G of order p. This is a consequence of the following.

Lemma 2.1. *Suppose that a non-trivial q-group Q acts by automorphisms on a non-trivial p-group P, where p and q are primes such that q divides $p-1$. Then there exists $x \in P$ of order p such that $|\mathbf{N}_Q(\langle x\rangle)| > 1$.*

Proof. We argue by induction on $|P|$. We may assume that $|Q| = q$. If $1 < N < P$ is Q-invariant, then we are done by induction. So we may assume that P is an irreducible Q-module. Also, we may assume that P is faithful. But in this case, it is well-known that P can be identified with $V = \mathbb{F}_{p^m}$ and that $Q \subseteq \mathbb{F}_{p^m}^\times$ acts by multiplications. (See for instance of [9, Thm. 2.1].) Let $Q = \langle y\rangle$. Hence, if $0 \neq v \in V$ then $v^y = \lambda v$ for some $\lambda \in \mathbb{F}_{p^m}^\times$. Now $\mathbb{F}_{p^m}^\times$ is cyclic, and it has a unique subgroup of order q that lies in $\mathbb{F}_p^\times$, using that q divides $p-1$. We deduce that $\lambda \in \mathbb{F}_p^\times$, and $v^y \in \langle v\rangle$. Thus $|\mathbf{N}_Q(\langle v\rangle)| > 1$. □

Of course, Lemma 2.1, is no longer true without the hypothesis of q dividing $p-1$, as the alternating group $\mathfrak{A}_4$ shows.

Now, assuming the result of Theorem 3.1 on almost simple groups whose proof we defer to the next section, we proceed to prove our Theorem A which we restate for the reader's convenience:

Theorem 2.2. *Let G be a finite group and let p and q be different primes dividing $|G|$. If G is not p-solvable, assume that $q|(p-1)$. If q does not divide $|\mathbf{N}_G(R)|$ for every p-subgroup $R > 1$ of G, then G has a p-defect zero character.*

Proof. We argue by induction on $|G|$. Clearly, $O_p(G) = 1$. Notice that our hypotheses are inherited by the subgroups of G of order divisible by q. In particular, if K is a proper normal subgroup of G of order divisible by q with p'-index, then we conclude that K has an irreducible p-defect zero character $\theta \in \mathrm{Irr}(K)$. Now, if $\chi \in \mathrm{Irr}(G)$ lies over θ, then χ has p-defect zero and G is not a counterexample.

Let M be a minimal normal subgroup of G.

Suppose first that M is a p'-group. If $R > 1$ is a p-subgroup of G, then $\mathbf{N}_{G/M}(RM/M) = \mathbf{N}_G(R)M/M$ is isomorphic to $\mathbf{N}_G(R)/\mathbf{N}_M(R)$, and therefore has order not divisible by q. Thus if q divides $|G/M|$, then by induction, G/M has a p-defect zero character, which is a p-defect zero character of G. Hence, we may assume in this case that G/M is a q'-group. Now, let $x \in M$ be of order q. Then by hypothesis, $|\mathbf{C}_G(x)|$ is p'. Hence G has a p-block of defect zero by [12, Thm. 1].

Hence, we may assume that M has order divisible by p. By the Frattini argument and our hypothesis, we have that G/M has order not divisible by q. So q divides $|M|$. Thus, M is isomorphic to a direct product of copies of a non-abelian simple group S

of order divisible by pq. Since G has no elements of order pq, M is simple, and is the unique minimal normal subgroup of G. Thus G is almost simple. Also, notice now that $O^{p'}(G) = G$, by the first paragraph of this proof. So we conclude that G/G' is a p-group. Now, let $Q \in \mathrm{Syl}_q(M)$ and $P \in \mathrm{Syl}_p(\mathbf{N}_G(Q))$. If P is not cyclic, then it has an elementary abelian p-subgroup A of order p^2, for instance, using [4, Thm. 5.4.10]. Then $Q = \langle \mathbf{C}_Q(x) \mid 1 \neq x \in A \rangle$, by [4, Thm. 5.3.16]. By hypothesis $\mathbf{C}_Q(x) = 1$ for all $x \neq 1 \in A$, and this is a contradiction. Hence G/M has cyclic Sylow p-subgroups. According to Theorem 3.1, G is then not a counterexample. □

Example 2.3. The condition that q divides $p-1$ is necessary, as shown by the Mathieu group M_{22}: None of its proper 2-local subgroups has order divisible by $q = 11$, but still it does not possess characters of 2-defect zero.

Theorem 3.7 gives further examples.

3. Almost Simple Groups

Theorem 3.1. *Let p be a prime and G be a finite almost simple group such that G/G' is cyclic of prime power order p^a. Then one of:*

(1) *G has an irreducible character of p-defect zero, or*
(2) *for every prime r dividing $p-1$ there is a cyclic subgroup $U \cong C_p$ of G with r dividing $|\mathbf{N}_G(U)|$.*

Note that (1) can fail to hold. See Theorem 3.7 below for a precise statement.

We will subdivide the proof of Theorem 3.1 into several steps. First note that the assertion (2) is vacuously satisfied when $p = 2$. So we may assume that $p \geq 3$. Now according to the classification of finite simple groups a finite non-abelian simple group S with an outer automorphism of odd prime order must be of Lie type.

Also note that for $p = 3$ we only need to consider $r = 2$, and in particular the result holds if G has an element of order 6. Finally note that Theorem 3.1 holds when $p^a = 1$, that is, when G is simple: Michler [10] showed this for groups of Lie type, Granville and Ono for alternating groups [5], and for sporadic groups it can be checked very easily on the ordinary character tables that only Suz and Co_3 at $p = 3$ don't have blocks of p-defect zero. But clearly condition (2) then holds.

There are two quite different cases which arise: if p is the defining prime of a group of Lie type, then we show that (2) holds (while (1) will fail in general, see Theorem 3.7). The same happens when G/G' is generated by diagonal automorphisms. In all the remaining cases when p is not the defining prime we argue that there always exists a p-defect zero character. This result may be of independent interest.

We start off with a preliminary reduction that restricts the type of almost simple groups we need to look at.

Lemma 3.2. *Let G be a finite almost simple group with simple socle S such that G/G' is cyclic of prime power order $p^a > 1$ for some odd prime p. Then S is of Lie type and one of the following holds:*

(1) *G/G' is generated by a field automorphism and*
   (a) *either $G' = S$, or*

(b) $S = \mathrm{L}_n(q)$ *or* $\mathrm{U}_n(q)$*, and* G'/S *is generated by a diagonal automorphism of order prime to* p*;*

(2) $S = \mathrm{L}_n(q)$ *or* $\mathrm{U}_n(q)$*,* $p|(n, q \pm 1)$*, and* G/S *is generated by a diagonal automorphism; or*

(3) $p = 3$ *and* S *is of type* D_4*,* E_6 *or* ${}^2E_6$*.*

Proof. As the outer automorphism groups of alternating and sporadic groups are 2-groups, S must be of Lie type. Let $H := \mathrm{Out}(S)$. Then H has a quotient isomorphic to the direct product of the group of field automorphisms times the group of graph automorphisms of S. Moreover, graph automorphisms have order at most 3, and 3 only occurs for type D_4. Hence, if G involves graph automorphisms, then $p = 3$ and we are in case (3).

Now assume that no graph automorphisms are present. The group of diagonal automorphisms is a 2-group unless we are in types A_n, ${}^2A_n$ or E_6, ${}^2E_6$. In the latter case, diagonal automorphisms have order dividing 3, hence again we are in case (3). In the first case, as we have no graph automorphisms in G, G/S is a semidirect product of the cyclic group of diagonal automorphisms with the cyclic group of field automorphisms. If no field automorphisms are present we arrive at case (2), if no diagonal automorphisms are present we get (1)(a), and if both occur, then since G/G' is cyclic of p-power order, the group of diagonal automorphisms must be prime to p, so we arrive at (1)(b). □

Proposition 3.3. *Let* G *be as in Lemma 3.2 with simple socle* S *of Lie type such that* p *is the defining prime for* S*. Then for every prime divisor* r *of* $p-1$ *there is a subgroup* $U \cong C_p$ *of* G *such that* r *divides* $|\mathbf{N}_G(U)|$*.*

In particular Theorem 3.1 holds in this case.

Proof. Let S be simple of Lie type in characteristic p. Let B be a Borel subgroup of S. Then the order of B is divisible by $p-1$, unless either $S \cong \mathrm{L}_2(q)$ with $q = p^f$ odd, in which case $|B|$ is still divisible by $(p-1)/2$, or $S \cong \mathrm{U}_3(q)$ with $q = p^f \equiv 2 \pmod 3$, in which case $|B|$ is divisible by $(p^2-1)/3$.

In the first case, only $r = 2$ may cause problems (if $q \equiv 3 \pmod 4$). But S has a unique class of involutions, so any extension of degree p contains elements of order $rp = 2p$. In the second case, only $r = 3$ may cause problems. But again S has a unique class of elements of order 3, whence any extension of degree p (which is at least 5 in this case) contains elements of order pr. □

We now consider the remaining possibilities according to Lemma 3.2 in the case that p is not the defining prime.

Proposition 3.4. *Let* G *be as in Theorem 3.1 with simple socle* $S = \mathrm{L}_n(q)$ *or* $S = \mathrm{U}_n(q)$*, and assume that* G/S *is generated by a diagonal automorphism of* S *of order* $p^a > 1$*. Then there is a subgroup* $U \cong C_p$ *of* S *with* $|\mathbf{N}_G(U)|$ *divisible by* $p-1$*.*

In particular Theorem 3.1 holds in this case.

Proof. By assumption S has a diagonal outer automorphism of prime order p, so $p|(n, q-1)$ if S is a linear group $\mathrm{L}_n(q)$, and $p|(n, q+1)$ if S is a unitary group $\mathrm{U}_n(q)$. In either case, p divides the order of the Weyl group $\mathfrak{S}_n$ of G. But elements of $\mathfrak{S}_n$ are rational, so there is a subgroup $U \cong C_p$ of G with normaliser order divisible by $p-1$. □

Lemma 3.5. *Theorem 3.1 holds when* $p = 3$ *and* G *is of type* D_4*,* E_6 *or* ${}^2E_6$*.*

Proof. This is immediate since here necessarily $r = 2$ and any simple group of the listed types contains elements of order $6 = pr$. □

Theorem 3.6. *Let G be as in Theorem 3.1 with simple socle S of Lie type, $p > 2$ not the defining prime for S, and G/G' generated by field automorphisms. Then G has an irreducible character of p-defect zero. In particular Theorem 3.1 holds in this case.*

Proof. Note G has a character of p-defect zero if S has a character χ of p-defect zero with inertia group $I_G(\chi)$ such that $|I_G(\chi) : S|$ is prime to p. So, according to the description of the structure of G/S in Lemma 3.2 our task is to find p-defect zero characters of S with trivial inertia group under field automorphisms. As already for the case that $G = S$ is simple in [10], the idea how to produce such characters is to use irreducible Deligne–Lusztig characters with respect to suitable maximal tori.

We work in the following setting. Let $\mathbf{H}$ be a simple algebraic group in characteristic not p of simply connected type with a Steinberg endomorphism $F : \mathbf{H} \to \mathbf{H}$ such that $H = \mathbf{H}^F$ is quasi-simple with $S = H/Z(H)$. Assume that γ is a field automorphism of S of order a positive power of p. Thus in particular the underlying prime power q of H is a pth power and our group S is not a group over the prime field (and hence in particular $S \neq {}^2F_4(2)'$).

Now let $\mathbf{T}_1, \mathbf{T}_2$ be F-stable maximal tori of the dual group $\mathbf{H}^*$ such that $T_i = \mathbf{T}_i^F$ are as in Tables 1 and 2 of [7]. It is argued in [7, Prop. 2.4 and 2.5] that both maximal tori T_i do contain regular elements of $\mathbf{H}^{*F}$ in most cases, in fact always if H is not over the prime field and not of types $A_1, A_2, {}^2A_2$ or B_2, and there even exist elements $s_i \in T_i$ whose dth power is regular, where $d = |Z(H)| = \gcd(|T_1|, |T_2|)$. In particular, we can take for s_i an element of maximal order in T_i, since if s_i is regular then so is any root of it. But then by order reasons s_i^d cannot lie in a subfield subgroup of $\mathbf{H}^{*F}$ (which is defined over a field of cardinality at most $\sqrt[3]{q}$). Hence s_i^d is not fixed by any field automorphism of G of order p. But then the corresponding Deligne–Lusztig character $R_{T_i}(s_i^d)$ is irreducible (up to sign), of p-defect zero, and not invariant under field automorphisms of order p and we are done.

For the four excluded series $A_1, A_2, {}^2A_2$ or B_2 the assertion is easily checked directly, again along the lines of the argument given in [7, Prop. 2.5]. □

Observe that the proof of Theorem 3.1 is now complete. We close by classifying those almost simple groups G as in Theorem 3.1 violating conclusion (1), so having no p-defect zero character for $p \geq 5$.

Theorem 3.7. *Let G be as in Theorem 3.1 with simple socle S. Assume that $p \geq 5$. Then G has no p-block of defect zero if and only if $G \neq S$ and one of the following holds:*

(1) *S is of Lie type in characteristic p, or*
(2) *$S = \mathrm{L}_n(q) \leq G \leq \mathrm{PGL}_n(q)$ or $S = \mathrm{U}_n(q) \leq G \leq \mathrm{PGU}_n(q)$ with p dividing $|G : S|$.*

Proof. The "only if" part is a consequence of Lemma 3.2 in conjunction with Theorem 3.6 and the theorems of Michler and Granville–Ono (when $G = S$). As for the "if" part, note that by [8, Thm. 1.1], S in characteristic p has a unique p-defect zero character, viz. the Steinberg character, which is hence invariant under all outer automorphisms, so G cannot have defect zero characters when $G > S$.

Now assume that $S \leq G \leq \mathrm{PGL}_n(q)$ with p dividing $|G:S|$. There is a simple algebraic group $\mathbf{H}$ of type A_{n-1} with a Frobenius map F such that $G \cong H/Z(H)$, where $H = \mathbf{H}^F$. Now assume that $\chi \in \mathrm{Irr}(G)$ has p-defect zero. Then we may consider χ as an irreducible character of H of central defect. In particular, the degree polynomial of χ is divisible by $(q-1)^n$ (as p divides $q-1$). By Lusztig's Jordan decomposition this means that χ lies in a Lusztig series of a semisimple element $s \in \mathbf{H}^{*F}$ such that the Φ_1-torus of $C_{\mathbf{H}^*}(s)$ lies in $Z(\mathbf{H}^*)$. But this implies that $C_{\mathbf{H}^*}(s)$ is a Coxeter torus $\mathbf{T}$ of $\mathbf{H}^*$, and thus χ is (up to sign) an irreducible Deligne–Lusztig character of H, of degree $|\mathbf{H}^{*F} : \mathbf{T}^F|$. This cannot be of central defect as the p-part $|\mathbf{T}^F|_p = (q^n-1)_p$ is larger than $|Z(H)|_p$ (because $p|(q^n-1)/(q-1)$ as p divides $|G:S|$ which in turn divides n).

The argument for the unitary case is completely analogous. □

Remark 3.8. We do not see how to extend Theorem 3.7 to arbitrary almost simple groups; in any case the list of exceptions in (2) would probably become much more complicated (consider for example $\mathrm{L}_5(11)$ extended by some arbitrary subgroup of its outer automorphism group of order $2 \cdot 5^2$). But again the only examples without p-blocks of defect zero would be as in (1) or with $S = \mathrm{L}_n(q)$ or $S = \mathrm{U}_n(q)$.

For $p = 3$ there are further examples without characters of 3-defect zero when $S = \mathrm{O}_8^+(q)$, $E_6(q)$ or ${}^2E_6(q)$. Even more examples exist when $p = 2$.

Example 3.9. It is well known that the symmetric group $\mathfrak{S}_n$ has a 2-block of defect zero if and only if n is a triangular number. Now if $q \leq n$ is an odd prime, the only time that all 2-locals of $\mathfrak{S}_n$ are of q'-order is when $n = q$, or $n = q+1$ if q is not a Mersenne prime.

But the only odd prime which is a triangular number is $q = 3$ and the only time that $q+1$ is a triangular number $m(m+1)/2$ is when $2q = (m+2)(m-1)$, which only happens for $m = 3$, so $q = 5$. Hence for any prime $q > 5$, the symmetric group $\mathfrak{S}_q$ (and $\mathfrak{S}_{q+1}$ if q is not a Mersenne prime) has no 2-block of defect zero, and no 2-local subgroup of order divisible by q. This gives infinitely many counterexamples to the conclusion of Theorem A if we drop the condition that $q|(p-1)$.

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