WEIGHTS FOR COMPACT CONNECTED LIE GROUPS

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Abstract. Let ℓ be a prime. If $\mathbf{G}$ is a compact connected Lie group, or a connected reductive algebraic group in characteristic different from ℓ, and ℓ is a good prime for $\mathbf{G}$, we show that the number of weights of the ℓ-fusion system of $\mathbf{G}$ is equal to the number of irreducible characters of its Weyl group. The proof relies on the classification of ℓ-stubborn subgroups in compact Lie groups.

1. Introduction

In this paper we prove the following identity, first conjectured in [15]:

Theorem 1. *Suppose that $\mathbf{G}$ is a compact connected Lie group. Let ℓ be a prime which is good for $\mathbf{G}$, S a Sylow ℓ-subgroup of $\mathbf{G}$, $\mathcal{F} = \mathcal{F}_S(\mathbf{G})$ the corresponding saturated fusion system and W the Weyl group of $\mathbf{G}$. Then,*

$$\mathbf{w}(\mathcal{F}) = |\mathrm{Irr}(W)|.$$

Here $\mathbf{w}(\mathcal{F})$ denotes the number of weights of the ℓ-fusion system attached to $\mathbf{G}$ (see Section 2 for precise definitions) and $\mathrm{Irr}(W)$ denotes the set of ordinary irreducible characters of W. We recall the definition of good primes later in this introduction. While the stated identity is rather easy to see for primes not dividing $|W|$, it seems totally mysterious for the other primes.

The origin of our interest in this theorem is Alperin's Weight Conjecture (AWC) in finite group representation theory which relates the number of simple modules of a block of a finite group algebra over a field of characteristic ℓ to the number of simple, projective modules of smaller "local" groups. Indeed, via some deep representation theory, it can be shown that for the principal block of the group algebra of a finite group of Lie type whose defining characteristic is different from ℓ, and under some extra conditions, AWC is equivalent to an equation of the above type (see [9, Prop. 4.1]). In [9, Thm 1], we proved this equality for finite groups of Lie type (under the relevant conditions) and made the surprising discovery that it continues to hold in a broader setting than that of finite groups, in particular for certain exotic fusion systems on finite ℓ-groups arising from homotopy fixed point spaces of connected ℓ-compact groups. We view Theorem 1, where the underlying ℓ-group is an infinite group, as additional evidence for AWC and a further indication that it is not restricted to the world of finite groups. We also note

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that Theorem 1 yields the analogous identity for ℓ-weights of fusion systems of connected reductive algebraic groups in characteristic different from ℓ:

Theorem 2. *Let ℓ and p be distinct primes. Suppose that $\mathbf{G}$ is a connected reductive algebraic group over $\overline{\mathbb{F}}_p$ such that ℓ is good for $\mathbf{G}$, let S be a Sylow ℓ-subgroup of $\mathbf{G}$, $\mathcal{F} = \mathcal{F}_S(\mathbf{G})$ the corresponding saturated fusion system and W the Weyl group of $\mathbf{G}$. Then,*

$$\mathbf{w}(\mathcal{F}) = |\mathrm{Irr}(W)|.$$

The proof of Theorem 1 proceeds by transferring from the discrete setting to the continuous one in which the role of centric, radical subgroups in fusion systems is taken over by ℓ-stubborn subgroups (in the sense of Jackowski–McClure–Oliver [7]) of compact Lie groups. This leads to a straightforward reduction to the case that the underlying compact group is simple, at which stage we are able to invoke known classifications of ℓ-stubborn subgroups [13], [8], [18]. The proof for the classical groups involves counting arguments similar to those used for [9, Thm 1] which originated in the work of Alperin–Fong [1].

Recall that a prime ℓ is *bad* for a root system Φ if $\mathbb{Z}\Phi/\mathbb{Z}\Psi$ has ℓ-torsion for some closed subsystem $\Psi \subset \Phi$ (see e.g. [12, Def. B.24]). Thus, it is bad for Φ if it is so for one of its indecomposable summands. The bad primes for indecomposable root systems are: none for type A_n, $\ell = 2$ for B_n, C_n and D_n, $\ell = 2, 3$ for G_2, F_4, E_6, E_7, and $\ell = 2, 3, 5$ for E_8 (see [12, Tab. 14.1]). In particular, all primes $\ell \geq 7$ are good (that is, not bad) for all types. In Proposition 4.1 we show that the conclusion of Theorem 1 does not hold for the compact symplectic groups at the bad prime $\ell = 2$, and in Propositions 4.5 and 4.6 that it does not hold for $\mathbf{G} = F_4$ at the bad prime $\ell = 3$, respectively for $\mathbf{G} = E_8$ at the bad prime $\ell = 5$. Still in both cases we obtain $\mathbf{w}(\mathcal{F}) \leq |\mathrm{Irr}(W)|$, as conjectured in [15]. In Example 4.3 we observe that for finite reductive groups in bad characteristic we may have $\mathbf{w}(\mathcal{F}) > |\mathrm{Irr}(W)|$.

Recall that by results of Friedlander (see [4, Thm 3.1] and Section 2.3), the ℓ-completed classifying space (and therefore the ℓ-fusion system) of a finite group of Lie type in non-describing characteristic can be recovered by taking homotopy fixed points under suitable unstable Adams operations on the ℓ-completed classifying space of the corresponding compact Lie group. It would be desirable to obtain a direct connection between the weight equation demonstrated in Theorems 1 and 2 and that shown in [9, Thm 1], but as yet we do not see such a connection either via unstable Adams operations or via Frobenius morphisms on algebraic groups. In particular, we do not see how to deduce the equation in the finite setting from the infinite one or vice-versa. Note that our results show that the number of weights for compact groups does not depend on the isogeny type, while for example the two isogenous groups $\mathrm{SL}_3(4)$ and $\mathrm{PGL}_3(4)$ have five respectively three 3-weights. In another direction, we believe that the purview of Theorem 1 could be expanded to also cover fusion systems of ℓ-compact groups as defined in [5, Sec. 10].

The paper is organised as follows. In Section 2, we recall the relevant background material. Section 3 contains the proofs of Theorems 1 and 2 and in Section 4, we present our calculations for bad primes.

2. Preliminaries

In this section we recall some aspects of the theory of saturated fusion systems associated to compact Lie groups as set up by Broto, Levi and Oliver in [5]. For subgroups Q, R of a group G, $\mathrm{Hom}_G(Q,R)$ consists of the group homomorphisms from Q to R induced by conjugation by elements of G, $\mathrm{Aut}_G(Q) := \mathrm{Hom}_G(Q,R) \cong N_G(Q)/C_G(Q)$ is the group of G-automorphisms of Q, and $\mathrm{Out}_G(Q) = \mathrm{Aut}_G(Q)/\mathrm{Inn}(Q) \cong N_G(Q)/QC_G(Q)$ is the corresponding group of outer automorphisms. For $H \leq G$ denote by $\mathcal{F}_H(G)$ the category with objects the subgroups of H, in which the $\mathcal{F}$-morphisms from Q to R are the elements of $\mathrm{Hom}_G(Q,R)$ for $Q, R \leq H$, and composition of morphisms is the usual composition of maps. Let ℓ be a prime number.

2.1. **Fusion systems on discrete ℓ-toral groups.** A *discrete ℓ-toral group* is a group P with normal subgroup P° such that P° is isomorphic to a finite product of copies of $\mathbb{Z}/\ell^\infty$ and P/P° is a finite ℓ-group. In particular, any finite ℓ-group is a discrete ℓ-toral group. Here P° is characterised as the subset of infinitely divisible elements of P as well as the minimal subgroup of finite index of P. In particular, P° is characteristic in P. It is called the *identity component of* P and P is *connected* if $P = P^\circ$.

Let S be a discrete ℓ-toral group and let $\mathcal{F}$ be a saturated fusion system on S as defined in [5, Sec. 2]. Following [15], the the number of weights of $\mathcal{F}$ is defined by

$$\mathbf{w}(\mathcal{F}) := \sum_{Q} z(\mathrm{Out}_{\mathcal{F}}(Q)),$$

where Q runs over a set of representatives of $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups of S, $\mathrm{Out}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Inn}(Q)$ is the group of $\mathcal{F}$-outer automorphisms of Q, and $z(\mathrm{Out}_{\mathcal{F}}(Q))$ is the number of ordinary irreducible characters of $\mathrm{Out}_{\mathcal{F}}(Q)$ characters of zero ℓ-defect. Note that by [5, Cor. 3.5, Defn. 2.2], S has only finitely many classes of $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups and moreover, the $\mathcal{F}$-outer automorphism group of any $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroup of S is finite. Thus, $\mathbf{w}(\mathcal{F})$ is well defined. Also, note that if G is a finite group, S is a Sylow ℓ-subgroup of G and $\mathcal{F} = \mathcal{F}_S(G)$ is the corresponding saturated fusion system, then $\mathbf{w}(\mathcal{F})$ is the number of Alperin weights associated to the principal block of kG, k an algebraically closed field of characteristic ℓ (see [10, Thm 8.14.4]).

Following [5, Sec. 8], we say a group G "has Sylow ℓ-subgroups" if there is a discrete ℓ-toral subgroup $S \leq G$ which contains all discrete ℓ-toral subgroups of G up to conjugacy. Such a subgroup, if it exists, is called a *Sylow ℓ-subgroup of* G. Note that the set of Sylow ℓ-subgroups of G is a single G-conjugacy class. An *ℓ-centric subgroup* of G is a discrete ℓ-toral subgroup $P \leq G$ such that $C_G(P)$ has a Sylow ℓ-subgroup, which is $Z(P)$ (and unique). Equivalently, a discrete ℓ-toral $P \leq G$ is ℓ-centric in G if $C_G(P)/Z(P)$ contains no elements of order ℓ.

2.2. **Fusion systems of compact Lie groups.** Let $\mathbf{G}$ be a compact Lie group. The following definitions and results are taken from [5, Sec. 9].

- An *ℓ-toral group* is a compact Lie group whose identity component is a torus and whose group of components is an ℓ-group. If P is a discrete ℓ-toral subgroup of $\mathbf{G}$, then the closure $\bar{P}$ of P in $\mathbf{G}$ is an ℓ-toral group.

- For any ℓ-toral group $\mathbf{P}$, $\mathrm{Syl}_\ell(\mathbf{P})$ denotes the set of all discrete ℓ-toral subgroups P of $\mathbf{P}$ such that $\mathbf{P}^\circ.P = \mathbf{P}$, where $\mathbf{P}^\circ$ is the identity component of $\mathbf{P}$ and P contains all ℓ-power torsion of $\mathbf{P}$. The elements of $\mathrm{Syl}_\ell(\mathbf{P})$ form a single $\mathbf{P}$-conjugacy class.
- We denote by $\overline{\mathrm{Syl}}_\ell(\mathbf{G})$ the set of all ℓ-toral subgroups $\mathbf{S}$ of $\mathbf{G}$ such that the identity component $\mathbf{S}^\circ$ is a maximal torus of $\mathbf{G}$ and $\mathbf{S}/\mathbf{S}^\circ$ is a Sylow ℓ-subgroup of $N_{\mathbf{G}}(\mathbf{S}^\circ)/\mathbf{S}^\circ$; $\mathrm{Syl}_\ell(\mathbf{G})$ is the set of all discrete ℓ-toral subgroups P of $\mathbf{G}$ such that $\bar{P} \in \overline{\mathrm{Syl}}_\ell(\mathbf{G})$ and $P \in \mathrm{Syl}_\ell(\bar{P})$. Any two elements of $\overline{\mathrm{Syl}}_\ell(\mathbf{G})$ are $\mathbf{G}$-conjugate and each ℓ-toral subgroup of $\mathbf{G}$ is contained in an element of $\overline{\mathrm{Syl}}_\ell(\mathbf{G})$. Any two elements of $\mathrm{Syl}_\ell(\mathbf{G})$ are $\mathbf{G}$-conjugate and each discrete ℓ-toral subgroup of $\mathbf{G}$ is contained in an element of $\mathrm{Syl}_\ell(\mathbf{G})$ (see [5, Prop. 9.3]). In particular, the elements of $\mathrm{Syl}_\ell(\mathbf{G})$ are Sylow ℓ-subgroups of $\mathbf{G}$.

If $S \le \mathbf{G}$ is a Sylow ℓ-subgroup then by [5, Lemma 9.5], $\mathcal{F} := \mathcal{F}_S(\mathbf{G})$ is a saturated fusion system on S.

- An *ℓ-stubborn subgroup* of $\mathbf{G}$ is an ℓ-toral subgroup $\mathbf{P}$ of $\mathbf{G}$ such that $N_{\mathbf{G}}(\mathbf{P})/\mathbf{P}$ is a finite group which satisfies $O_\ell(N_{\mathbf{G}}(\mathbf{P})/\mathbf{P}) = 1$.
- An ℓ-toral subgroup $\mathbf{P} \le \mathbf{G}$ is called *ℓ-centric* if $Z(\mathbf{P}) \in \overline{\mathrm{Syl}}_\ell(C_{\mathbf{G}}(\mathbf{P}))$.
- A discrete ℓ-toral subgroup P of a compact Lie group $\mathbf{G}$ is said to be *snugly embedded* in $\mathbf{G}$ if $P \in \mathrm{Syl}_\ell(\bar{P})$.

The following well-known result is the first step in the proof of Theorem 1 (see also [3, Sec. 4]).

Lemma 2.1. *Let $\mathbf{G}$ be a compact Lie group with Sylow ℓ-subgroup S and let $\mathcal{F} = \mathcal{F}_S(\mathbf{G})$.*

(a) *For any ℓ-stubborn subgroup $\mathbf{P}$ of $\mathbf{G}$, $C_{\mathbf{G}^\circ}(\mathbf{P}) \le Z(\mathbf{P})$ and if $\mathbf{G}/\mathbf{G}^\circ$ is an ℓ-group, then $C_{\mathbf{G}}(\mathbf{P}) \le Z(\mathbf{P})$, and consequently $\mathrm{Out}_{\mathbf{G}}(\mathbf{P}) \cong N_{\mathbf{G}}(\mathbf{P})/\mathbf{P}$.*
(b) *Suppose that $P \le S$ is $\mathcal{F}$-centric and $\mathcal{F}$-radical. Then $\mathbf{P} := \bar{P}$ is an ℓ-stubborn subgroup of $\mathbf{G}$ and $\mathrm{Out}_{\mathcal{F}}(P) \cong \mathrm{Out}_{\mathbf{G}}(\mathbf{P})$.*
(c) *The map $P \mapsto \bar{P}$ induces an injective map from the set of $\mathcal{F}$-classes of $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups of S to the set of $\mathbf{G}$-classes of ℓ-stubborn subgroups of $\mathbf{G}$.*
(d) *If $\mathbf{G}/\mathbf{G}^\circ$ is an ℓ-group or an ℓ'-group, then the map in (c) is a bijection.*
(e) *Suppose that $\mathbf{G}/\mathbf{G}^\circ$ is an ℓ-group. Let $\bar{\mathcal{F}}$ be the orbit category of $\mathcal{F}$ and $\bar{\mathcal{F}}^{cr}$ be the full subcategory whose objects are $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups of S. Let $\mathcal{R}_\ell(\mathbf{G})$ be the full subcategory of the orbit category of $\mathbf{G}$ whose objects are the ℓ-toral subgroups of $\mathbf{G}$. There is an equivalence of categories $\bar{\mathcal{F}}^{cr} \simeq \mathcal{R}_\ell(\mathbf{G})$ which for $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups P, Q of S sends P to $\mathbf{G}/\bar{P}$ and which sends the $\bar{\mathcal{F}}^{cr}$-morphism from P to Q induced by conjugation by $g \in \mathbf{G}$ to the $\mathcal{R}_\ell(\mathbf{G})$-morphism from $\mathbf{G}/\bar{P}$ to $\mathbf{G}/\bar{Q}$ defined by $x\bar{P} \to xg^{-1}\bar{Q}$, $x \in \mathbf{G}$.*

Proof. Part (a) is Lemma 7 of [13]. Let $P \le S$ be $\mathcal{F}$-centric and radical. By [5, Lemma 3.2 and Cor. 3.5], $P = P^\bullet$, where $P^\bullet$ is as defined in Section 3 of [5] and hence by [5, Lemma 9.9], P is snugly embedded in $\mathbf{G}$. By Lemma 9.4 of [5] and its proof we have $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Out}_{\mathbf{G}}(P) \cong \mathrm{Out}_{\mathbf{G}}(\mathbf{P})$. Since P is $\mathcal{F}$-centric, by [5, Lemma 8.4], P is ℓ-centric in $\mathbf{G}$ and hence by [5, Lemma 9.6(a),(b)], $\mathbf{P}$ is ℓ-centric in $\mathbf{G}$, $N_{\mathbf{G}}(\mathbf{P})/\mathbf{P}$ is finite and $\mathbf{P}C_{\mathbf{G}}(\mathbf{P})/\mathbf{P} \cong C_{\mathbf{G}}(\mathbf{P})/Z(\mathbf{P})$ is finite of order prime to ℓ. On the other hand, since $\mathrm{Out}_{\mathcal{F}}(P) \cong \mathrm{Out}_{\mathbf{G}}(\mathbf{P}) \cong N_{\mathbf{G}}(\mathbf{P})/\mathbf{P}C_{\mathbf{G}}(\mathbf{P})$ and P is $\mathcal{F}$-radical, $O_\ell(N_{\mathbf{G}}(\mathbf{P})/\mathbf{P}C_{\mathbf{G}}(\mathbf{P})) = 1$, hence $O_\ell(N_{\mathbf{G}}(\mathbf{P})/\mathbf{P}) \leq \mathbf{P}C_{\mathbf{G}}(\mathbf{P})/\mathbf{P}$. Since the latter is an ℓ'-group, $O_\ell(N_{\mathbf{G}}(\mathbf{P})/\mathbf{P}) = 1$, showing that $\mathbf{P}$ is ℓ-stubborn. This proves (b).

The assignment $P \mapsto \mathbf{P} := \bar{P}$ induces a map from the set of $\mathbf{G}$-classes of discrete ℓ-toral subgroups of $\mathbf{G}$ which are snugly embedded in $\mathbf{G}$ to the set of $\mathbf{G}$-classes of ℓ-toral subgroups of $\mathbf{G}$. By [5, Prop. 9.3], this map is a bijection with the inverse being the map which sends the $\mathbf{G}$-class of an ℓ-toral subgroup $\mathbf{P}$ of $\mathbf{G}$ to the $\mathbf{G}$-class of P, where $P \in \mathrm{Syl}_\ell(\mathbf{P})$. Now (c) follows from (b) and the fact that by the definition of $\mathcal{F}$, $P, P' \leq S$ are $\mathcal{F}$-conjugate if and only if P and P' are $\mathbf{G}$-conjugate.

Now suppose that $P \leq \mathbf{G}$ is a snugly embedded subgroup such that $\mathbf{P} := \bar{P} \leq \mathbf{G}$ is ℓ-stubborn. From the above, we see that in order to prove (d) it suffices to show that P is $\mathcal{F}$-centric and $\mathcal{F}$-radical provided that $\mathbf{G}/\mathbf{G}^\circ$ is an ℓ-group or an ℓ'-group. For this, note that as above since P is snugly embedded in $\mathbf{G}$, $\mathrm{Out}_{\mathbf{G}}(P) \cong \mathrm{Out}_{\mathbf{G}}(\mathbf{P})$ by [5, Lemma 9.4]. Suppose first that $\mathbf{G}/\mathbf{G}^\circ$ is an ℓ-group. By (a), $C_{\mathbf{G}}(\mathbf{P}) = Z(\mathbf{P})$, hence $\mathrm{Out}_{\mathbf{G}}(P) = \mathrm{Out}_{\mathcal{F}}(\mathbf{P}) \cong N_{\mathbf{G}}(\mathbf{P})/\mathbf{P}$ and by hypothesis $O_\ell(N_{\mathbf{G}}(\mathbf{P})/\mathbf{P}) = 1$. This shows that P is $\mathcal{F}$-radical. Also, since $C_{\mathbf{G}}(\mathbf{P}) \leq Z(\mathbf{P})$, $\mathbf{P}$ is ℓ-centric in $\mathbf{G}$. Hence by [5, Lemma 9.6], P is ℓ-centric in $\mathbf{G}$ and consequently by [5, Lemma 8.4], P is $\mathcal{F}$-centric. Now suppose that $\mathbf{G}/\mathbf{G}^\circ$ is an ℓ'-group. We claim that $\mathbf{P} \leq \mathbf{G}^\circ$. Indeed, we have $\mathbf{P} = \mathbf{P}^\circ.P$ and $\mathbf{P}^\circ \leq \mathbf{G}^\circ$. On the other hand, every element of P has finite order a power of ℓ and $\mathbf{G}/\mathbf{G}^\circ$ is an ℓ'-group, hence $P \leq \mathbf{G}^\circ$. This proves the claim. Since $\mathbf{G}^\circ$ is normal in $\mathbf{G}$, $\mathbf{P}$ is ℓ-stubborn in $\mathbf{G}^\circ$, hence again by [13, Lemma 7], $C_{\mathbf{G}^\circ}(\mathbf{P}) = Z(\mathbf{P})$ and hence $C_{\mathbf{G}}(\mathbf{P})/Z(\mathbf{P}) = C_{\mathbf{G}}(\mathbf{P})/C_{\mathbf{G}^\circ}(\mathbf{P}) \leq \mathbf{G}/\mathbf{G}^\circ$ is a finite ℓ'-group. This shows that $Z(\mathbf{P})$ contains all torsion elements of ℓ-power order of $C_{\mathbf{G}}(\mathbf{P})$ and hence that $\mathbf{P}$ is ℓ-centric in $\mathbf{G}$. Now it follows as in the previous case that P is $\mathcal{F}$-centric. This completes the proof of (d).

Now suppose that $P, Q \leq S$ are $\mathcal{F}$-centric and $\mathcal{F}$-radical. As above, P and Q are snugly embedded in $\mathbf{G}$. By [5, Lemma 9.4(c)], there is a bijection

$$\mathrm{Inn}(Q)\backslash\mathrm{Hom}_{\mathbf{G}}(P, Q) \longrightarrow \mathrm{Inn}(\mathbf{Q})\backslash\mathrm{Hom}_{\mathbf{G}}(\mathbf{P}, \mathbf{Q})$$

induced by the canonical map from $\mathrm{Hom}_{\mathbf{G}}(P, Q)$ to $\mathrm{Hom}_{\mathbf{G}}(\mathbf{P}, \mathbf{Q})$. On the one hand, $\mathrm{Hom}_{\bar{\mathcal{F}}^{cr}}(P, Q)$ may be identified with $\mathrm{Inn}(Q)\backslash\mathrm{Hom}_{\mathbf{G}}(P, Q)$. On the other hand, by (a), $C_{\mathbf{G}}(\bar{P}) \leq Z(\bar{P})$ from which it follows that $\mathrm{Inn}(\mathbf{Q})\backslash\mathrm{Hom}_{\mathbf{G}}(\mathbf{P}, \mathbf{Q})$ may be identified with $\mathrm{Mor}_{\mathcal{R}_\ell(\mathbf{G})}(\mathbf{G}/\mathbf{P}, \mathbf{G}/\mathbf{Q})$. Now (e) follows by (d). □

As an immediate consequence of the previous lemma we get:

Lemma 2.2. *Suppose that* $\mathbf{G}$ *is a compact connected Lie group,* ℓ *a prime,* $S \in \mathrm{Syl}_\ell(\mathbf{G})$ *a Sylow* ℓ*-subgroup of* $\mathbf{G}$*,* $\mathcal{F} = \mathcal{F}_S(\mathbf{G})$*. Then*

$$\mathbf{w}(\mathcal{F}) = \sum_{\mathbf{P}} z(N_{\mathbf{G}}(\mathbf{P})/\mathbf{P})$$

where $\mathbf{P}$ *runs over a set of representatives of* $\mathbf{G}$*-classes of* ℓ*-stubborn subgroups of* $\mathbf{G}$ *and* $z(N_{\mathbf{G}}(\mathbf{P})/\mathbf{P})$ *is the number of irreducible characters of* $N_{\mathbf{G}}(\mathbf{P})/\mathbf{P}$ *of zero* ℓ*-defect.*

2.3. **Fusion systems of connected reductive algebraic groups.** Let p, ℓ be distinct prime numbers and let $\mathbf{G}$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_p$. Then since $\mathbf{G}$ has a finite dimensional faithful representation over $\overline{\mathbb{F}}_p$, $\mathbf{G}$ is a linear torsion group in characteristic p (in the sense of [5, Sec. 8]). Hence, by [5, Thm 8.10], $\mathbf{G}$ has a Sylow ℓ-subgroup S and $\mathcal{F}_S(\mathbf{G})$ is a saturated fusion system. Also, $\mathbf{G}$ is the group of $\overline{\mathbb{F}}_p$-points of a connected split reductive algebraic group scheme $\mathbb{G}$ over $\mathbb{Z}$. By a theorem of Friedlander–Mislin [6, Thm 1.4], there is a homotopy equivalence of ℓ-completed classifying spaces

$$B\mathbb{G}(\mathbb{C})^\wedge_\ell \simeq (B\mathbf{G})^\wedge_\ell.$$

Further, by the Malcev–Iwasawa theorem (see [16, Thm 32.5]), $\mathbb{G}(\mathbb{C})$ has a maximal compact subgroup K and there is a homotopy equivalence $B\mathbb{G}(\mathbb{C}) \simeq BK$. Hence,

$$BK^\wedge_\ell \simeq B\mathbb{G}(\mathbb{C})^\wedge_\ell \simeq (B\mathbf{G})^\wedge_\ell$$

and as a consequence of [5, Thms 7.4, 8.10, and 9.10] we obtain the following.

Proposition 2.3. *With the above notation there is an isomorphism $S' \cong S$ between a Sylow ℓ-subgroup S' of K and a Sylow ℓ-subgroup S of $\mathbf{G}$ inducing an isomorphism $\mathcal{F}_{S'}(K) \cong \mathcal{F}_S(\mathbf{G})$ of the corresponding fusion systems.*

3. Proofs of Theorem 1 and Theorem 2.

We recall some notation from Section 7 of [9]. Denote by $\mathcal{C}$ the set of finite sequences $\mathfrak{c} = (c_1, \ldots, c_t)$ of strictly positive integers including the empty sequence (). For $\mathfrak{c} \in \mathcal{C}$, write $|\mathfrak{c}| := c_1 + \cdots + c_t$ and let $G_\mathfrak{c} := \mathrm{GL}_{c_1}(\ell) \times \cdots \times \mathrm{GL}_{c_t}(\ell)$. Denote by $\mathcal{A}(\mathfrak{c})$ the set of irreducible characters of $G_\mathfrak{c} = \mathrm{GL}_{c_1}(\ell) \times \cdots \times \mathrm{GL}_{c_t}(\ell)$ of the form $\chi_1 \cdots \chi_t$, where each χ_j is an extension to $\mathrm{GL}_{c_j}(\ell)$ of a Steinberg character of $\mathrm{SL}_{c_j}(\ell)$. Note that $|\mathcal{A}(\mathfrak{c})| = (\ell-1)^t$.

We recall the description of ℓ-stubborn subgroups in classical groups due to Oliver [13]. As noted already by Oliver this is akin to the Alperin–Fong type classification of ℓ-radical subgroups of finite general linear and symmetric groups in [1].

Proposition 3.1. *Suppose that $\mathbf{G}$ is the compact connected Lie group $\mathrm{U}(n)$, $\mathrm{Sp}(n)$, $\mathrm{SO}(2n+1)$ or $\mathrm{SO}(2n)$, $n \geq 1$. Let ℓ be a prime, assumed odd unless $\mathbf{G} = \mathrm{U}(n)$. If $\mathbf{G} \neq \mathrm{U}(n)$, let $\mathcal{X}$ denote the set of functions*

$$f : \mathbb{N} \times \mathcal{C} \to \mathbb{N}$$

such that $\sum_{(\gamma,\mathfrak{c})} \ell^{\gamma+|\mathfrak{c}|} f(\gamma, \mathfrak{c}) = n$. If $\mathbf{G} = \mathrm{U}(n)$, let $\mathcal{X}$ denote the subset of the above set of functions f which additionally have the property $f(0, ()) \neq 2, 4$ if $\ell = 2$ and $f(0, ()) \neq 3$ if $\ell = 3$. There is a bijection between the set of $\mathbf{G}$-conjugacy classes of ℓ-stubborn subgroups of $\mathbf{G}$ and $\mathcal{X}$ satisfying the following: Let $\mathbf{P} \leq \mathbf{G}$ be an ℓ-stubborn subgroup whose class corresponds to $f \in \mathcal{X}$.

(a) *If $\mathbf{G} = \mathrm{U}(n)$, then*

$$N_\mathbf{G}(\mathbf{P})/\mathbf{P} \cong \prod_{(\gamma,\mathfrak{c})} (\mathrm{Sp}_{2\gamma}(\ell) \times G_\mathfrak{c}) \wr \mathfrak{S}_{f(\gamma,\mathfrak{c})}.$$

(b) *If $\mathbf{G} = \mathrm{Sp}(n)$ or $\mathbf{G} = \mathrm{SO}(2n+1)$, then*

$$N_\mathbf{G}(\mathbf{P})/\mathbf{P} \cong \prod_{(\gamma,\mathfrak{c})} (C_2 \times \mathrm{Sp}_{2\gamma}(\ell) \times G_\mathfrak{c}) \wr \mathfrak{S}_{f(\gamma,\mathfrak{c})}.$$

(c) *If* $\mathbf{G} = \mathrm{SO}(2n)$*, then* $N_{\mathbf{G}}(\mathbf{P})/\mathbf{P}$ *is isomorphic to the subgroup of*

$$\prod_{(\gamma,\mathfrak{c})} (C_2 \times \mathrm{Sp}_{2\gamma}(\ell) \times G_{\mathfrak{c}}) \wr \mathfrak{S}_{f(\gamma,\mathfrak{c})}$$

consisting of those elements for which the number of non-trivial entries from the C_2 *components is even.*

Proof. For $\mathfrak{c} \in \mathcal{C}$ and a non-negative integer γ let $\mathbf{R}_{\gamma,\mathfrak{c}}$ denote the irreducible subgroup $\Gamma^U_{\ell^\gamma} \wr E_{\ell^{c_1}} \wr \cdots \wr E_{\ell^{c_t}}$ of $\mathrm{U}(\ell^{\gamma+|\mathfrak{c}|})$ defined in [13, Def. 2]. We regard $\mathbf{R}_{\gamma,\mathfrak{c}}$ as a subgroup of $\mathrm{Sp}(\ell^{\gamma+|\mathfrak{c}|})$ (respectively $\mathrm{SO}(2\ell^{\gamma+|\mathfrak{c}|})$ via a standard embedding $\mathrm{U}(\ell^{\gamma+|\mathfrak{c}|}) \leq \mathrm{Sp}(\ell^{\gamma+|\mathfrak{c}|})$ (respectively $\mathrm{U}(\ell^{\gamma+|\mathfrak{c}|}) \leq \mathrm{O}(2\ell^{\gamma+|\mathfrak{c}|})$. By [13, Thm 6],

$$N_{\mathrm{U}(\ell^{\gamma+|\mathfrak{c}|})}(\mathbf{R}_{\gamma,\mathfrak{c}})/\mathbf{R}_{\gamma,\mathfrak{c}} \cong \mathrm{Sp}_{2\gamma}(\ell) \times G_{\mathfrak{c}},$$
$$N_{\mathrm{Sp}(\ell^{\gamma+|\mathfrak{c}|})}(\mathbf{R}_{\gamma,\mathfrak{c}})/\mathbf{R}_{\gamma,\mathfrak{c}} \cong C_2 \times \mathrm{Sp}_{2\gamma}(\ell) \times G_{\mathfrak{c}},$$

and

$$N_{\mathrm{O}(2\ell^{\gamma+|\mathfrak{c}|})}(\mathbf{R}_{\gamma,\mathfrak{c}})/\mathbf{R}_{\gamma,\mathfrak{c}} \cong C_2 \times \mathrm{Sp}_{2\gamma}(\ell) \times G_{\mathfrak{c}}.$$

Moreover, from the proof of [13, Thm 6] it can be checked that the full inverse image of $\mathrm{Sp}_{2\gamma}(\ell) \times G_{\mathfrak{c}}$ in $N_{\mathrm{O}(2\ell^{\gamma+|\mathfrak{c}|})}(\mathbf{R}_{\gamma,\mathfrak{c}})$ lies in $\mathrm{SO}(2\ell^{\gamma+|\mathfrak{c}|})$, and that the generator of the C_2 factor lifts to an element of determinant -1 in $\mathrm{O}(2\ell^{\gamma+|\mathfrak{c}|})$, specifically to an element of order 2 with -1 as an eigenvalue of multiplicity $\ell^{\gamma+|\mathfrak{c}|}$.

If $\mathbf{R}$ is a direct product of m copies of $\mathbf{R}_{\gamma,\mathfrak{c}}$ diagonally embedded via

$$\mathrm{U}(\ell^{\gamma+|\mathfrak{c}|}) \times \cdots \times \mathrm{U}(\ell^{\gamma+|\mathfrak{c}|}) \leq \mathrm{U}(m\ell^{\gamma+|\mathfrak{c}|}),$$

then

$$N_{\mathrm{U}(m\ell^{\gamma+|\mathfrak{c}|})}(\mathbf{R})/\mathbf{R} \cong (N_{\mathrm{U}(\ell^{\gamma+|\mathfrak{c}|})}(\mathbf{R}_{\gamma,\mathfrak{c}})/\mathbf{R}_{\gamma,\mathfrak{c}}) \wr \mathfrak{S}_m \cong (\mathrm{Sp}_{2\gamma}(\ell) \times G_{\mathfrak{c}}) \wr \mathfrak{S}_m$$

where $\mathfrak{S}_m$ acts by usual permutation of factors. Similarly, regarding $\mathbf{R}$ as a diagonally embedded subgroup via $\mathrm{Sp}(\ell^{\gamma+|\mathfrak{c}|})^m \leq \mathrm{Sp}(m\ell^{\gamma+|\mathfrak{c}|})$, then

$$N_{\mathrm{Sp}(m\ell^{\gamma+|\mathfrak{c}|})}(\mathbf{R})/\mathbf{R} \cong N_{\mathrm{Sp}(\ell^{\gamma+|\mathfrak{c}|})}(\mathbf{R}_{\gamma,\mathfrak{c}})/\mathbf{R}_{\gamma,\mathfrak{c}} \wr \mathfrak{S}_m \cong (C_2 \times \mathrm{Sp}_{2\gamma}(\ell) \times G_{\mathfrak{c}}) \wr \mathfrak{S}_m$$

and regarding $\mathbf{R}$ as a diagonally embedded subgroup via $\mathrm{O}(2\ell^{\gamma+|\mathfrak{c}|})^m \leq \mathrm{O}(2m\ell^{\gamma+|\mathfrak{c}|})$, then

$$N_{\mathrm{O}(2m\ell^{\gamma+|\mathfrak{c}|})}(\mathbf{R})/\mathbf{R} \cong N_{\mathrm{O}(2\ell^{\gamma+|\mathfrak{c}|})}(\mathbf{R}_{\gamma,\mathfrak{c}})/\mathbf{R}_{\gamma,\mathfrak{c}} \wr \mathfrak{S}_m \cong (C_2 \times \mathrm{Sp}_{2\gamma}(\ell) \times G_{\mathfrak{c}}) \wr \mathfrak{S}_m.$$

Note that since the space underlying $\mathbf{R}_{\gamma,\mathfrak{c}}$ in the orthogonal case is even-dimensional, the elements of $\mathfrak{S}_m$ lift to elements of determinant 1 in $\mathrm{O}(2m\ell^{\gamma+|\mathfrak{c}|})$.

Suppose that $\mathbf{G} = \mathrm{U}(n)$. Then (a) follows from [13, Thms 6 and 8] (see also the last part of Definition 2 of [13]) with the bijection between $\mathcal{X}$ and the $\mathbf{G}$-conjugacy classes of ℓ-stubborn subgroups sending $f \in \mathcal{X}$ to the class of

$$\mathbf{P} = \prod_{(\gamma,\mathfrak{c})} \mathbf{R}^{f(\gamma,\mathfrak{c})}_{\gamma,\mathfrak{c}} \leq \prod_{(\gamma,\mathfrak{c})} \mathrm{U}(\ell^{\gamma+|\mathfrak{c}|})^{f(\gamma,\mathfrak{c})} \leq \prod_{(\gamma,\mathfrak{c})} \mathrm{U}(f(\gamma,\mathfrak{c})\ell^{\gamma+|\mathfrak{c}|}) \leq \mathrm{U}(n),$$

where the last inclusion is via a decomposition of the vector space underlying $\mathrm{U}(n)$, and where

$$N_{\mathbf{G}}(\mathbf{P})/\mathbf{P} \cong \prod_{(\gamma,\mathfrak{c})} N_{\mathrm{U}(f(\gamma,\mathfrak{c})\ell^{\gamma+|\mathfrak{c}|})}(\mathbf{R}^{f(\gamma,\mathfrak{c})}_{\gamma,\mathfrak{c}})/\mathbf{R}^{f(\gamma,\mathfrak{c})}_{\gamma,\mathfrak{c}} \cong \prod_{(\gamma,\mathfrak{c})} (\mathrm{Sp}_{2\gamma}(\ell) \times G_{\mathfrak{c}}) \wr \mathfrak{S}_{f(\gamma,\mathfrak{c})},$$

proving (a). The proof for the case that $\mathbf{G} = \mathrm{Sp}(n)$ is entirely similar.

Suppose that $\mathbf{G} = \mathrm{SO}(2n)$ and set $\hat{\mathbf{G}} = \mathrm{O}(2n)$. Since $\mathbf{G}$ is the connected component of $\hat{\mathbf{G}}$ of index 2, and ℓ is odd, the set of ℓ-stubborn subgroups of $\mathbf{G}$ coincides with that of $\hat{\mathbf{G}}$. By [13, Thms 6 and 8], the set of $\hat{\mathbf{G}}$-conjugacy classes of ℓ-stubborn subgroups is in one to one correspondence with $\mathcal{X}$ in the same way as above where

$$\mathbf{P} = \prod_{(\gamma,\mathfrak{c})} \mathbf{R}_{\gamma,\mathfrak{c}}^{f(\gamma,\mathfrak{c})} \leq \prod_{(\gamma,\mathfrak{c})} \mathrm{U}(\ell^{\gamma+|\mathfrak{c}|})^{f(\gamma,\mathfrak{c})} \leq \prod_{(\gamma,\mathfrak{c})} \mathrm{O}(2\ell^{\gamma+|\mathfrak{c}|})^{f(\gamma,\mathfrak{c})} \leq \prod_{(\gamma,\mathfrak{c})} \mathrm{O}(2f(\gamma,\mathfrak{c})\ell^{\gamma+|\mathfrak{c}|}) \leq \mathrm{O}(2n)$$

and

$$N_{\hat{\mathbf{G}}}(\mathbf{P})/\mathbf{P} \cong \prod_{(\gamma,\mathfrak{c})} (C_2 \times \mathrm{Sp}_{2\gamma}(\ell) \times G_{\mathfrak{c}}) \wr \mathfrak{S}_{f(\gamma,\mathfrak{c})}.$$

From the description above, it follows that $N_{\mathbf{G}}(\mathbf{P})$ is the index 2-subgroup of $N_{\hat{\mathbf{G}}}(\mathbf{P})$ described in the statement of (c), and in particular, the $\hat{\mathbf{G}}$-conjugacy class of $\mathbf{P}$ is the same as the $\mathbf{G}$-conjugacy class of $\mathbf{P}$. This proves (c).

Finally suppose that $\mathbf{G} = \mathrm{SO}(2n+1)$, and set $\hat{\mathbf{G}} = \mathrm{O}(2n+1)$. Then since $\hat{\mathbf{G}} = \mathbf{G} \times \{\pm I\}$ and ℓ is odd, the $\mathbf{G}$-conjugacy classes of ℓ-stubborn subgroups of $\mathbf{G}$ are the same as the $\hat{\mathbf{G}}$-conjugacy classes of ℓ-stubborn subgroups of $\hat{\mathbf{G}}$. By [13, Thms 6 and 8], the set of $\hat{\mathbf{G}}$-conjugacy classes of ℓ-stubborn subgroups of $\hat{\mathbf{G}}$ are in one to one correspondence with $\mathcal{X}$ in the same way as above where now

$$\mathbf{P} = \prod_{(\gamma,\mathfrak{c})} \mathbf{R}_{\gamma,\mathfrak{c}}^{f(\gamma,\mathfrak{c})} \leq \prod_{(\gamma,\mathfrak{c})} \mathrm{U}(\ell^{\gamma+|\mathfrak{c}|})^{f(\gamma,\mathfrak{c})} \leq \prod_{(\gamma,\mathfrak{c})} \mathrm{O}(2\ell^{\gamma+|\mathfrak{c}|})^{f(\gamma,\mathfrak{c})} \leq \mathrm{O}(2n) \times \mathrm{O}(1) \leq \mathrm{O}(2n+1)$$

and

$$N_{\hat{\mathbf{G}}}(\mathbf{P})/\mathbf{P} \cong \Big(\prod_{(\gamma,\mathfrak{c})} (C_2 \times \mathrm{Sp}_{2\gamma}(\ell) \times G_{\mathfrak{c}}) \wr \mathfrak{S}_{f(\gamma,\mathfrak{c})} \Big) \times \mathrm{O}(1).$$

Here, note that $\mathrm{O}(1) \cong C_2$. Now the restriction to $N_{\mathbf{G}}(\mathbf{P})/\mathbf{P}$ of the canonical surjection $N_{\hat{\mathbf{G}}}(\mathbf{P})/\mathbf{P} \to N_{\hat{\mathbf{G}}}(\mathbf{P})/(\mathbf{P} \times \mathrm{O}(1))$ is an isomorphism and this proves (b). □

Theorem 3.2. *Suppose that* $\mathbf{G}$ *is the compact connected Lie group* $\mathrm{U}(n)$, $\mathrm{Sp}(n)$, *or* $\mathrm{SO}(n)$, $n \geq 1$. *Let* ℓ *be a prime, assumed odd unless* $\mathbf{G} = \mathrm{U}(n)$, $S \in \mathrm{Syl}_\ell(\mathbf{G})$ *a Sylow* ℓ*-subgroup of* $\mathbf{G}$, $\mathcal{F} = \mathcal{F}_S(\mathbf{G})$ *the associated fusion system and* W *the Weyl group of* $\mathbf{G}$. *Then,*

$$\mathbf{w}(\mathcal{F}) = |\mathrm{Irr}(W)|.$$

Proof. By Lemma 2.2 we have

$$\mathbf{w}(\mathcal{F}) = \sum_{\mathbf{P}} z(N_{\mathbf{G}}(\mathbf{P})/\mathbf{P})$$

where $\mathbf{P}$ runs over representatives of $\mathbf{G}$-conjugacy classes of ℓ-stubborn subgroups of $\mathbf{G}$ and $z(N_{\mathbf{G}}(\mathbf{P})/\mathbf{P})$ is the number of irreducible characters of $N_{\mathbf{G}}(\mathbf{P})/\mathbf{P}$ of zero ℓ-defect.

Suppose first that $\mathbf{G} = \mathrm{U}(n)$. Let $\mathcal{X}$ be as in Proposition 3.1, let $f \in \mathcal{X}$ and let $\mathbf{P}$ be an ℓ-stubborn of $\mathbf{G}$ corresponding to f. The only characters of zero ℓ-defect of $\mathrm{GL}_c(\ell)$ or $\mathrm{Sp}_{2\gamma}(\ell)$ are Steinberg characters and $\mathrm{Sp}_{2\gamma}(\ell)$ has a unique Steinberg character which we denote by χ_γ. Thus by Proposition 3.1(i) and [9, Lemma 9.1], the weights contributed by $\mathbf{P}$ are in bijection with the set

$$\mathcal{W}_f := \Big\{ w : \bigcup_{(\gamma,\mathfrak{c})} \{\chi_\gamma\} \times \mathcal{A}(\mathfrak{c}) \to \{\ell\text{-cores}\} \quad \text{with} \sum_{\varphi \in \{\chi_\gamma\} \times \mathcal{A}(\mathfrak{c})} |w(\varphi)| = f(\gamma,\mathfrak{c}) \Big\}.$$

Letting f range over $\mathcal{X}$ and identifying $\{\chi_\gamma\} \times \mathcal{A}(\mathfrak{c})$ with $\mathcal{A}(\mathfrak{c})$, we see that the set of weights of $\mathcal{F}$ is in bijection with the set $\mathcal{W}$ of assignments

$$w : \mathbb{N} \times \bigcup_{\mathfrak{c}\in\mathcal{C}} \mathcal{A}(\mathfrak{c}) \to \{\ell\text{-cores}\}$$

such that

$$\sum_{\mathfrak{c}} \sum_{(\gamma,\varphi)\in\mathbb{N}\times\mathcal{A}(\mathfrak{c})} \ell^{\gamma+|\mathfrak{c}|}|w(\gamma,\varphi)| = n.$$

Here note that the restrictions on $\mathcal{X}$ in Proposition 3.1(a) do not play a role. By [9, Lemma 7.15], applied with $e = r = 1$ and by the argument of the proof of Theorem 4.2 at the end of Section 7 of [9] it follows that the number of $\mathcal{F}$-weights equals the number of ordinary irreducible characters of $G(1,1,n) \cong \mathfrak{S}_n$, the Weyl group of $\mathrm{U}(n)$. (Observe that the standing assumption $\ell > 2$ in [9, §7] is insubstantial for the combinatorial assertion of [9, Lemma 7.15].)

Suppose next that $\mathbf{G} = \mathrm{Sp}(n)$ or $\mathrm{SO}(2n+1)$. Let $\mathcal{X}$ be as in Proposition 3.1, let $f \in \mathcal{X}$ and let $\mathbf{P}$ be an ℓ-stubborn subgroup of $\mathbf{G}$ corresponding to f. Since ℓ is odd, all irreducible characters of C_2 are of ℓ-defect 0, we obtain from Proposition 3.1(b) and by [9, Lemma 9.1] that the weights contributed by $\mathbf{P}$ are in bijection with the set

$$\tilde{\mathcal{W}}_f := \Big\{w : \bigcup_{(\gamma,\mathfrak{c})} \{\chi_\gamma\} \times C_2 \times \mathcal{A}(\mathfrak{c}) \to \{\ell\text{-cores}\} \quad \text{with} \sum_{\varphi\in\{\chi_\gamma\}\times C_2\times\mathcal{A}(\mathfrak{c})} |w(\varphi)| = f(\gamma,\mathfrak{c})\Big\}.$$

Letting f range over $\mathcal{X}$ and identifying $\{\chi_\gamma\} \times C_2 \times \mathcal{A}(\mathfrak{c})$ with $C_2 \times \mathcal{A}(\mathfrak{c})$, it follows that the set of $\mathcal{F}$-weights is in bijection with the set $\tilde{\mathcal{W}}$ of assignments

$$w : \mathbb{N} \times C_2 \times \bigcup_{\mathfrak{c}\in\mathcal{C}} \mathcal{A}(\mathfrak{c}) \to \{\ell\text{-cores}\}$$

such that

$$\sum_{\mathfrak{c}} \sum_{(\gamma,x,\varphi)\in\mathbb{N}\times C_2\times\mathcal{A}(\mathfrak{c})} \ell^{\gamma+|\mathfrak{c}|}|w(\gamma,x,\varphi)| = n.$$

As in the previous case by [9, Lemma 7.15] now applied with $e = 2$, $r = 1$ and with $\tilde{\mathcal{W}}$ in place of $\mathcal{W}$ and by the argument of the proof of Theorem 4.2 of [9] we obtain that the number of $\mathcal{F}$-weights equals the number of ordinary irreducible characters of $G(2,1,n) \cong W(C_n) = W(B_n)$.

Now suppose that $\mathbf{G} = \mathrm{SO}(2n)$ and let f, $\tilde{\mathcal{W}}_f$ be as just defined above. Let C_2 act on $\tilde{\mathcal{W}}_f$ via $y.w(\gamma,x,\varphi) := w(\gamma, y+x, \varphi)$ for $y \in C_2$, $w \in \tilde{\mathcal{W}}_f$ and $(\gamma,x,\varphi) \in \mathbb{N} \times C_2 \times \mathcal{A}$. By Proposition 3.1(c) and [9, Lemma 9.1], applied with $N_{\mathrm{O}(2n)}(\mathbf{P})/\mathbf{P}$ in place of G and $N_{\mathrm{SO}(2n)}(\mathbf{P})/\mathbf{P}$ in place of M, the weights contributed by $\mathbf{P}$ are indexed by $\mathbb{Z}/2\mathbb{Z}$-orbits of $\tilde{\mathcal{W}}_f$ with each orbit contributing as many weights as the order of the stabiliser of a point of the orbit. Again, letting f range over all elements of $\mathcal{X}$, we obtain that the $\mathcal{F}$-weights are indexed by the $\mathbb{Z}/2\mathbb{Z}$-orbits of $\mathcal{W}$ with each orbit contributing as many weights as the order of the stabiliser of a point of the orbit, where C_2 acts on $\tilde{\mathcal{W}}_f$ as above, that is via $y.w(\gamma,x,\varphi) := w(\gamma,y+x,\varphi)$ for $y \in C_2$, $w \in \tilde{\mathcal{W}}_f$ and $(\gamma,x,\varphi) \in \mathbb{N} \times C_2 \times \mathcal{A}$. As before, by [9, Lemma 7.15] now applied with $e = 2$, $r = 2$ and with $\tilde{\mathcal{W}}$ in place of $\mathcal{W}$ and by the argument of the proof of Theorem 4.2 of [9] we obtain that the number of $\mathcal{F}$-weights equals the number of ordinary irreducible characters of $G(2,2,n) \cong W(D_n)$. □

Proof of Theorem 1. We use again Lemma 2.2. By [7, Prop. 1.6(i)], any ℓ-stubborn subgroup of $\mathbf{G}$ contains $Z(\mathbf{G})$ and a subgroup $\mathbf{P} \leq \mathbf{G}$ which contains $Z(\mathbf{G})$ is ℓ-stubborn if and only if $\bar{\mathbf{P}} := \mathbf{P}/Z(\mathbf{G})$ is an ℓ-stubborn subgroup of $\bar{\mathbf{G}} := \mathbf{G}/Z(\mathbf{G})$. For any $\mathbf{P} \leq \mathbf{G}$ containing $Z(\mathbf{G})$, $N_{\mathbf{G}}(\mathbf{P})/\mathbf{P} \cong N_{\bar{\mathbf{G}}}(\bar{\mathbf{P}})/\bar{\mathbf{P}}$ and the Weyl groups of $\mathbf{G}$ and $\bar{\mathbf{G}}$ are isomorphic. Thus, we may assume that $Z(\mathbf{G}) = 1$ and hence that $\mathbf{G}$ is a direct product of simple compact Lie groups. By [7, Prop. 1.6(ii)], the ℓ-stubborn subgroups of a direct product of compact connected Lie groups are the direct products of ℓ-stubborn subgroups of the factors and the decomposition into direct factors respects conjugacy, normalisers, and Weyl groups hence we may assume that $\mathbf{G}$ is simple.

Assume first that $\mathbf{G}$ is of classical type. Applying the above arguments again, we may assume that $\mathbf{G} = \mathrm{U}(n)$, or $\ell \geq 3$ and $\mathbf{G}$ is one of $\mathrm{Sp}(n)$, $\mathrm{SO}(2n+1)$ or $\mathrm{SO}(2n)$, and we are done by Theorem 3.2. Now assume that $\mathbf{G}$ is an exceptional group. Suppose first that the Weyl group of $\mathbf{G}$ is an ℓ'-group. Then every ℓ-toral subgroup of $\mathbf{G}$ is contained in a maximal torus of $\mathbf{G}$, and by Lemma 2.1(a) the only ℓ-stubborn subgroups of $\mathbf{G}$ are the maximal tori and the result is immediate. Thus the only cases left are $\mathbf{G} = E_6$ with $\ell = 5$, $\mathbf{G} = E_7$ with $\ell = 5, 7$ and $\mathbf{G} = E_8$ with $\ell = 7$. These are handled by the next proposition. Note that in all of these cases Sylow ℓ-subgroups of W are cyclic. □

Proposition 3.3. *Suppose* $\mathbf{G} = E_n$ *for* $n \in \{6, 7, 8\}$ *and let* ℓ *be a good prime dividing* $|W(\mathbf{G})|$. *Then* $\mathbf{w}(\mathcal{F}_\ell(\mathbf{G})) = |\mathrm{Irr}(W)|$ *where* $\mathcal{F}_\ell(\mathbf{G})$ *denotes the* ℓ*-fusion system of* $\mathbf{G}$.

Proof. By [14, Thm B(c)], $\mathcal{F}_\ell(\mathbf{G})$ is the fusion system coming from a simple ℓ-local compact group, and hence $\mathbf{w}(\mathcal{F}_\ell(\mathbf{G})) = |\mathrm{Irr}(W)|$ by [15, Thm 1.2]. □

Proof of Theorem 2. This is immediate from Proposition 2.3 and Theorem 1. □

4. Bad primes

Proposition 4.1. *Suppose that* $\mathbf{G} = \mathrm{Sp}(n)$ *with* $n \geq 2$ *and* $\ell = 2$. *Then* $\mathbf{w}(\mathcal{F}) < |\mathrm{Irr}(W)|$.

Proof. For $\gamma \in \mathbb{N}$ and $\mathfrak{c} \in \mathcal{F}$, let $\mathbf{S}_{\gamma,\mathfrak{c}}$ (respectively $\bar{\mathbf{S}}_{\gamma,\mathfrak{c}}$) denote the irreducible subgroup $\Gamma^{\mathrm{Sp}}_{2^\gamma} \wr E_{2^{c_1}} \wr \cdots \wr E_{2^{c_t}}$ (respectively $\bar{\Gamma}^{\mathrm{Sp}}_{2^\gamma} \wr E_{2^{c_1}} \wr \cdots \wr E_{2^{c_t}}$) of $\mathrm{Sp}(2^{\gamma+|\mathfrak{c}|})$ defined in [13, Def. 2]. By [13, Thms 6 and 8], the $\mathbf{G}$-classes of 2-stubborn subgroups of $\mathbf{G}$ are in bijection with the set of ordered pairs of functions (f, f') where

$$f : \mathbb{N} \times \mathcal{C} \to \mathbb{N} \quad \text{and} \quad f' : \mathbb{N} \times \mathcal{C} \to \mathbb{N}$$

are such that

$$\sum_{(\gamma,\mathfrak{c})} 2^{\gamma+|\mathfrak{c}|} f(\gamma, \mathfrak{c}) + \sum_{(\gamma',\mathfrak{c}')} 2^{\gamma'+|\mathfrak{c}'|} f'(\gamma', \mathfrak{c}') = n$$

and $f'((0, ())) \neq 2, 4$. If $\mathbf{P} \leq \mathbf{G}$ is a 2-stubborn subgroup corresponding to (f, f') then

$$\mathbf{P} \cong \prod_{(\gamma,\mathfrak{c})} \mathbf{S}_{\gamma,\mathfrak{c}}^{f(\gamma,\mathfrak{c})} \times \prod_{(\gamma',\mathfrak{c}')} \bar{\mathbf{S}}_{\gamma',\mathfrak{c}'}^{f(\gamma',\mathfrak{c}')}$$

and

$$N_{\mathbf{G}}(\mathbf{P})/\mathbf{P} \cong \prod_{(\gamma,\mathfrak{c})} (\mathrm{GO}^-_{2\gamma+2}(2) \times G_{\mathfrak{c}}) \wr \mathfrak{S}_{f(\gamma,\mathfrak{c})} \times \prod_{(\gamma',\mathfrak{c}')} (\mathrm{Sp}_{2\gamma'}(2) \times G_{\mathfrak{c}'}) \wr \mathfrak{S}_{f'(\gamma',\mathfrak{c}')}.$$

Now $\mathrm{GO}_2^-(2) \cong \mathfrak{S}_3$ has a unique irreducible character of 2-defect 0 while $\mathrm{GO}_{2\gamma+2}^-(2)$ has no irreducible characters of 2-defect 0 for $\gamma \geq 1$ so the group $\mathrm{GO}_{2\gamma+2}^-(2) \times G_{\mathfrak{c}}$ has a unique irreducible character of 2-defect 0 if $\gamma = 0$ and none if $\gamma \geq 1$. The group $\mathrm{Sp}_{2\gamma'}(2) \times G_{\mathfrak{c}'}$ has a unique irreducible character of 2-defect 0 for all $\gamma' \geq 0$. Further, the symmetric group $\mathfrak{S}_m$ has one block of 2-defect 0 if m is a triangular number and none otherwise. From this and [9, Lemma 9.1], it follows that the weight contribution of $\mathbf{P}$ is 1 if $\gamma = 0$ and the values of f and f' are triangular numbers (including 0), and is 0 otherwise. So, the number of $\mathcal{F}$-weights equals the number of pairs (f, f') of functions

$$f : \mathcal{C} \to \mathbb{N} \quad \text{and} \quad f' : \mathbb{N} \times \mathcal{C} \to \mathbb{N}$$

such that

$$\sum_{\mathfrak{c}} 2^{|\mathfrak{c}|} f(\mathfrak{c}) + \sum_{(\gamma', \mathfrak{c}')} 2^{\gamma' + |\mathfrak{c}'|} f'(\gamma', \mathfrak{c}') = n$$

and such that all values of f and f' are triangular numbers. On the other, hand using the analysis for the odd ℓ case and noting that when $\ell = 2$, $\mathcal{A}(\mathfrak{c})$ is a singleton and that the set of 2-cores is in bijection with the set of triangular numbers, we have that the number of bipartitions of n, i.e., $|\mathrm{Irr}(W)|$, equals the number of pairs (f, f') of functions

$$f : \mathbb{N} \times \mathcal{C} \to \mathbb{N} \quad \text{and} \quad f' : \mathbb{N} \times \mathcal{C} \to \mathbb{N}$$

such that

$$\sum_{(\gamma, \mathfrak{c})} 2^{|\gamma| + |\mathfrak{c}|} f(\gamma, \mathfrak{c}) + \sum_{(\gamma', \mathfrak{c}')} 2^{\gamma' + |\mathfrak{c}'|} f'(\gamma', \mathfrak{c}') = n$$

and such that all values of f and f' are triangular numbers. Thus, the number of weights is strictly less than $|\mathrm{Irr}(W)|$ for all $n \geq 2$. □

Example 4.2. For $n = 2$, we get 4 weights for $\mathrm{Sp}(2)$, namely for $((1); (0, ()))$ with $f = 1, f' = 0$, $((); (0, ()))$ with $f = f' = 1$, $((); (1, ()))$ and $((); (0, (1)))$ with $f = 0, f' = 1$, and so by the proof of Theorem 2, $4 = \mathbf{w}(\mathcal{F}) < |\mathrm{Irr}(W(\mathbf{G}))| = 5$, where $\mathbf{G}$ is a simple algebraic group over $\overline{\mathbb{F}}_p$ of type C_2 and $\mathcal{F}$ is its 2-fusion system. Note, however, that if $\mathcal{F}$ is the 2-fusion system of $\mathrm{CSp}_4(q)$ with $\nu_2(q-1) > 2$ there are five $\mathcal{F}$-centric radical subgroups each contributing a single weight (so $\mathbf{w}(\mathcal{F}) = 5$), but two of these subgroups are fused in $\mathrm{CSp}_4(q^2)$.

Example 4.3. Let $\mathbf{G}$ be a connected reductive group of symplectic or odd dimensional orthogonal type over an algebraic closed field of odd characteristic and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius map. Then the number $\mathbf{w}(\mathcal{F})$ of 2-weights for the principal 2-block of $\mathbf{G}^F$ equals the number of unipotent conjugacy classes of $\mathbf{G}^F$ (see [2, Prop. (2A)] in general). Now the Springer correspondence (defined by Lusztig [11, Thm 6.5; see also 10.5] for bad primes) defines an injective map from $\mathrm{Irr}(W)$ to the set of unipotent classes of $\mathbf{G}^F$, so we conclude $\mathbf{w}(\mathcal{F}) \geq |\mathrm{Irr}(W)|$. In fact, since the Springer correspondence is not surjective in general we have $\mathbf{w}(\mathcal{F}) > |\mathrm{Irr}(W)|$ for large enough rank. E.g., for $\mathbf{G}$ of type B_4 or type C_6 we have $\mathbf{w}(\mathcal{F}) = |\mathrm{Irr}(W)| + 1$.

Proposition 4.4. *Suppose* $\mathbf{G} = G_2$. *Then* $\mathbf{w}(\mathcal{F}) = |\mathrm{Irr}(W)| = 6$ *for all primes* ℓ.

Proof. If $\ell > 3$, then we are done by Theorem 3.2. The 2-stubborn and 3-stubborn subgroups of $\mathbf{G}$ are determined in [8, Lemma 3.2]. There are six classes of 2-stubborn subgroups $\mathbf{P}$, with $N(\mathbf{P})/\mathbf{P}$ isomorphic to 1, $\mathfrak{S}_3$ (for three of the classes), $\mathfrak{S}_3 \times \mathfrak{S}_3$, and $\mathrm{GL}_3(2)$, respectively. Each of these contributes one weight. There are two classes of 3-stubborn subgroups $\mathbf{P}$ with $N(\mathbf{P})/\mathbf{P}$ isomorphic to $C_2 \times C_2$ and $\mathrm{GL}_2(3)$, respectively. The first contributes 4 weights and the second contributes 2 weights. □

Proposition 4.5. *Suppose* $\mathbf{G} = F_4$ *and* $\ell = 3$. *Then* $\mathbf{w}(\mathcal{F}) = 22 < |\mathrm{Irr}(W)| = 25$.

Proof. The 3-stubborn subgroups of $\mathbf{G}$ are determined in [18, Prop. 3.6]: there are seven conjugacy classes, with respective automisers D_8, $(C_2 \times \mathrm{Sp}_2(3)).2$ (twice), $\mathrm{Sp}_2(3) \wr 2$, $\mathrm{GL}_2(3)$, $\mathrm{SL}_3(3)$ and $W(F_4)$. The respective number of weights for those is 5, 4 (twice), 2, 2, 1, 4, adding up to 22. □

Proposition 4.6. *Suppose* $\mathbf{G} = E_8$ *and* $\ell = 5$. *Then* $\mathbf{w}(\mathcal{F}) = 103 < |\mathrm{Irr}(W)| = 112$.

TABLE 1. 5-weights for E_8

$N(P)/P$	z	$N(P)/P$	z
$(C_4 \times C_4)\colon C_4$	28	$(\mathrm{SL}_2(5) \times \mathfrak{S}_5)\colon C_2$	4
$(C_4 \times \mathrm{SL}_2(5))\colon C_2$	5	$\mathrm{GL}_2(5)$	4
$(C_4 \times \mathfrak{S}_5)\colon C_2$	10	$\mathrm{SL}_3(5)$	1
$(\mathrm{SL}_2(5) \times \mathrm{SL}_2(5))\colon C_4$	4	$W(E_8)$	47

Proof. The 5-stubborn subgroups of $\mathbf{G}$ and their automisers are determined in [17, Prop. 2.5], see Table 1. From this the claim is straightforward. For example, the number of 5-weights for $(\mathrm{SL}_2(5) \times \mathfrak{S}_5)\colon 2$ equals 4. Indeed, $\mathrm{SL}_2(5)$ has a unique character of 5-defect 0, $\mathfrak{S}_5$ has two and they are stable under the action of C_2 as $\mathrm{Out}(\mathfrak{S}_5) = 1$. So each of the two defect 0 characters of $\mathrm{SL}_2(5) \times \mathfrak{S}_5$ extends in two different ways. □

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