JORDAN CORRESPONDENCE AND BLOCK DISTRIBUTION OF CHARACTERS

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Dedicated to Michel Enguehard

Abstract. We complete the determination of the ℓ-block distribution of characters for quasi-simple exceptional groups of Lie type up to some minor ambiguities relating to non-uniqueness of Jordan decomposition. For this, we first determine the ℓ-block distribution for finite reductive groups whose ambient algebraic group defined in characteristic different from ℓ has connected centre. As a consequence we derive a compatibility between ℓ-blocks, e-Harish-Chandra series and Jordan decomposition. Further we apply our results to complete the proof of Robinson's conjecture on defects of characters.

1. Introduction

A fundamental ingredient in the understanding of the modular representation theory of a finite (simple) group is the distribution of its irreducible complex characters into Brauer ℓ-blocks for the primes ℓ dividing the group order. For example, such information has recently been used in a crucial way in the proof of several deep conjectures, like the Alperin–McKay conjecture for the prime $\ell = 2$, or Brauer's height zero conjecture.

In this paper we contribute to the determination of these ℓ-blocks for quasi-simple groups, in particular to the case of finite exceptional groups of Lie type for bad primes ℓ. A parametrisation of these blocks (for simply connected types) in terms of e-cuspidal pairs had been completed in our previous paper [21]; here we describe the subdivision of the corresponding Lusztig series along those blocks, first for groups arising from algebraic groups with connected centre (Theorem 1), and then using Clifford theory for the quasi-simple groups themselves (Proposition 5.1)

For unipotent blocks this subdivision had already previously been obtained by Broué–Malle–Michel [3] for large primes ℓ, by Cabanes–Enguehard [5] for good primes and by Enguehard [11] in general (up to some indeterminacies), while for arbitrary blocks at good primes ℓ this question had been studied by Cabanes–Enguehard [6], and Enguehard [12, 13], albeit in a different formulation. Here, we settle the remaining cases, thereby arriving at the following result:

Date: January 10, 2025.

2010 *Mathematics Subject Classification.* 20C15, 20C20, 20C33.

Key words and phrases. Brauer ℓ-blocks, Lusztig's Jordan decomposition, Robinson's conjecture.

The work of the first author was partially supported by EPSRC grant EP/T004592/1. The second author gratefully acknowledges financial support by the DFG –Project-ID 286237555–TRR 195. The authors would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme *Groups, Representations and Applications* when part of the work on this paper was undertaken. This work was supported by: EPSRC grant number EP/R014604/1.

Theorem 1. *Let* $\mathbf{X}$ *be a connected reductive group in characteristic* p *with connected centre and simple, simply connected derived subgroup, with a Frobenius map* $F : \mathbf{X} \to \mathbf{X}$*. Let* $\mathbf{G}$ *be an* F*-stable Levi subgroup of* $\mathbf{X}$*, let* $\ell \neq p$ *be a prime and* s *be a semisimple* ℓ'*-element in the dual group* $\mathbf{G}^{*F}$*. Then for every* ℓ*-element* $t \in C_{\mathbf{G}^*}(s)^F$ *there exists a map* $\bar{J}_t^{\mathbf{G}}$ *from the set of unipotent* ℓ*-blocks of* $C_{\mathbf{G}^*}(st)^F$ *to the set of* t*-twin* ℓ*-blocks in* $\mathcal{E}_\ell(\mathbf{G}^F, s)$ *such that if* $\eta \in \mathcal{E}(C_{\mathbf{G}^*}(st)^F, 1) \cap \mathrm{Irr}(b)$ *for a unipotent* ℓ*-block* b *of* $C_{\mathbf{G}^*}(st)^F$*, then its Jordan correspondent in* $\mathcal{E}(\mathbf{G}^F, st)$ *belongs to* $\bar{J}_t^{\mathbf{G}}(b)$*.*

This shows that [5, Thm(iii)] and [11, Thm B] continue to hold for bad primes ℓ and non-unipotent blocks, up to the small ambiguities around twin blocks, only occurring in type E_8. Let us point out that those also arise in unipotent blocks for which it seems they have not been resolved in [11], either. Our map $\bar{J}_t^{\mathbf{G}}$ is similar to the one introduced in [11]; it will be explained in Proposition 4.1 and Section 4.2. The meaning of "t-twin block" will be defined in Section 4.2. In most cases, it is just a single ℓ-block, but in a few cases in $\mathbf{G}^F = E_8(q)$ with $q \equiv -1 \bmod \ell$ it consists of a union of two blocks, see Table 1.

As a consequence we obtain the following compatibility between ℓ-blocks, e-Harish-Chandra series and Jordan decomposition; here e is the order of q modulo ℓ, respectively modulo 4 when $\ell = 2$:

Corollary 2. *Let* $\mathbf{G}$ *and* ℓ *be as in Theorem* 1*. If* $\chi, \chi' \in \mathrm{Irr}(\mathbf{G}^F)$ *lie in the same Lusztig series say* $\mathcal{E}(\mathbf{G}^F, r)$*, for some semisimple* $r \in \mathbf{G}^{*F}$*, and the Jordan correspondents of* χ *and* χ' *in* $\mathcal{E}(C_{\mathbf{G}^*}(r)^F, 1)$ *lie in the same unipotent* e*-Harish-Chandra series, then* χ *and* χ' *lie in the same* r_ℓ*-twin* ℓ*-block of* $\mathbf{G}^F$*.*

On the way we also complete, correct and extend our results in [21]: we deal with the isolated 5-blocks of $E_8(q)$ in Lusztig series indexed by isolated 6-elements that had been omitted there, showing that all results from [21] carry over (Theorem 6.5), and we correct information on 3-blocks of $E_7(q)$ and $E_8(q)$ that arose from a misinterpretation of results in [11] (Proposition 6.9). In addition, in Proposition 6.3 we parametrise the isolated blocks in groups of adjoint types E_6 and E_7, and in Lemma 6.10 we settle the last open instances from [3] of the decomposition of Lusztig induction of unipotent characters.

Robinson's conjecture [28] asserts that for any ℓ-block B of a finite group G with defect group D we have
$$\ell^{\mathrm{def}(\chi)} \geq |Z(D)| \qquad \text{for all } \chi \in \mathrm{Irr}(B)$$
with equality only when D is abelian, where $\mathrm{def}(\chi) := \log_\ell(|G|_\ell/\chi(1)_\ell)$ is the *defect of* χ and $Z(D)$ denotes the centre of the defect group D. This can be considered as a block-wise analogue of the well-known fact that $\chi(1)$ divides $|G : Z(G)|$ for any irreducible character $\chi \in \mathrm{Irr}(G)$. We combine our results on block distribution with information on defect groups from [29] to verify Robinson's conjecture for isolated 2-blocks of exceptional type groups and thus complete the proof of this conjecture:

Theorem 3. *Robinson's conjecture holds for all blocks of all finite groups.*

The paper is built up as follows. In Section 2 we recall and collect some background results in particular on Jordan decomposition of characters. In Section 3 we extend some of our earlier results to the present setting, based upon which, in Section 4 we prove our main results Theorem 1 and Corollary 2. In Section 5 we apply Clifford theory to describe the blocks of quasi-simple groups of types E_6 and E_7. In Section 6 we correct and extend results in [21]; in particular we parametrise the isolated ℓ-blocks of simple groups of adjoint exceptional type and prove the theorems stated in Section 3. Finally, the proof of Robinson's conjecture is given in Section 7.

Acknowledgements: Our general approach and several of our proofs are heavily inspired by the pioneering impressive work of Michel Enguehard, to whom this paper is dedicated. We thank the anonymous referee for their thorough reading of our paper and the helpful comments and questions.

2. On Jordan decomposition and block distribution

We refer to [17] for basic notions from Deligne–Lusztig character theory.

2.1. **Notation and background results.** Throughout this subsection, $\mathbf{G}$ is a connected reductive linear algebraic group over an algebraic closure of a finite field, and $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius endomorphism endowing $\mathbf{G}$ with an $\mathbb{F}_q$-structure for some prime power q. By $\mathbf{G}^*$ we denote a group in duality with $\mathbf{G}$ with respect to some fixed F-stable maximal torus of $\mathbf{G}$, with corresponding Frobenius endomorphism also denoted by F. The notions recalled here make sense independently of whether $\mathbf{G}$ has connected centre or not.

For e a positive integer and any F-stable torus $\mathbf{T} \leq \mathbf{G}$, let $\mathbf{T}_e$ denote its Sylow e-torus (see e.g. [3] for terminology on Sylow e-theory). An F-stable Levi subgroup $\mathbf{L} \leq \mathbf{G}$ is called *e-split* if $\mathbf{L} = C_{\mathbf{G}}(Z^\circ(\mathbf{L})_e)$ or equivalently if $\mathbf{L} = C_{\mathbf{G}}(\mathbf{T})$ for some e-torus $\mathbf{T}$ of $\mathbf{G}$. A character $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$ is called *e-cuspidal* if ${}^*R^{\mathbf{L}}_{\mathbf{M}\leq\mathbf{P}}(\lambda) = 0$ for all proper e-split Levi subgroups $\mathbf{M} < \mathbf{L}$ and any parabolic subgroup $\mathbf{P}$ of $\mathbf{L}$ containing $\mathbf{M}$ as Levi complement, where ${}^*R^{\mathbf{L}}_{\mathbf{M}\leq\mathbf{P}}$ denotes Lusztig restriction. It is known that this property is independent of the chosen parabolic subgroup $\mathbf{P}$ unless possibly if $\mathbf{G}^F$ has a component of type ${}^2E_6(2)$ or $E_8(2)$ (see [17, Thm 3.3.8]).

Let $s \in \mathbf{G}^{*F}$ be semisimple. Choose some Jordan decomposition for $\mathbf{G}^F$ (as in [10, Thm 11.5.1]). Then $\chi \in \mathcal{E}(\mathbf{G}^F, s)$ is *e-Jordan-cuspidal* if $Z^\circ(C^\circ_{\mathbf{G}^*}(s))_e = Z^\circ(\mathbf{G}^*)_e$ and χ corresponds under Jordan decomposition to the $C_{\mathbf{G}^*}(s)^F$-orbit of an e-cuspidal unipotent character of $C^\circ_{\mathbf{G}^*}(s)^F$. If $\mathbf{L} \leq \mathbf{G}$ is e-split and $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$ is e-Jordan-cuspidal, then $(\mathbf{L}, \lambda)$ is called an *e-Jordan-cuspidal pair* of $\mathbf{G}$. By [6, Prop. 1.10(ii)], e-cuspidality implies e-Jordan-cuspidality; a list of situations where the converse is true is given in Remark 2.2 and Section 4 of [22] (see also Theorem 3.1(f)).

2.2. **Generalities on Jordan decomposition.** Suppose now that $\mathbf{G}$ is connected reductive with connected centre. Let $(\mathbf{G}^*, F)$ be dual to $(\mathbf{G}, F)$ with respect to a fixed duality. Fix a semisimple element $s \in \mathbf{G}^{*F}$, and denote as usual by $\mathcal{E}(C_{\mathbf{G}^*}(s)^F, 1)$ the set of unipotent characters of $C_{\mathbf{G}^*}(s)^F$ and deviating temporarily from the standard notation, denote by $\mathcal{E}(\mathbf{G}^F, \mathbf{G}^{*F}, s)$ (usually denoted $\mathcal{E}(\mathbf{G}^F, s)$) the Lusztig series of $\mathrm{Irr}(\mathbf{G}^F)$ corresponding to s. Denote by

$$\Psi_{\mathbf{G},\mathbf{G}^*,s} : \mathcal{E}(C_{\mathbf{G}^*}(s)^F, 1) \to \mathcal{E}(\mathbf{G}^F, \mathbf{G}^{*F}, s)$$

the inverse of the Jordan decomposition map of [9, Thm 7.1] (see also [31, Thm 2.1]).

For each pair $\mathbf{L} \leq \mathbf{G}$, $\mathbf{L}^* \leq \mathbf{G}^*$ of F-stable Levi subgroups in dual conjugacy classes, we fix a duality between $(\mathbf{L}, F)$ and $(\mathbf{L}^*, F)$ induced by the duality between $(\mathbf{G}, F)$ and

$(\mathbf{G}^*, F)$ as described for example in [10, Prop. 11.4.1] and the ensuing discussion). Note that if $\mathbf{L}_i$ (respectively $\mathbf{L}_i^*$), $i = 1, 2$, are $\mathbf{G}^F$-conjugate (respectively $\mathbf{G}^{*F}$-conjugate) F-stable Levi subgroups of $\mathbf{G}$ (respectively $\mathbf{G}^*$) in dual conjugacy classes, then for any $g^* \in \mathbf{G}^{*F}$ such that $\mathbf{L}_1^* = {}^{g^*}\mathbf{L}_2^*$, there exists $g \in \mathbf{G}^F$ such that $\mathbf{L}_2 = {}^g\mathbf{L}_1$ and such that conjugation by g and g^* are dual isomorphisms (in the sense of [17, Sec. 1.7.11] or [31, Def. 4.4]) from $\mathbf{L}_1$ to $\mathbf{L}_2$ and from $\mathbf{L}_2^*$ to $\mathbf{L}_1^*$, respectively. In this situation, if $s \in \mathbf{L}_1^{*F}$, then setting $t = g^{*-1} s g^*$, [9, Thm 7.1(vi)] gives

$$(\dagger) \qquad \Psi_{\mathbf{L}_2,\mathbf{L}_2^*,t}(\chi) = {}^g(\Psi_{\mathbf{L}_1,\mathbf{L}_1^*,s}({}^{g^*}\chi)) \qquad \text{for all } \chi \in \mathcal{E}(C_{\mathbf{L}_2^*}(t)^F, 1).$$

Here ${}^{g^*}\chi$ is the character of $C_{\mathbf{L}_1^*}(s)^F$ defined by ${}^{g^*}\chi(x) = \chi(g^{*-1}xg^*)$, $x \in C_{\mathbf{L}_1^*}(s)^F$, and similarly, for any $\tau \in \mathrm{Irr}(\mathbf{L}_1^F)$, ${}^g\tau$ is the character of $\mathbf{L}_2^F$ defined by ${}^g\tau(y) = \tau(g^{-1}yg)$, $y \in \mathbf{L}_2^F$. Further, if $\mathbf{L} = \mathbf{L}_1 = \mathbf{L}_2$, $\mathbf{L}^* = \mathbf{L}_1^* = \mathbf{L}_2^*$, and $g^* \in \mathbf{L}^{*F}$ then g can be chosen in $\mathbf{L}^F$. In this case, since inner automorphisms act trivially on characters, the above equation yields $\Psi_{\mathbf{L},\mathbf{L}^*,t}(\chi) = \Psi_{\mathbf{L},\mathbf{L}^*,s}({}^{g^*}\chi)$ for all $\chi \in \mathcal{E}_\ell(C_{\mathbf{L}^*}(t)^F, 1)$.

We say that a pair $(\mathbf{L}, \lambda)$, where $\mathbf{L} \le \mathbf{G}$ is an F-stable Levi subgroup and $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$, *lies below* $(\mathbf{G}^F, s)$ if $\lambda \in \mathcal{E}(\mathbf{L}^F, \mathbf{L}^{*F}, s)$ for some $\mathbf{L}^*$ containing s and in duality with $\mathbf{L}$. We say that a $\mathbf{G}^F$-class of pairs $(\mathbf{L}, \lambda)$ lies below $(\mathbf{G}^F, s)$ if some element of the class lies below $(\mathbf{G}^F, s)$. Let $\mathbf{C}^* := C_{\mathbf{G}^*}(s)$.

Lemma 2.1. *There is a bijection between $\mathbf{C}^{*F}$-classes of pairs $(\mathbf{L}^*, \mu)$, for $\mathbf{L}^* \le \mathbf{G}^*$ an F-stable Levi subgroup with $s \in \mathbf{L}^*$ and $\mu \in \mathcal{E}(C_{\mathbf{L}^*}(s)^F, 1)$, and $\mathbf{G}^F$-classes of pairs below $(\mathbf{G}^F, s)$. It sends the $\mathbf{C}^{*F}$-class of $(\mathbf{L}^*, \mu)$ to the $\mathbf{G}^F$-class of $(\mathbf{L}, \Psi_{\mathbf{L},\mathbf{L}^*,s}(\mu))$ where $\mathbf{L}$ is dual to $\mathbf{L}^*$.*

Proof. The proof is a consequence of $(\dagger)$. We give the details. First note that for any dual pair $\mathbf{L}, \mathbf{L}^*$ of F-stable Levi subgroups with $s \in \mathbf{L}^*$ and any $\mu \in \mathcal{E}(C_{\mathbf{L}^*}(s)^F, 1)$, the pair $(\mathbf{L}, \Psi_{\mathbf{L},\mathbf{L}^*,s}(\mu))$ lies below $(\mathbf{G}^F, s)$. Suppose that $(\mathbf{L}_i, \lambda_i)$, $i = 1, 2$ are two pairs such that $\mathbf{L}_i$ is dual to $\mathbf{L}^*$ and $\lambda_i = \Psi_{\mathbf{L}_i,\mathbf{L}^*,s}(\mu)$. Applying $(\dagger)$ with $\mathbf{L}_2^* = \mathbf{L}_1^*$ and $g^* = 1$ yields that $(\mathbf{L}_1, \lambda_1)$ and $(\mathbf{L}_2, \lambda_2)$ are $\mathbf{G}^F$-conjugate. Thus there is a well-defined and surjective map from the set of pairs $(\mathbf{L}^*, \mu)$, $\mathbf{L}^*$ an F-stable Levi subgroup of $\mathbf{G}^*$ with $s \in \mathbf{L}^*$ and $\mu \in \mathcal{E}(C_{\mathbf{L}^*}(s)^F, 1)$, to the set of $\mathbf{G}^F$-classes of pairs below $(\mathbf{G}^F, s)$ sending the pair $(\mathbf{L}^*, \mu)$ to the class of $(\mathbf{L}, \Psi_{\mathbf{L},\mathbf{L}^*,s}(\mu))$ where $\mathbf{L}$ is dual to $\mathbf{L}^*$.

Now let $(\mathbf{L}_i^*, \mu_i)$, $i = 1, 2$, be such that $\mathbf{L}_i^*$ is an F-stable Levi subgroup of $\mathbf{G}^*$, $s \in \mathbf{L}_i^*$, $\mu_i \in \mathcal{E}(C_{\mathbf{L}_i^*}(s)^F, 1)$. If $(\mathbf{L}_1^*, \mu_1)$ and $(\mathbf{L}_2^*, \mu_2)$ are $\mathbf{C}^{*F}$-conjugate, say $(\mathbf{L}_2^*, \mu_2) = {}^x(\mathbf{L}_1^*, \mu_1)$, $x \in \mathbf{C}^{*F}$, then applying $(\dagger)$ with $g^* = x^{-1}$ and any $\mathbf{L}_1 = \mathbf{L}_2$ dual to the $\mathbf{L}_i^*$ yields that $(\mathbf{L}_1^*, \mu_1)$ and $(\mathbf{L}_2^*, \mu_2)$ have the same image under the above map. Conversely, suppose that $(\mathbf{L}_1^*, \mu_1)$ and $(\mathbf{L}_2^*, \mu_2)$ have the same image. To complete the proof it suffices to show they are $\mathbf{C}^{*F}$-conjugate. Let $\mathbf{L}$ be dual to the $\mathbf{L}_i^*$s and set $\lambda_i = \Psi_{\mathbf{L},\mathbf{L}_i^*,s}(\mu_i)$. By hypothesis, there exists $g \in N_{\mathbf{G}}(\mathbf{L})^F$ such that $\lambda_2 = {}^g\lambda_1 = {}^g\Psi_{\mathbf{L},\mathbf{L}_1^*,s}(\mu_1)$. So, by $(\dagger)$, there exists $g^* \in \mathbf{G}^{*F}$ such that $\mathbf{L}_1^* = {}^{g^*}\mathbf{L}_2^*$ and

$$\lambda_2 = {}^g(\Psi_{\mathbf{L},\mathbf{L}_1^*,s}(\mu_1)) = \Psi_{\mathbf{L},\mathbf{L}_2^*,t}({}^{g^{*-1}}\mu_1)$$

with $t = g^{*^{-1}} s g^*$. In particular, $\lambda_2 \in \mathcal{E}(\mathbf{L}^F, \mathbf{L}_2^{*F}, t)$. Since by assumption $\lambda_2 \in \mathcal{E}(\mathbf{L}^F, \mathbf{L}_2^{*F}, s)$, t and s are $\mathbf{L}_2^{*F}$-conjugate, say $t = h^{*-1} s h^*$ with $h^* \in \mathbf{L}_2^{*F}$. Thus, by the remarks after $(\dagger)$

applied with h^* in place of g^*, $\mathbf{L}_2^*$ in place of $\mathbf{L}^*$ and ${}^{g^{*-1}}\mu_1$ in place of χ,

$$\lambda_2 = \Psi_{\mathbf{L},\mathbf{L}_2^*,t}({}^{g^{*-1}}\mu_1) = \Psi_{\mathbf{L},\mathbf{L}_2^*,s}({}^{h^*g^{*-1}}\mu_1).$$

Since we also have $\lambda_2 = \Psi_{\mathbf{L},\mathbf{L}_2^*,s}(\mu_2)$, it follows that $\mu_2 = {}^{h^*g^{*-1}}\mu_1$. As $h^* \in \mathbf{L}_2^{*F}$ we obtain

$$(\mathbf{L}_2^*, \mu_2) = {}^{h^*g^{*-1}}(\mathbf{L}_1^*, \mu_1)$$

with $h^*g^{*-1} \in \mathbf{C}^{*F}$, as required. □

From now on for any pair of dual F-stable Levi subgroups $\mathbf{L}, \mathbf{L}^*$ with $s \in \mathbf{L}^*$, we revert to the notation $\mathcal{E}(\mathbf{L}^F, s)$ for $\mathcal{E}(\mathbf{L}^F, \mathbf{L}^{*F}, s)$ and if $s \in \mathbf{L}^{*F}$ we denote by

$$\pi_s^{\mathbf{L}} : \mathcal{E}(\mathbf{L}^F, s) \longrightarrow \mathcal{E}(C_{\mathbf{L}^*}(s)^F, 1)$$

the Jordan decomposition inverse to $\Psi_{\mathbf{L},\mathbf{L}^*,s}$.

We are going to define some maps between e-Jordan cuspidal pairs which will then induce maps between ℓ-blocks.

Proposition 2.2. *For any $e \geq 1$, there is a natural bijection between the $\mathbf{C}^{*F}$-classes of unipotent e-cuspidal pairs in $\mathbf{C}^*$ and the $\mathbf{G}^F$-classes of e-Jordan cuspidal pairs below $(\mathbf{G}^F, s)$. It sends the class of $(\mathbf{L}_s^*, \lambda_s)$ to the class of $(\mathbf{L}, \lambda)$, with $\mathbf{L}^* = C_{\mathbf{G}^*}(Z^\circ(\mathbf{L}_s^*)_e)$ and $\lambda_s = \pi_s^{\mathbf{L}}(\lambda)$, where $\mathbf{L}, \mathbf{L}^*$ are in duality.*

Proof. First of all, we note that the maps $\mathbf{M}^* \mapsto C_{\mathbf{G}^*}(Z^\circ(\mathbf{M}^*)_e)$, $\mathbf{L}^* \mapsto C_{\mathbf{L}^*}(s)$ are mutually inverse bijections between the set of e-split Levi subgroups $\mathbf{M}^*$ of $\mathbf{C}^*$ and the set of e-split Levi subgroups $\mathbf{L}^*$ of $\mathbf{G}^*$ which contain s and for which additionally $Z^\circ(C_{\mathbf{L}^*}(s))_e = Z^\circ(\mathbf{L}^*)_e$. These bijections are compatible with the action of $\mathbf{C}^{*F}$ and hence induce inverse bijections between the set of $\mathbf{C}^{*F}$-classes of e-split Levi subgroups of $\mathbf{C}^*$ and the set of $\mathbf{C}^{*F}$-classes of e-split Levi subgroups $\mathbf{L}^*$ of $\mathbf{G}^*$ which contain s and for which additionally $Z^\circ(C_{\mathbf{L}^*}(s))_e = Z^\circ(\mathbf{L}^*)_e$.

Now the composition of the bijection which sends the $\mathbf{C}^{*F}$-class of $(\mathbf{L}_s, \lambda_s)$ to the $\mathbf{C}^{*F}$-class of $(C_{\mathbf{G}^*}(Z^\circ(\mathbf{L}_s)_e), \lambda_s)$ with the bijection of Lemma 2.1 yields the result. □

2.3. **Compatibility with central products.** We state a compatibility result of central products with various constructions which surely is well-known to the experts. We begin with a general fact. Recall that an isotypy between algebraic groups $\mathbf{X}_0$, $\mathbf{X}$ is a morphism of algebraic groups $f : \mathbf{X}_0 \to \mathbf{X}$ with central kernel and with image containing $[\mathbf{X}, \mathbf{X}]$.

Lemma 2.3. *Let $f : \mathbf{X}_0 \to \mathbf{X}$ be an isotypy of connected reductive algebraic groups.*

(a) *The map $\mathbf{L} \mapsto \mathbf{L}_0 := f^{-1}(\mathbf{L})$ induces a bijection between the sets of Levi subgroups of $\mathbf{X}$ and of $\mathbf{X}_0$. If $\mathbf{L}$ and $\mathbf{L}_0$ correspond, then $\mathbf{L} = Z(\mathbf{X})f(\mathbf{L}_0)$ and $[\mathbf{L}, \mathbf{L}] = f([\mathbf{L}_0, \mathbf{L}_0])$. In particular, $f : \mathbf{L}_0 \to \mathbf{L}$ is an isotypy.*

(b) *Let $s \in \mathbf{X}_0$ be a semisimple element such that $C_{\mathbf{X}_0}(s)$ and $C_{\mathbf{X}}(f(s))$ are both connected. Then the induced map $C_{\mathbf{X}_0}(s) \to C_{\mathbf{X}}(f(s))$ is an isotypy. Suppose that $\mathbf{L}_0$ is a Levi subgroup of $\mathbf{X}_0$ containing s and let $\mathbf{L} \leq \mathbf{X}$ be the Levi subgroup corresponding to $\mathbf{L}_0$ via f. Then the Levi subgroups $C_{\mathbf{L}_0}(s) \leq C_{\mathbf{X}_0}(s)$ and $C_{\mathbf{L}}(f(s)) \leq C_{\mathbf{X}}(f(s))$ correspond via the induced isotypy between $C_{\mathbf{X}_0}(s)$ and $C_{\mathbf{X}}(f(s))$.*

Proof. Part (a) is well-known. For the first assertion of (b), it suffices to show that $f(C_{\mathbf{X}_0}(s))$ contains $[C_{\mathbf{X}}(f(s)), C_{\mathbf{X}}(f(s))]$. Let $\mathbf{X}_1 := f^{-1}(C_{\mathbf{X}}(f(s)))$. Since $[\mathbf{X},\mathbf{X}] \leq f(X_0)$, $C_{[\mathbf{X},\mathbf{X}]}(f(s)) \leq f(\mathbf{X}_1)$. On the other hand, since $\mathbf{X} = Z(\mathbf{X})[\mathbf{X},\mathbf{X}]$, we have $C_{\mathbf{X}}(f(s)) = Z(\mathbf{X})C_{[\mathbf{X},\mathbf{X}]}(f(s))$. Thus, $C_{\mathbf{X}}(f(s)) = Z(\mathbf{X})f(\mathbf{X}_1)$. Since $\ker(f)$ is central in $\mathbf{X}_0$, $[\mathbf{X}_1, s] \leq Z(\mathbf{X}_0)$ and the map $\mathbf{X}_1 \to Z(\mathbf{X}_0)$ sending $x \in \mathbf{X}_1$ to $[x, s]$ is a group homomorphism with kernel $C_{\mathbf{X}_0}(s)$. In particular, $\mathbf{X}_1/C_{\mathbf{X}_0}(s)$ is abelian, hence so is $f(\mathbf{X}_1)/f(C_{\mathbf{X}_0}(s))$ and we obtain

$$f(C_{\mathbf{X}_0}(s)) \geq [f(\mathbf{X}_1), f(\mathbf{X}_1)] = [C_{\mathbf{X}}(f(s)), C_{\mathbf{X}}(f(s))].$$

The second assertion of (b) follows from part (a). $\square$

Suppose that $\mathbf{G} = \mathbf{G}_1\mathbf{G}_2$ is a central product of connected reductive, connected centre and F-stable subgroups $\mathbf{G}_i$, $i = 1, 2$. Let $\tilde{\mathbf{G}} = \mathbf{G}_1 \times \mathbf{G}_2$ and let $\varphi : \tilde{\mathbf{G}} \to \mathbf{G}$ be the canonical epimorphism given by multiplication. We assume $\ker(\varphi)$ is a (central) torus of $\tilde{\mathbf{G}}$. In particular, φ is an isotypy. Let $\varphi^* : \mathbf{G}^* \to \tilde{\mathbf{G}}^*$ denote a dual isotypy and note that $\tilde{\mathbf{G}}^* = \mathbf{G}_1^* \times \mathbf{G}_2^*$, with $\mathbf{G}_i^*$ dual to $\mathbf{G}_i$. For an F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ with dual Levi subgroup $\mathbf{L}^* \leq \mathbf{G}^*$, let $\tilde{\mathbf{L}} := \varphi^{-1}(\mathbf{L})$ be the corresponding Levi subgroup of $\tilde{\mathbf{G}}$. Then $\tilde{\mathbf{L}} = \mathbf{L}_1 \times \mathbf{L}_2$ with $\mathbf{L}_i$ an F-stable Levi subgroup of $\mathbf{G}_i$, with dual $\tilde{\mathbf{L}}^* := Z^\circ(\tilde{\mathbf{G}}^*)\varphi^*(\mathbf{L}^*)$ of the form $\mathbf{L}_1^* \times \mathbf{L}_2^*$, with $\mathbf{L}_i^*$ dual to $\mathbf{L}_i$. Note that the Levi subgroups $\mathbf{L}^* \leq \mathbf{G}^*$ and $\tilde{\mathbf{L}}^* \leq \tilde{\mathbf{G}}^*$ correspond to each other via φ^*.

Let $s \in \mathbf{L}^{*F}$ be semisimple. Then $\varphi^*(s) = (s_1, s_2)$ with $s_i \in \mathbf{L}_i^{*F}$ and $C_{\tilde{\mathbf{L}}^*}(\varphi^*(s)) = C_{\mathbf{L}_1^*}(s_1) \times C_{\mathbf{L}_2^*}(s_2)$. Further, $C_{\mathbf{L}^*}(s)$ and $C_{\tilde{\mathbf{L}}^*}(\varphi^*(s))$ are corresponding Levi subgroups under the isotypy $\varphi^* : C_{\mathbf{G}^*}(s) \to C_{\tilde{\mathbf{G}}^*}(\varphi^*(s))$ (see Lemma 2.3). So, φ^* induces a canonical bijection (see [10, Prop. 11.3.8])

$$\hat{\ } : \mathcal{E}(C_{\mathbf{L}^*}(s)^F, 1) \to \mathcal{E}(C_{\tilde{\mathbf{L}}^*}(\varphi^*(s))^F, 1);$$

we write $\hat{\alpha} = \alpha_1 \otimes \alpha_2$ with $\alpha_i \in \mathcal{E}(C_{\mathbf{L}_i^*}(s_i)^F, 1)$. For $\lambda \in \mathcal{E}(\mathbf{L}^F, s)$, set $\tilde{\lambda} := \lambda \circ \varphi$. Then $\tilde{\lambda} \in \mathcal{E}(\tilde{\mathbf{L}}^F, \varphi^*(s))$ and is of the form $\lambda_1 \otimes \lambda_2$, $\lambda_i \in \mathcal{E}(\mathbf{L}_i^F, s_i)$. By properties of Jordan decomposition [9, Thm 7.1(vi),(vii)], we have that if $\alpha = \pi_s^{\mathbf{L}}(\lambda)$, then $\hat{\alpha} = \pi_{\varphi^*(s)}^{\tilde{\mathbf{L}}}(\tilde{\lambda})$ and $\alpha_i = \pi_s^{\mathbf{L}_i}(\lambda_i)$, $i = 1, 2$.

Proposition 2.4. *Let $\mathbf{L} \leq \mathbf{G}$ be an F-stable Levi subgroup with dual Levi subgroup $\mathbf{L}^* \leq \mathbf{G}^*$ and let $s \in \mathbf{L}^{*F}$ be semisimple. With the notation above, and denoting by X' the derived subgroup of a group X, we have the following.*

(a) *$(\mathbf{L}, \lambda)$ is an e-Jordan cuspidal pair for $\mathbf{G}$ if and only if $(\tilde{\mathbf{L}}, \tilde{\lambda})$ is an e-Jordan cuspidal pair for $\tilde{\mathbf{G}}$, if and only if $(\mathbf{L}_i, \lambda_i)$ are e-Jordan cuspidal pairs for $\mathbf{G}_i$, $i = 1, 2$.*

(b) *Suppose that $(\mathbf{L}, \lambda)$ is an e-Jordan cuspidal pair for $\mathbf{G}$. If $(\mathbf{L}, \lambda)$ corresponds to the unipotent e-cuspidal pair $(C_{\mathbf{L}^*}(s), \alpha)$ of $C_{\mathbf{G}^*}(s)$ via Proposition 2.2, then $(\tilde{\mathbf{L}}, \tilde{\lambda})$ corresponds to the unipotent e-cuspidal pair $(C_{\tilde{\mathbf{L}}^*}(\varphi^*(s)), \hat{\alpha})$ of $C_{\tilde{\mathbf{G}}^*}(\varphi^*(s))$ and $(\mathbf{L}_i, \lambda_i)$ correspond to the unipotent e-cuspidal pairs $(C_{\mathbf{L}_i^*}(s_i), \alpha_i)$ of $C_{\mathbf{G}_i^*}(s_i)$ for $i = 1, 2$.*

(c) *Let $\mathbf{M} \leq \mathbf{G}$ be an F-stable Levi subgroup with $\mathbf{L} \leq \mathbf{M}$ and let $\chi \in \mathrm{Irr}(\mathbf{M}^F)$. Then, keeping the notational convention as above, χ is a constituent of $R_{\mathbf{L}}^{\mathbf{M}}(\lambda)$ if and only if $\tilde{\chi} = \chi_1 \otimes \chi_2$ with χ_i a constituent of $R_{\mathbf{L}_i}^{\mathbf{M}_i}(\lambda_i)$.*

(d) *Let* $\mathbf{M} \leq \mathbf{G}$ *be an* F*-stable Levi subgroup with dual Levi subgroup* $\mathbf{M}^* \leq \mathbf{G}^*$ *containing* s *and let* $\beta \in \mathcal{E}(C_{\mathbf{M}^*}(s)^F, 1)$. *Suppose that*

$$(C_{\mathbf{M}^*}(s)', \beta|_{C_{\mathbf{M}^*}(s)'^F}) \qquad \textit{and} \qquad (C_{\mathbf{L}^*}(s)', \alpha|_{C_{\mathbf{L}^*}(s)'^F})$$

are $C_{\mathbf{G}^*}(s)^F$*-conjugate. Then,*

$$(C_{\tilde{\mathbf{M}}^*}(\varphi^*(s))', \hat{\beta}|_{C_{\tilde{\mathbf{M}}^*}(\varphi^*(s))'^F}) \qquad \textit{and} \qquad (C_{\tilde{\mathbf{L}}^*}(\varphi^*(s))', \hat{\alpha}|_{C_{\tilde{\mathbf{L}}^*}(\varphi^*(s))'^F})$$

are $C_{\tilde{\mathbf{G}}^*}(\varphi^*(s))^F$*-conjugate and consequently,*

$$(C_{\mathbf{M}_i^*}(s_i)', \beta_i|_{C_{\mathbf{M}_i^*}(s_i)'^F}) \qquad \textit{and} \qquad (C_{\mathbf{L}_i^*}(s_i)', \alpha_i|_{C_{\mathbf{L}_i^*}(s_i)'^F})$$

are $C_{\mathbf{G}_i^*}(s_i)^F$*-conjugate for* $i = 1, 2$.

Proof. Parts (a) and (b) are implicit in [13, Prop. 2.1.5] and part (c) is a consequence of the commutation of Lusztig induction with φ, see [9, Cor. 9.2]. The first assertion of part (d) follows from Lemma 2.3 and the bijection between unipotent characters induced by isotypies. The second assertion follows from the first through properties of direct products. □

2.4. **Generalities on block distribution.** Let ℓ be a prime not dividing q and let $e := e_\ell(q)$ denote the order of q modulo ℓ if ℓ is odd and the order of q modulo 4 if $\ell = 2$. We denote by $\mathcal{E}(\mathbf{G}^F, \ell')$ the set of irreducible characters of $\mathbf{G}^F$ lying in a Lusztig series $\mathcal{E}(\mathbf{G}^F, s)$ for some semisimple ℓ'-element $s \in \mathbf{G}^{*F}$. Recall from [21, Def. 2.4] that a character $\chi \in \mathcal{E}(\mathbf{G}^F, \ell')$ is said to be of *central* ℓ*-defect* if $|\chi(1)|_\ell |Z(\mathbf{G})^F|_\ell = |\mathbf{G}^F|_\ell$ or equivalently if the ℓ-block of $\mathbf{G}^F$ containing χ has a central defect group (see [21, Prop. 2.5(c)]) and χ is said to be of *quasi-central* ℓ*-defect* if some (and hence any) character of $[\mathbf{G}, \mathbf{G}]^F$ covered by χ is of central ℓ-defect. By [21, Prop. 2.5(b)], if χ is of central ℓ-defect, then it is of quasi-central ℓ-defect. For unipotent characters, the converse holds and the terms "quasi-central ℓ-defect" and "central ℓ-defect" are used interchangeably:

Lemma 2.5. *Any* $\chi \in \mathcal{E}(\mathbf{G}^F, 1)$ *of quasi-central* ℓ*-defect is of central* ℓ*-defect and is also* e*-cuspidal.*

Proof. Let b be the ℓ-block of $\mathbf{G}^F$ containing χ. By [21, Prop. 2.5(f)], χ is the unique unipotent character in its ℓ-block. On the other hand, by [11, Thms A and A.bis], $\mathrm{Irr}(b) \cap \mathcal{E}(\mathbf{G}^F, 1)$ contains the e-Harish-Chandra series of $\mathbf{G}^F$ above some e-cuspidal unipotent pair $(\mathbf{L}, \lambda)$ with λ of central ℓ-defect. But this means that $\mathbf{L} = \mathbf{G}$ and $\lambda = \chi$. □

We record here another fact that will be used later.

Lemma 2.6. *Suppose that all components of* $[\mathbf{G}, \mathbf{G}]$ *are of classical type and let* $\chi \in \mathcal{E}(\mathbf{G}^F, 1)$. *If* χ *is of (quasi)-central* 2*-defect, then* $\mathbf{G}$ *is a torus.*

Proof. By [4, Thm 13] and the table in [11, p. 348], the principal 2-block is the only unipotent 2-block of $\mathbf{G}^F$. Hence the hypothesis implies that the principal 2-block of $\mathbf{G}^F$ has central defect groups, that is, the Sylow 2-subgroups of $\mathbf{G}^F$ are contained in $Z(\mathbf{G}^F)$. The result follows. □

Now fix $s \in \mathbf{G}^{*F}$ a semisimple ℓ'-element. The following is an extension of the inductive argument given on Page 367 of [11]. For an F-stable subgroup $\mathbf{H}$ of $\mathbf{G}$ we denote by $d^{1,\mathbf{H}}$ the decomposition map on the set of class functions of $\mathbf{H}^F$ defined by $d^{1,\mathbf{H}}(\chi)(x) := \chi(x)$ if $x \in \mathbf{H}^F$ is ℓ-regular and $d^{1,\mathbf{H}}(\chi)(x) := 0$ otherwise. Note that we are diverging from the more customary (but also more clumsy) notation $d^{1,\mathbf{H}^F}$.

Lemma 2.7. *Let $t \in C_{\mathbf{G}^*}(s)^F$ be an ℓ-element and $\chi \in \mathcal{E}(\mathbf{G}^F, st)$. Then there exists $\gamma \in \mathcal{E}(\mathbf{G}^F, s)$ with $\langle d^{1,\mathbf{G}}(\chi), \gamma\rangle \neq 0$.*

Proof. The regular character of $\mathbf{G}^F$ is a linear combination of class functions of the form $R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{reg}_{\mathbf{T}^F})$, where $\mathbf{T}$ runs over the F-stable maximal tori of $\mathbf{G}$ and $\mathrm{reg}_{\mathbf{T}^F}$ is the regular character of $\mathbf{T}^F$ (see [10, Cor. 10.2.6]). Hence, there exists some F-stable maximal torus $\mathbf{T} \leq \mathbf{G}$ such that

$$a := {}^*R_{\mathbf{T}}^{\mathbf{G}}(\chi)(1) = \big\langle {}^*R_{\mathbf{T}}^{\mathbf{G}}(\chi), \mathrm{reg}_{\mathbf{T}^F}\big\rangle = \big\langle \chi, R_{\mathbf{T}}^{\mathbf{G}}(\mathrm{reg}_{\mathbf{T}^F})\big\rangle \neq 0.$$

Let ${}^*R_{\mathbf{T}}^{\mathbf{G}}(\chi) = \sum_{\sigma,\tau} a_{\sigma,\tau}\sigma\tau$ where σ runs over $\mathrm{Irr}(\mathbf{T}^F)_{\ell'}$ and τ runs over $\mathrm{Irr}(\mathbf{T}^F)_\ell$ and note that $a = \sum_{\sigma,\tau} a_{\sigma,\tau}$. Let $\sigma \in \mathrm{Irr}(\mathbf{T}^F)_{\ell'}$ with $\sum_\tau a_{\sigma,\tau} \neq 0$. For any $\tau \in \mathrm{Irr}(\mathbf{T}^F)_\ell$, we have $d^{1,\mathbf{T}}(\sigma\tau) = d^{1,\mathbf{T}}(\sigma)$ and hence

$$\big\langle d^{1,\mathbf{G}}(\chi), R_{\mathbf{T}}^{\mathbf{G}}(\sigma)\big\rangle = \big\langle d^{1,\mathbf{T}}({}^*R_{\mathbf{T}}^{\mathbf{G}}(\chi)), \sigma\big\rangle = \frac{1}{|\mathbf{T}_\ell^F|}\sum_\tau a_{\sigma,\tau} \neq 0.$$

Thus, there is a constituent γ of $R_{\mathbf{T}}^{\mathbf{G}}(\sigma)$ with $\big\langle d^{1,\mathbf{G}}(\chi), \gamma\big\rangle \neq 0$. By the above displayed equation, we have $a_{\sigma,\tau} \neq 0$ for some $\tau \in \mathrm{Irr}(\mathbf{T}^F)_\ell$, implying $\gamma \in \mathcal{E}(\mathbf{G}^F, s)$ as desired. $\square$

Now assume $\mathbf{G}$ has connected centre. For any ℓ-element $t \in \mathbf{C}^{*F}$ recall the Digne–Michel Jordan decomposition

$$\pi_{st}^{\mathbf{G}} : \mathcal{E}(\mathbf{G}^F, st) \longrightarrow \mathcal{E}(C_{\mathbf{G}^*}(st)^F, 1)$$

discussed above. For any maximal torus $\mathbf{T}^* \leq C_{\mathbf{G}^*}(s)$ denote by $\hat{s} \in \mathrm{Irr}(\mathbf{T}^F)$ the linear character associated to s by duality [7, (8.14)].

Lemma 2.8. *Let $t \in C_{\mathbf{G}^*}(s)^F$ be an ℓ-element and $\chi \in \mathcal{E}(\mathbf{G}^F, st)$. Let $\mathbf{T}^* \leq C_{\mathbf{G}^*}(st)$ be a maximal torus such that $\langle \pi_{st}^{\mathbf{G}}(\chi), R_{\mathbf{T}^*}^{C_{\mathbf{G}^*}(st)}(1)\rangle \neq 0$ and $\mathbf{T}^*$ is $C_{\mathbf{G}^*}(st)^F$-conjugate to any of its $\mathbf{G}^{*F}$-conjugates contained in $C_{\mathbf{G}^*}(st)$. Then χ lies in the same ℓ-block of $\mathbf{G}^F$ as some constituent of $R_{\mathbf{T}}^{\mathbf{G}}(\hat{s})$.*

Proof. Since $\langle \pi_{st}^{\mathbf{G}}(\chi), R_{\mathbf{T}^*}^{C_{\mathbf{G}^*}(st)}(1)\rangle \neq 0$, by the properties of Jordan decomposition we have $\langle \chi, R_{\mathbf{T}}^{\mathbf{G}}(\widehat{st})\rangle \neq 0$, for $\mathbf{T}$ dual to $\mathbf{T}^*$. Now under our assumption on $\mathbf{T}^*$, by [11, Lemma 21] we have

$${}^*R_{\mathbf{T}}^{\mathbf{G}}(\chi) = m\sum_w \widehat{st}^w$$

for w ranging over certain elements of the relative Weyl group of $\mathbf{T}$, including $w = 1$, and where $m \neq 0$. Then

$$\big\langle d^{1,\mathbf{G}}(\chi), R_{\mathbf{T}}^{\mathbf{G}}(\hat{s})\big\rangle = \big\langle d^{1,\mathbf{T}}({}^*R_{\mathbf{T}}^{\mathbf{G}}(\chi)), d^{1,\mathbf{T}}(\hat{s})\big\rangle = m \cdot \sum_w \big\langle d^{1,\mathbf{T}}(\hat{s})^w, d^{1,\mathbf{T}}(\hat{s})\big\rangle \neq 0.$$

Thus χ lies in the ℓ-block of some constituent of $R_{\mathbf{T}}^{\mathbf{G}}(\hat{s})$. $\square$

The next result uses the notation $\mathbf{G_a}$, $\mathbf{G_b}$ introduced in [5, Not. 2.3]. Recall that $Z^\circ(\mathbf{G}) \le \mathbf{G_a}$ while $\mathbf{G_b}$ is semisimple.

Lemma 2.9. *Suppose that $Z(\mathbf{G})$ is connected, $\mathbf{G} = \mathbf{G_a}$ and that ℓ is odd. Let $s \in \mathbf{G}^{*F}$ be a semisimple ℓ'-element. Then there is a unique $\mathbf{G}^F$-conjugacy class of e-Jordan cuspidal pairs below $(\mathbf{G}^F, s)$ and a unique ℓ-block in $\mathcal{E}_\ell(\mathbf{G}^F, s)$, say b. Further, denoting by $(\mathbf{L}, \lambda)$ an e-Jordan cuspidal pair below $(\mathbf{G}^F, s)$, every element of $\mathrm{Irr}(b) \cap \mathcal{E}(\mathbf{G}^F, s)$ is a constituent of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$.*

Proof. We have $C_{\mathbf{G}^*}(s)_{\mathbf{a}} = C_{\mathbf{G}^*}(s)$ (see for instance remark after [5, Not. 2.3]), hence by [5, Prop. 3.3], there is only one class of unipotent e-cuspidal pairs in $C_{\mathbf{G}^*}(s)$. The first assertion now follows from Proposition 2.2. The second assertion is a consequence of the first, the main theorem of [6], and the fact that an e-cuspidal pair is also e-Jordan cuspidal (see [6, Sec. 1.3]). Here we note that in [6] it is assumed that $\ell \ge 5$, but this is not necessary for the case that we are considering (the second assertion can also be obtained from the main theorem of [5] in combination with the Bonnafé–Rouquier Jordan decomposition theorem since $C_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$ in our case). The final assertion follows from the main theorem of [5] applied to $C_{\mathbf{G}^*}(s)$ and the fact that Jordan decomposition commutes with Lusztig induction in groups of type A (see [17, Thm 4.7.2]). □

For later use, we record here the following structural result.

Proposition 2.10. *Let $\mathbf{H}$ be connected reductive with $[\mathbf{H}, \mathbf{H}]$ simple, $F : \mathbf{H} \to \mathbf{H}$ a Frobenius map, $\mathbf{G} \le \mathbf{H}$ an F-stable Levi subgroup and let $s \in \mathbf{G}^*$ be semisimple. If $C_{\mathbf{G}^*}(s)$ has an e-split Levi subgroup, $e \in \{1, 2\}$, whose F-fixed points have a component of type ${}^3D_4$, then one of the following hold:*

(1) *$[\mathbf{H}, \mathbf{H}] = [\mathbf{G}, \mathbf{G}]$ is of type D_4, F induces triality and s is central; or*
(2) *$\mathbf{H}$ is of exceptional type, and either $\mathbf{G}$ is also of exceptional type E_n, $6 \le n \le 8$, or $\mathbf{G}^F$ has a component of type ${}^3D_4$ and s is central in $\mathbf{G}^*$ or non-isolated.*

Proof. Clearly we may replace $\mathbf{H}$ by $[\mathbf{H}, \mathbf{H}]$ and $\mathbf{G}$ by $\mathbf{G} \cap [\mathbf{H}, \mathbf{H}]$. Now first assume $\mathbf{H}$ is of classical type. Let $\mathbf{T}_0$ be a maximally split torus of $\mathbf{H}$, with Weyl group W, set of simple reflections S, and F acting by σ on W. Then there is some subset $I \subseteq S$ and $w \in N_W(W_I)$ such that $\mathbf{G}$ has Weyl group W_I and F acts by $w\sigma$ on W_I. Now by the explicit description in [20, p. 71] the normalisers of parabolic subgroups of W of type D_n do not induce a triality automorphism on any simple factor, so nor does $w\sigma$ unless σ itself is triality. Thus, except we are in the excluded case, $\mathbf{G}$ is a product of groups of classical type such that $\mathbf{G}^F$ has no component of type ${}^3D_4$. But the centralisers of semisimple elements in finite classical groups only possess classical components (see e.g. [16, Sect. 1]), and their e-split Levi subgroups then have the same property.

Now assume $\mathbf{H}$ is of exceptional type and $C_{\mathbf{G}^*}(s)$ has an e-split Levi subgroup whose F-fixed points have a component of type ${}^3D_4$. If $\mathbf{G}$ itself is not of exceptional type, then by rank considerations it has at most one factor of type D_n, where $4 \le n \le 8$, and type A-factors otherwise. In fact, there must be exactly one type D_n-factor since Levi subgroups of element centralisers in type A-groups do not have ${}^3D_4$-components. The F-fixed points of the unique D_n-factor are either finite orthogonal groups, but as seen above these do not possess semisimple elements with suitable Levi subgroups, or we have $n = 4$ and F

induces triality, so $\mathbf{G}^F$ has a component of type ${}^3D_4$. As all other components of $\mathbf{G}$ are of type A, their only isolated elements are central, whence our claim. □

Lemma 2.11. *Let $\mathbf{G}$ be simple of type D_4 in characteristic different from 3 and $F : \mathbf{G} \to \mathbf{G}$ with $\mathbf{G}^F = {}^3D_4(q)$. Let $\mathbf{T} \leq \mathbf{G}$ be the centraliser of a Sylow $e_3(q)$-torus. Then for all 3-elements $1 \neq t \in \mathbf{G}^{*F}$, $(\mathbf{T}, 1)$ is the unique unipotent $e_3(q)$-cuspidal pair of $C_{\mathbf{G}^*}(t)$ (up to conjugation).*

Proof. By inspection of [8, Tab. 2.2], for all 3-elements $t \neq 1$, $C_{\mathbf{G}^*}(t) = C_{\mathbf{G}^*}(t)_{\mathbf{a}}$ contains a conjugate of $\mathbf{T}$. The claim follows. □

3. On the ℓ-block distribution of ℓ'-characters

In this section we formulate some straightforward extensions of results obtained in our predecessor papers [21] and [22] to the following setting: let $\mathbf{X}$ be connected reductive with connected centre such that $[\mathbf{X}, \mathbf{X}]$ is simple of simply connected type, with a Frobenius map $F : \mathbf{X} \to \mathbf{X}$ with respect to an $\mathbb{F}_q$-structure. Let $\mathbf{G} \leq \mathbf{X}$ be an F-stable Levi subgroup. Thus, in particular, all centralisers of semisimple elements in $\mathbf{G}^*$ are connected. Let $\ell \not| q$ be a prime and set $e = e_\ell(q)$.

Recall that a character $\chi \in \mathcal{E}(\mathbf{G}^F, \ell')$ is called *e-Jordan quasi-central cuspidal* if χ is e-Jordan cuspidal and its Jordan correspondent is of quasi-central ℓ-defect (see [22, Def. 2.12]). An *e-Jordan quasi-central cuspidal pair of* $\mathbf{G}$ is a pair $(\mathbf{L}, \lambda)$ such that $\mathbf{L}$ is an e-split Levi subgroup of $\mathbf{G}$ and $\lambda \in \mathcal{E}(\mathbf{L}^F, \ell')$ is an e-Jordan quasi-central cuspidal character of $\mathbf{L}^F$.

3.1. **Parametrisation of ℓ-blocks in connected centre groups.** Parts (a), (b) and (c) of the following extend Theorems 1.2(a) and 1.4 of [21] whilst parts (d) and (e) were implicit from the computations made in [21] (where $\mathbf{X}$ was assumed simple of simply connected exceptional type):

Theorem 3.1. *Assume $\mathbf{G} = \mathbf{X}$ above and that ℓ is bad for $\mathbf{G}$, and let $s \in \mathbf{G}^{*F}$ be an isolated semisimple ℓ'-element. Then:*

(a) *There is a natural bijection $b_{\mathbf{G}^F}(\mathbf{L}, \lambda) \longleftrightarrow (\mathbf{L}, \lambda)$ between ℓ-blocks of $\mathbf{G}^F$ in $\mathcal{E}_\ell(\mathbf{G}^F, s)$ and e-cuspidal pairs $(\mathbf{L}, \lambda)$ below $(\mathbf{G}^F, s)$ of quasi-central ℓ-defect.*
(b) *The sets $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$, where $(\mathbf{L}, \lambda)$ runs over a set of representatives of the $\mathbf{G}^F$-classes of e-cuspidal pairs below $(\mathbf{G}^F, s)$, partition $\mathcal{E}(\mathbf{G}^F, s)$.*
(c) *$\mathbf{G}^F$ satisfies an e-Harish-Chandra theory above each e-cuspidal pair $(\mathbf{L}, \lambda)$ below $(\mathbf{G}^F, s)$.*
(d) *If two e-cuspidal pairs below $(\mathbf{G}^F, s)$ define the same block of $\mathbf{G}^F$, then their Jordan correspondents define the same unipotent block of $C_{\mathbf{G}^*}(s)^F$, up to twins.*
(e) *For any e-cuspidal pair $(\mathbf{L}, \lambda)$ below $(\mathbf{G}^F, s)$, Jordan decomposition commutes with $R_{\mathbf{L}}^{\mathbf{G}}$ up to twins.*
(f) *Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ such that $s \in \mathbf{L}^*$ and let $\lambda \in \mathcal{E}(\mathbf{L}^F, s)$. Then $(\mathbf{L}, \lambda)$ is an e-Jordan cuspidal pair of $\mathbf{G}$ if and only if $(\mathbf{L}, \lambda)$ is an e-cuspidal pair of $\mathbf{G}$. Further, if $(\mathbf{L}, \lambda)$ is an e-Jordan cuspidal pair of $\mathbf{G}$, then $(\mathbf{L}, \lambda)$ is e-Jordan quasi-central cuspidal if and only if λ is of quasi-central ℓ-defect.*

We will explain how this can be derived from the results of [21] in Section 6.4. The definition of twins will be given in Section 4.2.

We also need the following extension of Theorem A(a) and (b) and Theorem 3.4 of [22]:

Theorem 3.2. *Let $\mathbf{X}$ be as above, and $\mathbf{G} \leq \mathbf{X}$ an F-stable Levi subgroup.*

(a) *For any e-split Levi subgroup $\mathbf{M}$ of $\mathbf{G}$ and any ℓ-block c of $\mathbf{M}^F$, there exists a block b of $\mathbf{G}^F$ such that for every $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\mathbf{M}^F, \ell')$, all irreducible constituents of $R_{\mathbf{M}}^{\mathbf{G}}(\mu)$ lie in b.*

(b) *For any e-Jordan-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ such that $\lambda \in \mathcal{E}(\mathbf{L}^F, \ell')$, there exists a unique ℓ-block $b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ of $\mathbf{G}^F$ such that all irreducible constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ lie in $b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$.*

(c) *The map $\Xi : (\mathbf{L}, \lambda) \mapsto b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ induces a surjection from the set of $\mathbf{G}^F$-classes of e-Jordan-cuspidal pairs $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ with $\lambda \in \mathcal{E}(\mathbf{L}^F, \ell')$ to the set of ℓ-blocks of $\mathbf{G}^F$.*

(d) *The map Ξ restricts to a surjection from the set of $\mathbf{G}^F$-classes of e-Jordan quasi-central cuspidal pairs $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ with $\lambda \in \mathcal{E}(\mathbf{L}^F, \ell')$ to the set of ℓ-blocks of $\mathbf{G}^F$.*

Again, the proof will be given in Section 6.4. For future use, we also note:

Proposition 3.3. *Let $\mathbf{X}, \mathbf{G}$ be as above and let $s \in \mathbf{G}^{*F}$ be a semisimple ℓ'-element. If there is a unique class of unipotent e-cuspidal pairs of $\mathbf{C} := C_{\mathbf{G}^*}(s)$ of central defect, then $\mathcal{E}_\ell(\mathbf{G}^F, s)$ is a single ℓ-block. In particular, if $\ell = 2$ and $\mathbf{C}$ has only components of classical type then $\mathcal{E}_2(\mathbf{G}^F, s)$ is a single 2-block.*

Proof. Suppose that there is a unique class of unipotent e-cuspidal pairs of central defect of $\mathbf{C} = C_{\mathbf{G}^*}(s)$. Then by Lemma 2.5 and Proposition 2.2, there is a unique $\mathbf{G}^F$-class of e-Jordan quasi-central cuspidal pairs below $(\mathbf{G}^F, s)$. By Theorem 3.2(d), for every ℓ-block b in $\mathcal{E}_\ell(\mathbf{G}^F, s)$, there is a $\mathbf{G}^F$-class of e-Jordan quasi-central cuspidal pairs $(\mathbf{L}, \lambda)$ of $\mathbf{G}^F$ with $\lambda \in \mathcal{E}(\mathbf{L}^F, \ell')$ such that $b = b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$. Since $b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ contains the irreducible constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ and since Lusztig induction preserves Lusztig series (see, e.g., [17, Prop. 3.3.20]), the $\mathbf{G}^F$-class of $(\mathbf{L}, \lambda)$ lies below s. This proves the first assertion. If $\ell = 2$ and all components of $\mathbf{C}$ are of classical type, the principal block is the only unipotent 2-block of $\mathbf{C}^F$ (see for instance [7, Thm 21.14])). Hence, by Theorems A and A.bis of [11] there is only one $\mathbf{C}^F$-class of unipotent e-cuspidal pairs of $\mathbf{C}$ of central 2-defect. □

3.2. **e-Cuspidal pairs below $(\mathbf{G}^F, s)$.** In this subsection and elsewhere, we will freely use the fact if $\mathbf{G}$ is as above, $s \in \mathbf{G}^*$ is an ℓ'-element and $t \in C_{\mathbf{G}^*}(s)$ is an ℓ-element, then $C_{\mathbf{G}^*}(st) = C_{C_{\mathbf{G}^*(s)}}(t)$, and therefore also that $C_{C_{\mathbf{G}^*(s)}}(t)$ is connected if in addition s is semisimple.

Lemma 3.4. *Let $\mathbf{X}, \mathbf{G}$ be as above, $s \in \mathbf{G}^{*F}$ a semisimple ℓ'-element and assume that $[\mathbf{X}, \mathbf{X}]$ is of exceptional type. Then for any ℓ-element $t \in C_{\mathbf{G}^*}(s)^F$, if $(\mathbf{L}_t^*, \lambda_t)$ is a unipotent e-cuspidal pair of $C_{\mathbf{G}^*}(st)$ then $\mathbf{L}_t^*$ is a Levi subgroup of $C_{\mathbf{G}^*}(s)$.*

Proof. We argue by induction on $\dim \mathbf{G}$. Let $\mathbf{G}_1^* \leq \mathbf{G}^*$ be a minimal (F-stable) Levi subgroup containing $C_{\mathbf{G}^*}(s)$. If $\mathbf{G}_1^*$ is proper, then by induction $\mathbf{L}_t^*$ is a Levi subgroup of $C_{\mathbf{G}_1^*}(s)$ and thus of $C_{\mathbf{G}^*}(s)$. Thus we may assume s is isolated. Moreover, we may assume that ℓ is bad for $C_{\mathbf{G}^*}(s)$, since otherwise $C_{\mathbf{G}^*}(st)$ is already a Levi subgroup of $C_{\mathbf{G}^*}(s)$ (see [7, Prop. 13.16]) whence so is $\mathbf{L}_t^*$. Then from the list of isolated elements (see [1, Prop. 4.9 and Tab. 3] it transpires that for s non-central only two configurations in $\mathbf{G}$ of type E_8 remain to be considered: either $\ell = 3$ and $C_{\mathbf{G}^*}(s)$ is of type E_7A_1, or $\ell = 2$ and $C_{\mathbf{G}^*}(s)$ is of type E_6A_2. For these, we may conclude using the list of unipotent e-cuspidal pairs (see [3, Tab. 1]). If s is central, again by induction we may assume t is isolated in $\mathbf{G}^*$. Again from the list of isolated elements, no case arises. □

Following Enguehard [11] we introduce a relationship between unipotent e-cuspidal pairs.

Definition 3.5. Let $\mathbf{H}$ denote a connected reductive group with a Frobenius map F with respect to an $\mathbb{F}_q$-structure and let $\mathbf{H}'$ be an F-stable connected reductive subgroup of $\mathbf{H}$ of maximal rank. Let $(\mathbf{L},\lambda)$, $(\mathbf{L}',\lambda')$ be unipotent e-cuspidal pairs of $\mathbf{H}$, $\mathbf{H}'$ respectively. We write

$$(\mathbf{L}',\lambda') \sim (\mathbf{L},\lambda)$$

if $([\mathbf{L},\mathbf{L}],\lambda|_{[\mathbf{L},\mathbf{L}]^F})$ and $([\mathbf{L}',\mathbf{L}'],\lambda'|_{[\mathbf{L}',\mathbf{L}']^F})$ are $\mathbf{H}^F$-conjugate.

The following is a slight variation on [11, Prop. 17] (for which no proof was given). Note that by the table of [11, p. 348], ${}^3D_4[-1]$ (respectively $\phi_{2,1}$) is the unique unipotent 1-cuspidal (respectively 2-cuspidal) character of ${}^3D_4(q)$ of quasi-central 3-defect.

Proposition 3.6. *Let $\mathbf{X}$ and $\mathbf{G}$ be as above. Let $s \in \mathbf{G}^{*F}$ be a semisimple ℓ'-element and let $\mathbf{C} = C_{\mathbf{G}^*}(s)$. Let $t \in \mathbf{C}^F$ be an ℓ-element and let $(\mathbf{L}_t,\lambda_t)$ be a unipotent e-cuspidal pair in $C_{\mathbf{C}}(t)$ of quasi-central ℓ-defect.*

(a) *If there exists an e-split Levi subgroup $\mathbf{M}$ of $\mathbf{C}$ with $[\mathbf{M},\mathbf{M}] = [\mathbf{L}_t,\mathbf{L}_t]$, then there exists a (unique) $\mathbf{C}^F$-class of unipotent e-cuspidal pairs $(\mathbf{L},\lambda)$ in $\mathbf{C}$ with $(\mathbf{L}_t,\lambda_t) \sim (\mathbf{L},\lambda)$ as in Definition* 3.5.

(b) *If there exists no e-split Levi subgroup $\mathbf{M}$ of $\mathbf{C}$ with $[\mathbf{M},\mathbf{M}] = [\mathbf{L}_t,\mathbf{L}_t]$, then $\ell = 3$ (so $e \in \{1,2\}$), $[\mathbf{L}_t,\mathbf{L}_t]^F$ is of type ${}^3D_4$ and there exists a unique $\mathbf{C}^F$-class of e-split Levi subgroups $\mathbf{L}$ of $\mathbf{C}$ with $[\mathbf{L},\mathbf{L}]^F = D_4(q)$. Define a unipotent e-cuspidal character λ of $\mathbf{L}^F$ by*
- *$\lambda = D_4$ when $e = 1$ and $\lambda_t = {}^3D_4[-1]$,*
- *$\lambda = \phi_{13,02}$ when $e = 2$ and $\lambda_t = \phi_{2,1}$.*

All pairs $(\mathbf{L},\lambda)$ are also of quasi-central ℓ-defect.

Definition 3.7. In either case of Proposition 3.6 we write $(\mathbf{L}_t,\lambda_t) \to_t (\mathbf{L},\lambda)$.

Proof. Suppose that $(\mathbf{L},\lambda)$ and $(\mathbf{L}',\lambda')$ are unipotent e-cuspidal pairs of $\mathbf{C}$ such that $([\mathbf{L},\mathbf{L}],\lambda|_{[\mathbf{L},\mathbf{L}]^F})$ and $([\mathbf{L}',\mathbf{L}'],\lambda'|_{[\mathbf{L}',\mathbf{L}']^F})$ are $\mathbf{C}^F$-conjugate. We claim that $(\mathbf{L},\lambda)$ and $(\mathbf{L}',\lambda')$ are $\mathbf{C}^F$-conjugate. Indeed, let $x \in \mathbf{C}^F$ with $([\mathbf{L}',\mathbf{L}'],\lambda'|_{[\mathbf{L}',\mathbf{L}']^F}) = {}^x([\mathbf{L},\mathbf{L}],\lambda|_{[\mathbf{L},\mathbf{L}]^F})$. By [5, Prop. 1.7(iii)], there exists $c \in C^\circ_{\mathbf{C}}({}^x\mathbf{L}\cap\mathbf{L}')^F$ with $\mathbf{L}' = {}^{cx}\mathbf{L}$. Since $[\mathbf{L}',\mathbf{L}'] = {}^x[\mathbf{L},\mathbf{L}] \le {}^x\mathbf{L}\cap\mathbf{L}'$, we have ${}^{cx}\lambda|_{[\mathbf{L}',\mathbf{L}']^F} = {}^x\lambda|_{[\mathbf{L}',\mathbf{L}']^F} = \lambda'|_{[\mathbf{L}',\mathbf{L}']^F}$, and consequently by [5, Prop. 3.1], ${}^{cx}\lambda = \lambda'$. It follows that $(\mathbf{L}',\lambda') = {}^{cx}(\mathbf{L},\lambda)$, proving the claim. We note that this argument is essentially lifted from the discussion after Definition 3.4 of [5].

Now, suppose that we are in case (a). Let $\mathbf{L}$ be an e-split Levi subgroup of $\mathbf{C}$ with $[\mathbf{L},\mathbf{L}] = [\mathbf{L}_t,\mathbf{L}_t]$ and let λ be the (unique) unipotent character of $\mathbf{L}^F$ with $\lambda|_{[\mathbf{L},\mathbf{L}]^F} = \lambda_t|_{[\mathbf{L}_t,\mathbf{L}_t]^F}$. Then λ is e-cuspidal (see [5, Prop. 3.1] and the paragraph following it) and clearly $(\mathbf{L},\lambda) \sim (\mathbf{L}_t,\lambda_t)$. The uniqueness assertion of (a) follows from the paragraph above.

Suppose that ℓ is odd, good for $\mathbf{C}$ and $\ell > 3$ if $\mathbf{C}^F$ has no component of type ${}^3D_4$. Then we are in case (a) by [5, Prop. 3.5]. Note that in loc. cit. the element t is in the dual of the group about which the assertion is being made, but one can check that the proof works exactly in the same way in the situation we are considering.

Suppose that $\ell = 3$ and $\mathbf{C}^F$ has a component of type ${}^3D_4$. Then by Proposition 2.10, $\mathbf{X}$ is of exceptional type or $\mathbf{X}^F$ is of type ${}^3D_4$, and by rank considerations $\mathbf{C}$ has a single component of type D_4 and all other components of $\mathbf{C}$ are of type A. Write $\mathbf{C} = \mathbf{C}_1\mathbf{C}_2$ where $\mathbf{C}_1$ has type D_4 and $\mathbf{C}_2$ is the product of all other components of $[\mathbf{C},\mathbf{C}]$ with $Z^\circ(\mathbf{C})$. Since $Z(\mathbf{C}_1)$ is a 2-group, $t = t_1t_2$ with $t_i \in \mathbf{C}_i^F$ and $C_\mathbf{C}(t) = C_{\mathbf{C}_1}(t_1)C_{\mathbf{C}_2}(t_2)$. Moreover $\mathbf{L}_t = \mathbf{M}_1\mathbf{M}_2$ with each $\mathbf{M}_i$ being e-split in $C_{\mathbf{C}_i}(t_i)$, and λ_t covers the irreducible character $\mu_1\mu_2$ of $C_{\mathbf{C}_1}(t_1)^F C_{\mathbf{C}_2}(t_2)^F$ where μ_i is a unipotent e-cuspidal character of $\mathbf{M}_i$ (this follows for instance from [5, Sect. 3.1] and the fact that unipotent e-cuspidal pairs behave well under taking direct products (see for instance Proposition 2.4 and note that e-Jordan cuspidality and e-cuspidality coincide in the unipotent case). By Lemma 2.11, there is an e-split Levi subgroup of $C_{\mathbf{C}_1}(t_1)$, say $\mathbf{L}_1$ with $[\mathbf{L}_1,\mathbf{L}_1] = [\mathbf{M}_1,\mathbf{M}_1]$ (we take $\mathbf{L}_1 = \mathbf{M}_1 = \mathbf{T}$ where $\mathbf{T}$ is as in the lemma). By the argument in the preceding paragraph, there is an e-split Levi subgroup of $C_{\mathbf{C}_2}(t_2)$, say $\mathbf{L}_2$ with $[\mathbf{L}_2,\mathbf{L}_2] = [\mathbf{M}_2,\mathbf{M}_2]$. Then $\mathbf{L} = \mathbf{L}_1\mathbf{L}_2$ is an e-split Levi subgroup of $\mathbf{C}$ with $[\mathbf{L},\mathbf{L}] = [\mathbf{L}_t,\mathbf{L}_t]$ and we are in case (a).

Now suppose that $\ell = 2$ and all components of $\mathbf{L}_t$ are of classical type. By Lemma 2.6, $\mathbf{L}_t$ is a torus and we may take $\mathbf{L}$ to be the centraliser of the corresponding Sylow e-torus of $\mathbf{C}^F$ (see [22, Lemma 3.17]).

In the remaining cases, recall that we need to prove the following: if either $\ell \neq 3$ or $[\mathbf{L}_t,\mathbf{L}_t]^F$ is not of type ${}^3D_4$, then there exists an e-split Levi subgroup $\mathbf{L}$ of $\mathbf{C}$ with $[\mathbf{L},\mathbf{L}] = [\mathbf{L}_t,\mathbf{L}_t]$, and that if $\ell = 3$ and $[\mathbf{L}_t,\mathbf{L}_t]^F$ is of type ${}^3D_4$, then $\mathbf{C}$ has an e-split Levi subgroup $\mathbf{L}$ with $[\mathbf{L},\mathbf{L}]^F$ of type D_4. If $\mathbf{C}$ is contained in a proper Levi subgroup $\mathbf{G}_1$ of $\mathbf{G}^*$ then by induction on $\dim \mathbf{G}$ there exists an e-split Levi subgroup $\mathbf{L}_{\mathbf{G}_1}$ of $C_{\mathbf{G}_1^*}(s) = \mathbf{C}$ as wanted, and we can take $\mathbf{L} := \mathbf{L}_{\mathbf{G}_1}$ in $\mathbf{C}$. Hence, we may assume that s is isolated in $\mathbf{G}^*$.

Let $\mathbf{G}_1 := C_{\mathbf{G}^*}(Z(\mathbf{L}_t)_e)$, an e-split Levi subgroup of $\mathbf{G}^*$ containing $\mathbf{L}_t$. Then, $C_{\mathbf{G}_1}(s)$ is e-split in $\mathbf{C}$. If $\mathbf{G}_1$ is proper in $\mathbf{G}^*$, then by induction on $\dim \mathbf{G}$ there exists an e-split Levi subgroup $\mathbf{L}_{\mathbf{G}_1}$ of $C_{\mathbf{G}_1}(s)$, (which is then also e-split in $\mathbf{C}$) as wanted, and we can take $\mathbf{L} = \mathbf{L}_{\mathbf{G}_1}$. Now assume $\mathbf{G}_1 = \mathbf{G}^*$. Then since $(\mathbf{L}_t, \lambda_t)$ is e-cuspidal in $C_\mathbf{C}(t)$, so in particular $\mathbf{L}_t$ is e-split in $C_\mathbf{C}(t)$, we have

$$\mathbf{L}_t = C_{C_\mathbf{C}(t)}(Z(\mathbf{L}_t)_e) = C_\mathbf{C}(t) \cap C_{\mathbf{G}^*}(Z(\mathbf{L}_t)_e) = C_\mathbf{C}(t) \cap \mathbf{G}^* = C_\mathbf{C}(t).$$

By the discussion above, we may also assume now that either $\ell = 2$, or $\ell = 3$ and $(\mathbf{C}, F)$ has a factor of exceptional type, or $\ell = 5$, $\mathbf{G}$ is of type E_8 and $s = 1$. So we have $e \in \{1, 2, 4\}$. Using Chevie [27] we can enumerate all possible candidates for $\mathbf{L}_t$, that is, all rational types of Levi subgroups $\mathbf{M}$ of the various $\mathbf{C}$ having e-cuspidal unipotent characters of quasi-central ℓ-defect. Here, note that by Lemma 3.4, $\mathbf{L}_t$ is a Levi subgroup of $\mathbf{C}$. It turns out that we are in one of three cases when $e = 1$ (the cases $e = 2$, and $e = 4$ in E_8, being entirely similar):

- The only non-trivial ℓ-elements in $Z(\mathbf{M})^F$ are involutions, but $\mathbf{M}$ is not the centraliser of an involution in any of the possible $\mathbf{C}$.
- $Z(\mathbf{M})_e > Z(\mathbf{C})_e$ (here, as earlier, an index e on an F-stable torus denotes its Sylow e-subtorus) and so $C_{\mathbf{G}^*}(Z(\mathbf{M})_e) < \mathbf{G}^*$, and we may conclude by induction, see above.
- $\mathbf{M}$ has rational type ${}^3D_4(q)\Phi_3^k$ for $k = 1, 2$, where $Z(\mathbf{M})^F$ contains non-trivial ℓ-elements only when $\ell = 3$.

In the last case, $\mathbf{C}$ has a factor of type E_n, $n \geq 6$, and so does possess a 1-split Levi subgroup of type D_4, unique up to conjugacy, whence we end up in case (b). By explicit enumeration, for $\ell = 3$ whenever $(\mathbf{L}_t, \lambda_t)$ is a unipotent e-cuspidal pair in $C_{\mathbf{C}}(t)$ such that $\mathbf{L}_t$ has a factor of type ${}^3D_4$, then $[\mathbf{L}_t, \mathbf{L}_t]^F$ is simple of type ${}^3D_4$. Also, it turns out that none of the relevant $\mathbf{C}$ do possess an e-split Levi subgroup, $e \in \{1, 2\}$, with a component of type ${}^3D_4$. The statement about quasi-central defect is obvious if we are not in situation (b); in the latter case it can be checked directly. □

Remark 3.8. In [11, Prop. 17] it is claimed that if one of $(\mathbf{L}_t, \lambda_t)$, $(\mathbf{L}, \lambda)$ is of central ℓ-defect, then so is the other. This is easily seen to not hold in two of the four exceptional cases listed there. It is also stated that $\mathbf{L}_t$ is "deployé" (split), but this seems to be a misprint as it can easily be seen to be wrong in general. Let us add that neither is $\mathbf{L}_t$ an e-split Levi subgroup of $\mathbf{C}$, in general.

4. The block distribution

In this section we prove Theorem 1. Several of the results presented here, or variants thereof, were stated in or are inspired by the work of Enguehard [11] on unipotent blocks for bad primes. Since [11] does not always give proofs, and some of its statements are obviously inaccurate, we have decided to provide some of the missing proofs (and indicate where we believe [11] is incorrect). We take the opportunity to point out that, in particular, Theorems A and A.bis of [11] seem not correct for $\ell = 2$ when e is *not* the order of q modulo 4.

Throughout this section, we fix the following notation:

- Let $\mathbf{X}$ be a connected reductive group in characteristic p with connected centre and simply connected simple derived subgroup with a Frobenius map $F : \mathbf{X} \to \mathbf{X}$, and let $\mathbf{G}$ be an F-stable Levi subgroup of $\mathbf{X}$;
- let ℓ be a prime not dividing q and $e := e_\ell(q)$; and
- let $s \in \mathbf{G}^{*F}$ be a semisimple ℓ'-element and set $\mathbf{C}^* := C_{\mathbf{G}^*}(s)$.

Note that $\mathbf{C}^*$ is connected, as all centralisers of semisimple elements in $\mathbf{G}^*$, since $\mathbf{G}$ has connected centre. For any ℓ-element $t \in \mathbf{C}^{*F}$ recall the Digne–Michel Jordan decomposition

$$\pi_{st}^{\mathbf{G}} : \mathcal{E}(\mathbf{G}^F, st) \longrightarrow \mathcal{E}(C_{\mathbf{G}^*}(st)^F, 1) = \mathcal{E}(C_{\mathbf{C}^*}(t)^F, 1)$$

already discussed in Section 2. For $(\mathbf{L}, \lambda)$ an e-Jordan cuspidal pair of $\mathbf{G}$ below $(\mathbf{G}^F, s)$ we write $b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ for the corresponding ℓ-block of $\mathbf{G}^F$ containing all constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ (see Theorem 3.2).

4.1. **The map $\bar{J}_t^{\mathbf{G}}$.** We start by defining the map $\bar{J}_t^{\mathbf{G}}$ in Theorem 1, based on the relation $\to_t$ introduced in Proposition 3.6 (see Definition 3.7).

Proposition 4.1. *Let $t \in \mathbf{C}^{*F}$ be an ℓ-element.*

(a) *The relationship $\to_t$ on e-cuspidal pairs defined in Proposition* 3.6 *induces a map $J_t^{\mathbf{G}}$ from the set of $C_{\mathbf{G}^*}(st)^F$-classes of unipotent e-cuspidal pairs of quasi-central ℓ-defect in $C_{\mathbf{G}^*}(st)$ to the set of $\mathbf{G}^F$-classes of e-Jordan quasi-central cuspidal pairs in $\mathbf{G}$ below $(\mathbf{G}^F, s)$.*

(b) *This induces a map* $\bar{J}_t^{\mathbf{G}}$ *from the set of unipotent* ℓ*-blocks of* $C_{\mathbf{G}^*}(st)^F$ *to the set of* ℓ*-blocks of* $\mathbf{G}^F$ *in* $\mathcal{E}_\ell(\mathbf{G}^F, s)$.

Proof. Let $(\mathbf{L}_t^*, \lambda_t)$ be a unipotent e-cuspidal pair of quasi-central ℓ-defect in $\mathbf{C}_t^* := C_{\mathbf{G}^*}(st) = C_{\mathbf{C}^*}(t)$. By Proposition 3.6 the $\mathbf{C}_t^{*F}$-class of $(\mathbf{L}_t^*, \lambda_t)$ gives rise to a unique $\mathbf{C}^{*F}$-class of unipotent e-cuspidal pairs $(\mathbf{L}_s^*, \lambda_s)$ in $\mathbf{C}^*$, of quasi-central defect. Then the bijection in Proposition 2.2 provides a $\mathbf{G}^F$-class of e-Jordan quasi-central cuspidal pairs $(\mathbf{L}, \lambda)$ in $\mathbf{G}$ below $(\mathbf{G}^F, s)$ and we define the image of the class of $(\mathbf{L}_t^*, \lambda_t)$ under $J_t^{\mathbf{G}}$ to be the class of $(\mathbf{L}, \lambda)$.

For (b) let b be a unipotent ℓ-block of $\mathbf{C}_t^{*F}$. By [11, Thm A and A.bis] there exists a unipotent e-cuspidal pair $(\mathbf{L}_t^*, \lambda_t)$ in $\mathbf{C}_t^*$, of quasi-central ℓ-defect and unique up to $\mathbf{C}_t^{*F}$-conjugacy such that $b = b_{\mathbf{C}_t^{*F}}(\mathbf{L}_t^*, \lambda_t)$. The map $J_t^{\mathbf{G}}$ from (a) provides an e-Jordan quasi-central cuspidal pair $(\mathbf{L}, \lambda)$ in $\mathbf{G}$ below $(\mathbf{G}^F, s)$, unique up to $\mathbf{G}^F$-conjugacy, and by Theorem 3.2 this determines an ℓ-block $\bar{J}_t^{\mathbf{G}}(b) := b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ of $\mathbf{G}^F$ in series s. □

We note that when $s = 1$, the map $\bar{J}_t^{\bar{G}}$ above coincides with the map $\bar{J}_t^{\mathbf{G},F}$ of [11, Thm B] (where our semisimple ℓ-element t is denoted s).

Remark 4.2. In the setting of Proposition 4.1 let $(\mathbf{L}_t^*, \lambda_t)$ be a unipotent e-cuspidal pair in $C_{\mathbf{G}^*}(st) = C_{\mathbf{C}^*}(t)$. If $\mathbf{L}_t^*$ is an e-split Levi subgroup of $\mathbf{G}^*$ with dual $\mathbf{L}_t \leq \mathbf{G}$, then by Proposition 3.6(a), $J_t^{\mathbf{G}}$ sends the $C_{\mathbf{G}^*}(st)^F$-class of $(\mathbf{L}_t^*, \lambda_t)$ to the $\mathbf{G}^F$-class of $(\mathbf{L}_t, \lambda')$ for some $\lambda' \in \mathcal{E}(\mathbf{L}_t^F, s)$.

In the following, by *semisimple block* in the union of Lusztig series $\mathcal{E}_\ell(\mathbf{G}^F, s)$ we mean the block containing the semisimple character of $\mathcal{E}(\mathbf{G}^F, s)$. Note that Lusztig induction of a semisimple character contains a semisimple character. This follows from the fact that Jordan decomposition preserves uniform functions and the fact that the trivial character is a constituent of any Lusztig induction of the trivial character (see e.g. [10, proof of Cor. 10.1.7]).

Proposition 4.3. *Let* $t \in C_{\mathbf{G}^*}(s)^F$ *be an* ℓ*-element and* b *the principal* ℓ*-block of* $C_{\mathbf{G}^*}(st)^F$. *Then* $\bar{J}_t^{\mathbf{G}}(b)$ *is the semisimple block in* $\mathcal{E}_\ell(\mathbf{G}^F, s)$.

Proof. The principal block of $C_{\mathbf{G}^*}(st)^F$ is labelled by the e-Harish-Chandra series of $(\mathbf{L}_t^*, 1)$ for $\mathbf{L}_t^*$ the centraliser of a Sylow e-torus of $C_{\mathbf{G}^*}(st)$. Thus we are not in one of the exceptional cases of Proposition 3.6 whence the corresponding unipotent e-cuspidal pair of $C_{\mathbf{G}^*}(s)$ is of the form $(\mathbf{L}_s^*, 1)$. Let $(\mathbf{L}, \lambda)$ be associated to $(\mathbf{L}_s^*, 1)$ as in Proposition 2.2, so λ is the semisimple character in $\mathcal{E}(\mathbf{L}^F, s)$. Hence the semisimple block in $\mathcal{E}_\ell(\mathbf{G}^F, s)$ lies above $(\mathbf{L}, \lambda)$, so equals $\bar{J}_t^{\mathbf{G}}(b)$. □

By Theorem 3.2, for any e-split Levi $\mathbf{M}$ of $\mathbf{G}$ there is a map $R_{\mathbf{M}}^{\mathbf{G}}$ from the set of ℓ-blocks in $\mathcal{E}_\ell(\mathbf{M}^F, s)$ to the set of ℓ-blocks in $\mathcal{E}_\ell(\mathbf{G}^F, s)$ such that if c is an ℓ-block of $\mathbf{M}^F$ in series s, then every constituent of $R_{\mathbf{M}}^{\mathbf{G}}(\mu)$ for any $\mu \in \mathrm{Irr}(c) \cap \mathcal{E}(\mathbf{M}^F, s)$, lies in $R_{\mathbf{M}}^{\mathbf{G}}(c)$.

The next statement mirrors [11, Prop. 16]:

Proposition 4.4. *Let* $\mathbf{M} \leq \mathbf{G}$ *be* e*-split with* $s \in \mathbf{M}^*$. *Let* $(\mathbf{L}_i, \lambda_i)$, $i = 1, 2$, *be two* e*-Jordan cuspidal pairs below* $(\mathbf{M}^F, s)$. *Then these are* e*-Jordan cuspidal pairs below* $(\mathbf{G}^F, s)$, *and if* $b_{\mathbf{M}^F}(\mathbf{L}_1, \lambda_1) = b_{\mathbf{M}^F}(\mathbf{L}_2, \lambda_2)$ *then also* $b_{\mathbf{G}^F}(\mathbf{L}_1, \lambda_1) = b_{\mathbf{G}^F}(\mathbf{L}_2, \lambda_2)$.

Further, if $(\mathbf{L}, \lambda)$ *is an* e*-Jordan quasi-central cuspidal pair below a block* b *in* $\mathcal{E}_\ell(\mathbf{M}^F, s)$ *then it is so below* $R_{\mathbf{M}}^{\mathbf{G}}(b)$.

Proof. The first statement is clear from the definition of e-Jordan cuspidal pairs and the fact that if $\mathbf{L}$ is e-split in $\mathbf{M}$ and $\mathbf{M}$ is e-split in $\mathbf{G}$, then $\mathbf{L}$ is e-split in $\mathbf{G}$. Suppose that $b_{\mathbf{M}^F}(\mathbf{L}_1, \lambda_1) = b_{\mathbf{M}^F}(\mathbf{L}_2, \lambda_2) =: c$. Then every irreducible constituent of $R_{\mathbf{L}_i}^{\mathbf{M}}(\lambda_i)$, $i = 1, 2$, lies in c. On the other hand, if χ is an irreducible constituent of $R_{\mathbf{L}_i}^{\mathbf{G}}(\lambda_i)$ for any $i = 1, 2$, then χ is a constituent of $R_{\mathbf{M}}^{\mathbf{G}}(\mu_i)$ for some irreducible constituent μ_i of $R_{\mathbf{L}_i}^{\mathbf{M}}(\lambda_i)$ and as observed before μ_i belongs to $\mathrm{Irr}(c)$. This proves the result. □

The next statement extends [11, Cor. 19].

Corollary 4.5. *Let $t \in \mathbf{C}^{*F}$ be an ℓ-element. If $\mathbf{M}^* \le \mathbf{G}^*$ is an e-split Levi subgroup such that $C_{\mathbf{G}^*}(st) \le \mathbf{M}^*$, with dual $\mathbf{M} \le \mathbf{G}$, then $R_{\mathbf{M}}^{\mathbf{G}} \circ \bar{J}_t^{\mathbf{M}} = \bar{J}_t^{\mathbf{G}}$ on the set of unipotent ℓ-blocks of $C_{\mathbf{M}^*}(st)^F$.*

Proof. Let c be a unipotent ℓ-block of $C_{\mathbf{G}^*}(st)$ and $(\mathbf{L}_t, \lambda_t)$ be a unipotent e-cuspidal pair of quasi-central ℓ-defect defining c. Let $(\mathbf{L}', \lambda')$ be a unipotent e-cuspidal pair (also of quasi-central ℓ-defect) of $C_{\mathbf{M}^*}(s)$ such that $(\mathbf{L}_t, \lambda_t) \to_t (\mathbf{L}', \lambda')$ in $\mathbf{M}^*$. Then, $C_{\mathbf{M}^*}(s)$ is e-split in $\mathbf{C}^*$, hence $(\mathbf{L}', \lambda')$ is an e-cuspidal pair of $\mathbf{C}^*$ and $(\mathbf{L}_t, \lambda_t) \to_t (\mathbf{L}', \lambda')$ in $\mathbf{G}^*$ by Proposition 3.6. Further, since $Z^\circ(\mathbf{M}^*)_e \le Z^\circ(C_{\mathbf{M}^*}(s))_e \le Z^\circ(\mathbf{L}')_e$ and $\mathbf{M}^* = C_{\mathbf{G}}^*(Z^\circ(\mathbf{M}^*)_e)$, it follows that $C_{\mathbf{G}^*}(Z^\circ(\mathbf{L}')_e) = C_{\mathbf{M}^*}(Z^\circ(\mathbf{L}')_e)$. So $(\mathbf{L}, \lambda) := J_t^{\mathbf{G}}(\mathbf{L}_t, \lambda_t) = J_t^{\mathbf{M}}(\mathbf{L}_t, \lambda_t)$. Now it follows from Proposition 4.4 that

$$\bar{J}_t^{\mathbf{G}}(c) = b_{\mathbf{G}^F}(\mathbf{L}, \lambda) = R_{\mathbf{M}}^{\mathbf{G}}(b_{\mathbf{M}^F}(\mathbf{L}', \lambda')) = R_{\mathbf{M}}^{\mathbf{G}}(\bar{J}_t^{\mathbf{M}}(c)). \tag{1}$$

□

4.2. **t-Twin blocks.** Our main theorem relies on the compatibility between Lusztig induction and Jordan decomposition. In order to deal with certain ambiguities in groups of type E_8, we group together certain unipotent 2-cuspidal pairs of exceptional groups into *twins* as follows:

- $(E_6, E_6[\theta])$ and $(E_6, E_6[\theta^2])$,
- $({}^2E_6, {}^2E_6[\theta])$ and $({}^2E_6, {}^2E_6[\theta^2])$,
- $(E_7, \phi_{512,11})$ and $(E_7, \phi_{512,12})$.

Now for any Levi subgroup of $\mathbf{G}$ having one of the above as a component (note that there can be at most one such component), we call *twin 2-Harish-Chandra series* the corresponding unions of 2-Harish-Chandra series of $\mathbf{G}$ lying above these (see also [17, p. 350]). Note that the two members in each of the first two pairs are Galois conjugate over $\mathbb{Q}(\theta)$, for θ a primitive third root of unity, while those in the last one are Galois conjugate over $\mathbb{Q}(\sqrt{q})$ when q is not a square, and rational otherwise.

For $s \in \mathbf{G}^{*F}$ a semisimple ℓ'-element, we define *t-twin blocks*, for $t \in C_{\mathbf{G}^*}(s)$ an ℓ-element, as follows: If $\mathbf{G}$ is not of type E_8, or if $e_\ell(q) \neq 2$, then t-twin blocks are blocks. Suppose that $\mathbf{G} = E_8$ and $e_\ell(q) = 2$. Let $(\mathbf{L}_t^*, \lambda_t)$ be a unipotent e-cuspidal pair of $C_{\mathbf{G}^*}(st) = C_{\mathbf{C}^*}(t)$ of quasi-central ℓ-defect and let $b = \bar{J}_t^{\mathbf{G}}(b_{C_{\mathbf{G}^*}(st)^F}(\mathbf{L}_t^*, \lambda_t))$. Then the t-twin block containing b is the pair consisting of b and its "twin" corresponding to the twin $(\mathbf{L}_t^*, \lambda_t')$ of $(\mathbf{L}_t^*, \lambda_t)$ if $(\mathbf{L}_t^*, \lambda_t)$ is as above; see the list shown in Table 1. Otherwise, the t-twin block of b contains only b. With this, the map $\bar{J}_t^{\mathbf{G}}$ from Proposition 4.1 can and will be considered as a map from the set of unipotent blocks of $C_{\mathbf{G}^*}(st)^F$ to the set of t-twin blocks of $\mathbf{G}^F$.

TABLE 1. t-Twin blocks, $\ell|(q+1)$

$\mathbf{G}$	$C_{\mathbf{G}^*}(s)^F$	ℓ	$\mathbf{L}_t^{*F}$	λ_t, λ'_t
E_8	$\mathbf{G}^{*F}$	$\neq 5$	$\Phi_2^2.{}^2E_6(q)$	${}^2E_6[\theta], {}^2E_6[\theta^2]$
E_8	${}^2E_6(q).{}^2A_2(q)$	$\neq 3$	$\Phi_2^2.{}^2E_6(q)$	${}^2E_6[\theta], {}^2E_6[\theta^2]$
E_8	$E_7(q).A_1(q)$	$\neq 2$	$\Phi_2^2.{}^2E_6(q)$	${}^2E_6[\theta], {}^2E_6[\theta^2]$
E_8	$\mathbf{G}^{*F}$	$\neq 2, 3$	$\Phi_2.E_7(q)$	$\phi_{512,11}, \phi_{512,12}$
E_8	$E_7(q).A_1(q)$	$\neq 2$	$\Phi_2.E_7(q)$	$\phi_{512,11}, \phi_{512,12}$

4.3. **Inductive arguments.** The following statement will allow us to inductively deal with many cases of Theorem 1.

Proposition 4.6. *Let $t \in \mathbf{C}^{*F}$ be an ℓ-element and $\chi \in \mathcal{E}(\mathbf{G}^F, st)$. Assume that $C_{\mathbf{G}^*}(st)$ is contained in an e-split proper Levi subgroup $\mathbf{M}^*$ of $\mathbf{G}^*$ and that Theorem 1 holds for its dual $\mathbf{M}$. Let b_t be the unipotent ℓ-block of $C_{\mathbf{G}^*}(st)^F$ containing $\pi_{st}^{\mathbf{G}}(\chi)$. Then $b_{\mathbf{G}^F}(\chi)$ is contained in $\bar{J}_t^{\mathbf{G}}(b_t)$.*

Proof. By Lemma 2.7, there is $\gamma \in \mathcal{E}(\mathbf{G}^F, s)$ with $\langle d^{1,\mathbf{G}}(\chi), \gamma\rangle \neq 0$, so in particular lying in the ℓ-block $b_{\mathbf{G}^F}(\chi) = b_{\mathbf{G}^F}(\gamma)$. Let $\mathbf{M} \leq \mathbf{G}$ be dual to $\mathbf{M}^*$ and let $\chi' \in \mathcal{E}(\mathbf{M}^F, st)$ with $\chi = \pm R_{\mathbf{M}}^{\mathbf{G}}(\chi')$ and $\pi_{st}^{\mathbf{G}}(\chi) = \pi_{st}^{\mathbf{M}}(\chi')$ (see [9, Thm 7.1]). Then

$$\big\langle d^{1,\mathbf{M}}(\chi'), {}^*R_{\mathbf{M}}^{\mathbf{G}}(\gamma)\big\rangle = \big\langle d^{1,\mathbf{G}}(R_{\mathbf{M}}^{\mathbf{G}}(\chi')), \gamma\big\rangle = \pm\langle d^{1,\mathbf{G}}(\chi), \gamma\rangle \neq 0,$$

whence there is a constituent $\gamma' \in \mathcal{E}(\mathbf{M}^F, s)$ of ${}^*R_{\mathbf{M}}^{\mathbf{G}}(\gamma)$ with

$$\big\langle d^{1,\mathbf{M}}(\chi'), \gamma'\big\rangle\big\langle {}^*R_{\mathbf{M}}^{\mathbf{G}}(\gamma), \gamma'\big\rangle \neq 0.$$

By assumption and since $\big\langle d^{1,\mathbf{M}}(\chi'), \gamma'\big\rangle \neq 0$ we know that $b_{\mathbf{M}^F}(\gamma') = b_{\mathbf{M}^F}(\chi')$ is contained in $\bar{J}_t^{\mathbf{M}}(b_t)$. Application of Proposition 4.4 and Corollary 4.5 then yields

$$b_{\mathbf{G}^F}(\chi) = b_{\mathbf{G}^F}(\gamma) = R_{\mathbf{M}}^{\mathbf{G}}(b_{\mathbf{M}^F}(\gamma')) \quad \text{lies in} \quad R_{\mathbf{M}}^{\mathbf{G}}(\bar{J}_t^{\mathbf{M}}(b_t)) = \bar{J}_t^{\mathbf{G}}(b_t),$$

as claimed. □

Another useful reduction is the following:

Proposition 4.7. *Let $\mathbf{M}$ be a proper F-stable Levi subgroup of $\mathbf{G}$ whose dual in $\mathbf{G}^*$ contains $C_{\mathbf{G}^*}(s)$ and let $t \in C_{\mathbf{G}^*}(s)^F$ be an ℓ-element. If Theorem 1 holds for $\mathbf{M}$ and t, then it holds for $\mathbf{G}$ and t.*

Proof. Recall that by the results of Bonnafé–Rouquier, $\pm R_{\mathbf{M}}^{\mathbf{G}}$ induces a bijection (referred to as Jordan correspondence in [22]), which we will abbreviate *BR-correspondence* here, between the ℓ-blocks in $\mathcal{E}_\ell(\mathbf{M}^F, s)$ and the ℓ-blocks in $\mathcal{E}_\ell(\mathbf{G}^F, s)$. By [22, Prop. 2.4, 2.6 and Thm A], there exists a bijection $\Psi_{\mathbf{M}}^{\mathbf{G}}$ between the set of e-Jordan cuspidal pairs below $(\mathbf{M}^F, s)$ and the set of e-Jordan cuspidal pairs below $(\mathbf{G}^F, s)$ such that for any e-Jordan cuspidal pair $(\mathbf{L}', \lambda')$ below $(\mathbf{M}^F, s)$, $b_{\mathbf{G}^F}(\Psi_{\mathbf{M}}^{\mathbf{G}}(\mathbf{L}', \lambda'))$ is the BR-correspondent of $b_{\mathbf{M}^F}(\mathbf{L}', \lambda')$. The bijection $\Psi_{\mathbf{M}}^{\mathbf{G}}$ is described as follows: If $(\mathbf{L}', \lambda')$ is an e-Jordan cuspidal pair below $(\mathbf{M}^F, s)$, then $\Psi_{\mathbf{M}}^{\mathbf{G}}(\mathbf{L}', \lambda') = (\mathbf{L}, \lambda)$, where $\mathbf{L} = C_{\mathbf{G}}(Z^\circ(\mathbf{L}')_e)$ and $\lambda = \pm R_{\mathbf{L}'}^{\mathbf{L}}(\lambda')$.

Passing to duals we have that $\mathbf{L}^* = C_{\mathbf{G}^*}(Z^\circ(\mathbf{L}'^*)_e)$. Since $\mathbf{L}'^*$ is e-split in $\mathbf{M}^*$, $\mathbf{L}'^* = C_{\mathbf{M}^*}(Z^\circ(\mathbf{L}'^*)_e)$ and hence $\mathbf{L}'^* = \mathbf{L}^* \cap \mathbf{M}^*$. Consequently, $C_{\mathbf{L}^*}(s) = C_{\mathbf{L}'^*}(s)$ and by properties of Digne–Michel's Jordan decomposition, $\pi_s^{\mathbf{L}}(\lambda) = \pi_s^{\mathbf{L}'}(\lambda')$. Thus, if $(\mathbf{L}', \lambda')$ corresponds to the unipotent e-cuspidal pair $(\mathbf{L}_s^*, \lambda_s)$ of $C_{\mathbf{M}^*}(s)$ via Proposition 2.2 (for the group $\mathbf{M}$), then $(\mathbf{L}, \lambda)$ also corresponds to $(\mathbf{L}_s^*, \lambda_s)$ via Proposition 2.2.

Let $t \in C_{\mathbf{G}^*}(s)^F$ be an ℓ-element. It follows from the above that for any unipotent e-cuspidal pair $(\mathbf{L}_t^*, \lambda_t)$ of $C_{\mathbf{G}^*}(st)$, if $(\mathbf{L}', \lambda')$ is in the $\mathbf{M}^F$-class of $J_t^{\mathbf{M}}((\mathbf{L}_t^*, \lambda_t))$, then $(\mathbf{L}, \lambda)$ is in the $\mathbf{G}^F$-class of $J_t^{\mathbf{G}}((\mathbf{L}_t, \lambda_t))$. Consequently, for any unipotent block b of $C_{\mathbf{G}^*}(st)$, $\bar{J}_t^{\mathbf{G}}(b)$ is the BR-correspondent block of $\bar{J}_t^{\mathbf{M}}(b)$.

Now let $\chi \in \mathcal{E}(\mathbf{G}^F, st)$. Since $C_{\mathbf{G}^*}(st) \leq \mathbf{M}^*$, there exists $\chi' \in \mathcal{E}(\mathbf{M}^F, st)$ with $\chi = \pm R_{\mathbf{M}}^{\mathbf{G}}(\chi')$ and $\gamma := \pi_{st}^{\mathbf{G}}(\chi) = \pi_{st}^{\mathbf{M}}(\chi')$ (see [9, Thm 7.1]). Denoting by b the block containing γ, by hypothesis we have $\chi' \in \mathrm{Irr}(\bar{J}_t^{\mathbf{M}}(b))$ (here we note that as $\mathbf{M}$ is proper in $\mathbf{G}$, t-twin blocks of $\mathbf{M}$ are the same as blocks). Thus, χ belongs to the BR-correspondent of $\bar{J}_t^{\mathbf{M}}(b)$, that is to $\bar{J}_t^{\mathbf{G}}(b)$. □

Proposition 4.8. *The conclusion of Theorem* 1 *holds whenever* $s \in Z(\mathbf{G}^{*F})$.

Proof. Let $s \in Z(\mathbf{G}^{*F})$, so $\mathcal{E}(C_{\mathbf{G}^*}(t)^F, 1) = \mathcal{E}(C_{\mathbf{G}^*}(st)^F, 1)$. Let $\hat{s} \in \mathrm{Irr}(\mathbf{G}^F)$ be the linear character of ℓ'-order determined by s. Then $\_ \otimes \hat{s} : \mathcal{E}(\mathbf{G}^F, t) \to \mathcal{E}(\mathbf{G}^F, st)$ preserves the partition into ℓ-blocks. Now by Enguehard [11, Cor. 19, Thm B], the conclusion of Theorem 1 holds whenever $s = 1$. The claim follows since tensoring with $\hat{s}$ commutes with Lusztig induction (see [7, Prop. 9.6] and note that all unipotent elements of $\mathbf{G}^F$ are contained in the kernel of any linear character of $\mathbf{G}^F$). □

4.4. Theorem 1 when t is central.

Proposition 4.9. *Suppose that* $s \in \mathbf{G}^{*F}$ *is isolated and non-central. Let* $(\mathbf{L}_s^*, \lambda_s)$ *be a unipotent* e*-cuspidal pair of* $\mathbf{C}^*$ *and let* $(\mathbf{L}, \lambda)$ *be an associated* e*-Jordan cuspidal pair below* $(\mathbf{G}^F, s)$ *via Proposition* 2.2*. Then* $b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ *is in* $\bar{J}_1^{\mathbf{G}}(b_{\mathbf{C}^{*F}}(\mathbf{L}_s^*, \lambda_s))$.

Proof. If $(\mathbf{L}_s^*, \lambda_s)$ is of quasi-central ℓ-defect then the result follows from the definition of $J_1^{\mathbf{G}}$ and the fact that by Proposition 3.6(a), the relation $\to_1$ is the identity. Since every unipotent block has an associated unipotent e-cuspidal pair of quasi-central ℓ-defect, it remains to show that if $(\mathbf{L}_s^*, \lambda_s)$ and $(\mathbf{L}'^*_s, \lambda'_s)$ are two unipotent e-cuspidal pairs of $\mathbf{C}^*$, corresponding via Proposition 2.2 to pairs $(\mathbf{L}, \lambda)$, $(\mathbf{L}', \lambda')$ respectively, of $\mathbf{G}^F$, and if $b_{\mathbf{C}^{*F}}(\mathbf{L}_s^*, \lambda_s) = b_{\mathbf{C}^{*F}}(\mathbf{L}'^*_s, \lambda'_s)$, then $b_{\mathbf{G}^F}(\mathbf{L}, \lambda) = b_{\mathbf{G}^F}(\mathbf{L}', \lambda')$.

Let b be a unipotent block of $\mathbf{C}^*$ containing two distinct unipotent e-Harish-Chandra series. By the main result of [5] we have that either $\ell = 2$, or ℓ is bad for $\mathbf{C}^*$ (since ${}^3D_4(q)$ is not a component of $\mathbf{C}^{*F}$ by Lemma 2.10).

Suppose first that all components of $\mathbf{G}$ (and hence of $\mathbf{C}^*$) are of classical type. Then by the above $\ell = 2$ and $\mathcal{E}_2(\mathbf{G}^F, s)$ is a single 2-block (see for instance [7, Thm 21.14]). Thus we may assume $\mathbf{G}$ has a component of exceptional type and ℓ is bad for $\mathbf{G}$.

If $[\mathbf{G}, \mathbf{G}]$ is simple, then the result follows from Theorem 3.1. Thus we may assume that $[\mathbf{G}, \mathbf{G}]$ is not simple and therefore also that the semisimple rank of $\mathbf{G}$ is at most 7, as $\mathbf{X}$ is of exceptional type and $\mathbf{G}$ is proper in $\mathbf{X}$. Since $\mathbf{G}$ has a component of exceptional type, it follows that $[\mathbf{G}, \mathbf{G}]$ is of type E_6A_1 and $\ell = 2$ or 3. Since groups of type A have no non-central isolated elements, the E_6-component of s in $\mathbf{G}^*$ is non-central and isolated.

Thus, by [1, Tab. 3], one sees that all components of $\mathbf{C}^*$ are of type A. Since 3 is good for groups of type A, this forces $\ell = 2$. Then the result follows by Proposition 3.3. □

Proposition 4.10. *Suppose that $s \in \mathbf{G}^{*F}$ is isolated and non-central. Let $(\mathbf{L}_s^*, \lambda_s)$ be a unipotent e-cuspidal pair of $\mathbf{C}^*$ and let $(\mathbf{L}, \lambda)$ be an associated e-Jordan cuspidal pair below $(\mathbf{G}^F, s)$ via Proposition 2.2. Then there exists a bijection*

$$\tilde{\pi}_s^{\mathbf{G}} : \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) \to \mathcal{E}(\mathbf{C}^{*F}, (\mathbf{L}_s^*, \lambda_s))$$

such that any constituent χ of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ has Jordan correspondent $\tilde{\pi}_s^{\mathbf{G}}(\chi)$, unless possibly if $\mathbf{G} = E_8$, and $\mathbf{C}^*$ and $(\mathbf{L}_s^*, \lambda_s)$ belong to a "twin", in which case the Jordan correspondent could be a constituent of $R_{\mathbf{L}_s^*}^{\mathbf{C}^*}(\lambda'_s)$ for $(\mathbf{L}_s^*, \lambda'_s)$ the twin of $(\mathbf{L}_s^*, \lambda_s)$.

In particular, $\tilde{\pi}_s^{\mathbf{G}}(\chi)$ and the Jordan correspondent of χ have the same degree, so $\tilde{\pi}_s^{\mathbf{G}}$ preserves ℓ-defects.

Proof. Suppose first that $[\mathbf{G}, \mathbf{G}]^F \neq E_8(2)$. Then the Mackey formula holds for $\mathbf{G}^F$ and the result follows by [17, Cor. 4.7.7]. Here, we note that in the statement of [17, Cor. 4.7.7], the assumption on the Mackey formula is on the ambient group, and that this group is itself simple, but it can be easily checked that the proof and result carry over with our assumptions. Now suppose that $[\mathbf{G}, \mathbf{G}]^F = E_8(2)$. This implies that there is an F-stable decomposition $\mathbf{G} = E_8 \times \mathbf{T}$ with a torus $\mathbf{T}$. Using the compatibility of Lusztig induction with direct products the result then follows from [19, Prop. 3.11] when $\ell \geq 7$, and from [21, Prop. 6.11] when $\ell \in \{3, 5\}$. □

Proposition 4.11. *The conclusion of Theorem 1 holds for any $t \in Z(\mathbf{G}^{*F})$.*

Proof. Let $t \in Z(\mathbf{G}^{*F})$ and $\hat{t} \in \mathrm{Irr}(\mathbf{G}^F)$ be the linear character of ℓ-power order determined by t. Then $\_\otimes \hat{t} : \mathcal{E}(\mathbf{G}^F, s) \to \mathcal{E}(\mathbf{G}^F, st)$ is a bijection preserving the partition into ℓ-blocks, and $\mathcal{E}_\ell(C_{\mathbf{G}^*}(s)^F, 1) = \mathcal{E}_\ell(C_{\mathbf{G}^*}(st)^F, 1)$ so it suffices to prove the result for $t = 1$.

Suppose that $t = 1$. By Proposition 4.7 we may assume s is isolated in $\mathbf{G}^*$ and by Proposition 4.8 we may assume that s is not central. Let $\eta \in \mathcal{E}(\mathbf{C}^{*F}, 1)$ and $\chi \in \mathcal{E}(\mathbf{G}^F, s)$ with $\eta = \pi_s^{\mathbf{G}}(\chi)$. We are required to show that $b_{\mathbf{G}^F}(\chi)$ belongs to $\bar{J}_1^{\mathbf{G}}(b_{\mathbf{C}^{*F}}(\eta))$. Let $(\mathbf{L}_s^*, \lambda_s)$ be a unipotent e-cuspidal pair of $\mathbf{C}^{*F}$ with $\eta \in \mathcal{E}(\mathbf{C}^{*F}, (\mathbf{L}_s^*, \lambda_s))$ and let $(\mathbf{L}, \lambda)$ be an e-Jordan cuspidal pair below $(\mathbf{G}^F, s)$ associated to $(\mathbf{L}_s^*, \lambda_s)$ via Proposition 2.2 (note that the existence of $(\mathbf{L}_s^*, \lambda_s)$ is guaranteed by generalised e-Harish-Chandra theory). By Proposition 4.10, $\chi \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) \cup \mathcal{E}(\mathbf{G}^F, (\mathbf{L}', \lambda'))$, where $(\mathbf{L}', \lambda')$ is an e-Jordan cuspidal pair below $(\mathbf{G}^F, s)$ associated to the twin of $(\mathbf{L}_s^*, \lambda_s)$ via Proposition 2.2. Consequently, $b_{\mathbf{G}^F}(\chi)$ is either $b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ or $b_{\mathbf{G}^F}(\mathbf{L}', \lambda')$. Since also $b_{\mathbf{C}^{*F}}(\eta) = b_{\mathbf{C}^{*F}}(\mathbf{L}_s^*, \lambda_s)$, the claim follows from Proposition 4.9. □

Corollary 4.12. *Suppose $[\mathbf{X}, \mathbf{X}]$ is of exceptional type. Assume $\ell > 2$ is good for $\mathbf{C}^*$ and $\ell \neq 3$ if $\mathbf{C}^{*F}$ has a component of type ${}^3D_4$. Suppose $\mathbf{C}^* = \mathbf{C}_{\mathbf{b}}^*$ and that Theorem 1 holds for all proper F-stable Levi subgroups of $\mathbf{G}$. Then the conclusion of Theorem 1 holds for every ℓ-element $t \in \mathbf{C}^{*F}$.*

Proof. Let $t \in \mathbf{C}^{*F}$ be an ℓ-element. The assumptions imply that $\ell \in \Gamma(\mathbf{C}, F)$ in the notation of [5]. If $t \notin Z(\mathbf{C}^*)$, then (the proof of) [5, Prop. 2.5] shows that $C_{\mathbf{G}^*}(st) = C_{\mathbf{C}^*}(t)$ is contained in a proper e-split Levi subgroup $\mathbf{M}^*$ of $\mathbf{C}^*$ and thus also in the proper e-split Levi subgroup $C_{\mathbf{G}^*}(Z^\circ(\mathbf{M}^*)_e)$ of $\mathbf{G}^*$, and hence the conclusion follows inductively by Proposition 4.6. Thus $t \in Z(\mathbf{C}^*)$, so $C_{\mathbf{G}^*}(st) = C_{\mathbf{G}^*}(s)$. By Proposition 4.7, we may assume s is isolated in $\mathbf{G}^*$. By inspection of the lists of isolated elements in [1, Prop. 4.9, Tab. 3], if $t \in Z(\mathbf{C}^*)$ is an ℓ-element then even $t \in Z(\mathbf{G}^*)$, and we conclude by Proposition 4.11. □

4.5. **Theorem 1 for good primes.** Suppose that ℓ is a good prime for $\mathbf{G}$. Let $t \in \mathbf{C}^{*F} = C_{\mathbf{G}^*}(s)^F$ be an ℓ-element and let $\mathbf{G}(t) \le \mathbf{G}$ be a Levi subgroup of $\mathbf{G}$ in duality with the Levi subgroup $C_{\mathbf{G}^*}(t)$ of $\mathbf{G}^*$.

Let $\chi \in \mathcal{E}(\mathbf{G}^F, st)$. By properties of Digne–Michel's Jordan decomposition, $\chi = R_{\mathbf{G}(t)}^{\mathbf{G}}(\hat{t}\chi_t)$ where $\chi_t \in \mathcal{E}(\mathbf{G}(t)^F, s)$ satisfies $\pi_{st}^{\mathbf{G}}(\chi) = \pi_s^{\mathbf{G}(t)}(\chi_t)$. By the usual commutation property of the decomposition map with Lusztig induction (apply [7, Prop. 9.6] with $f = d^{1,\mathbf{G}}(1)$), χ lies in the same ℓ-block of $\mathbf{G}^F$ as some constituent of $R_{\mathbf{G}(t)}^{\mathbf{G}}(\chi_t)$. By results of Cabanes–Enguehard, we have the following.

Proposition 4.13. *With the notation above, every irreducible constituent of $R_{\mathbf{G}(t)}^{\mathbf{G}}(\chi_t)$ lies in $b_{\mathbf{G}^F}(\chi)$. Moreover, for every block c of $\mathbf{G}(t)^F$ in $\mathcal{E}_\ell(\mathbf{G}(t)^F, s)$, there exists a unique block $R_{\mathbf{G}(t)}^{\mathbf{G}}(c)$ of $\mathbf{G}^F$ in $\mathcal{E}_\ell(\mathbf{G}^F, s)$ such that every irreducible constituent of $R_{\mathbf{G}(t)}^{\mathbf{G}}(\mu)$, for $\mu \in \mathcal{E}(\mathbf{G}(t)^F, s) \cap \mathrm{Irr}(c)$, lies in $R_{\mathbf{G}(t)}^{\mathbf{G}}(c)$.*

Proof. By [6, Prop. 2.4], $C_{\mathbf{G}^*}(t)$ is $E_{q,\ell}$-split in $\mathbf{G}^*$ and therefore $\mathbf{G}(t)$ is $E_{q,\ell}$-split in $\mathbf{G}$. The assertion follows from [6, Thms 2.5 and 2.8]. □

Proposition 4.14. *Assume ℓ is odd and good for $\mathbf{G}$ and $\ell > 3$ if $\mathbf{G}^F$ has a component of type ${}^3D_4$. Let $(\mathbf{L}_t^*, \lambda_t)$ be a unipotent e-cuspidal pair of $C_{\mathbf{G}^*}(st)$ of (quasi-)central ℓ-defect and $(\mathbf{L}, \lambda) \in J_t^{\mathbf{G}}((\mathbf{L}_t^*, \lambda_t))$, $(\mathbf{L}(t), \lambda(t)) \in J_t^{\mathbf{G}(t)}((\mathbf{L}_t^*, \lambda_t))$. Suppose that $\mathbf{G}_\mathbf{b} \le \mathbf{G}(t)$. Then*

$$R_{\mathbf{G}(t)}^{\mathbf{G}}(b_{\mathbf{G}(t)^F}(\mathbf{L}(t), \lambda(t))) = b_{\mathbf{G}^F}(\mathbf{L}, \lambda).$$

Proof. Set $b := b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$, $\mathbf{H} := \mathbf{G}(t)$, $\mathbf{L}_\mathbf{H} := \mathbf{L}(t)$, and $\lambda_\mathbf{H} := \lambda(t)$. Set $\mathbf{G}_1 := \mathbf{G}_\mathbf{a}$ and $\mathbf{G}_2 := Z^\circ(\mathbf{G})\mathbf{G}_\mathbf{b}$. Then $\mathbf{G} = \mathbf{G}_1\mathbf{G}_2$, the kernel of the multiplication map from $\tilde{\mathbf{G}} := \mathbf{G}_1 \times \mathbf{G}_2$ to $\mathbf{G}$ is a central torus, isomorphic to $Z(\mathbf{G}) = Z^\circ(\mathbf{G})$, and we may put ourselves in the setting (and use the notation) of Proposition 2.4. So, letting $\tilde{\mathbf{H}}$ be the inverse image of $\mathbf{H}$ in $\tilde{\mathbf{G}}$, we have $\tilde{\mathbf{H}} = \mathbf{H}_1 \times \mathbf{H}_2$ with $\mathbf{H}_i$ an F-stable Levi subgroup of $\mathbf{G}_i$. Note that by hypothesis $\mathbf{H}_2 = \mathbf{G}_2$.

Let $(\mathbf{L}_s^*, \alpha)$ be a unipotent e-cuspidal pair of $C_{\mathbf{G}^*}(s)^F$ such that $\mathbf{L}^* = C_{\mathbf{G}^*}(Z^\circ(\mathbf{L}_s^*)_e)$ is dual to $\mathbf{L}$ in $\mathbf{G}$, $\pi_s^{\mathbf{L}}(\lambda) = \alpha$ and $(\mathbf{L}_s^*, \alpha) \to_t (\mathbf{L}_t^*, \lambda_t)$. Similarly, let $(\mathbf{L}_{s,\mathbf{H}}^*, \alpha_\mathbf{H})$ be a unipotent e-cuspidal pair of $C_{\mathbf{H}^*}(s)^F$ such that $\mathbf{L}_\mathbf{H}^* = C_{\mathbf{H}^*}(Z^\circ(\mathbf{L}_{s,\mathbf{H}}^*)_e)$ is dual to $\mathbf{L}_\mathbf{H}$ in $\mathbf{H}$, $\pi_s^{\mathbf{L}_\mathbf{H}}(\lambda_\mathbf{H}) = \alpha_\mathbf{H}$ and $(\mathbf{L}_{s,\mathbf{H}}^*, \alpha_\mathbf{H}) \to_t (\mathbf{L}_t^*, \lambda_t)$ in $C_{\mathbf{H}^*}(s)$. Note that by Proposition 2.10, applied with st in place of s, $\mathbf{L}_t^{*F}$ does not have a component of type ${}^3D_4$ and hence $\to_t$ equals $\sim$ in $C_{\mathbf{G}^*}(s)$ as well as in $C_{\mathbf{H}^*}(s)$. Here, we set $\mathbf{H}^* = \mathbf{G}(t)^* = C_{\mathbf{G}^*}(t)$. Further, since $C_{C_{\mathbf{H}^*}(s)}(t) = C_{\mathbf{H}^*}(s)$, we may in fact assume that $(\mathbf{L}_{s,\mathbf{H}}^*, \alpha_\mathbf{H}) = (\mathbf{L}_t^*, \lambda_t)$. So, in particular, $(\mathbf{L}_s^{*\prime}, \alpha|_{\mathbf{L}_s^{*\prime F}})$ and $(\mathbf{L}_{s,\mathbf{H}}^{*}{}', \alpha_\mathbf{H}|_{\mathbf{L}_{s,\mathbf{H}}^*{}'^F})$ are $C_{\mathbf{G}^*}(s)^F$-conjugate, where again we denote by X' the derived subgroup of X. By Proposition 2.4(d), it follows that $((\mathbf{L}_s^*)_2', \alpha_2|_{(\mathbf{L}_s^*)_2'^F})$ and $((\mathbf{L}_{s,\mathbf{H}}^*)_2', (\alpha_\mathbf{H})_2|_{(\mathbf{L}_{s,\mathbf{H}}^*)_2'^F})$ are $C_{\mathbf{G}_2^*}(s_2)^F$-conjugate. Further, by Proposition 2.4(a), $((\mathbf{L}_s^*)_2, \alpha_2)$ and $((\mathbf{L}_{s,\mathbf{H}}^*)_2, (\alpha_\mathbf{H})_2)$ are unipotent e-cuspidal pairs of $C_{\mathbf{G}_2^*}(s_2) =$

$C_{\mathbf{H}_2^*}(s_2)$. Therefore, by the argument at the beginning of the proof of Proposition 3.6 and up to conjugation in $C_{\mathbf{G}_2^*}(s_2)^F$, we may assume that

$$((\mathbf{L}_s^*)_2, \alpha_2) = ((\mathbf{L}_{s,\mathbf{H}}^*)_2, (\alpha_\mathbf{H})_2),$$

and hence by Proposition 2.4(b) that

$$(\mathbf{L}_2, \lambda_2) = ((\mathbf{L}_\mathbf{H})_2, (\lambda_\mathbf{H})_2).$$

Now let $\chi_\mathbf{H} \in \mathcal{E}(\mathbf{H}^F, (\mathbf{L}_\mathbf{H}, \lambda_\mathbf{H}))$ and let χ be a constituent of $R_\mathbf{H}^\mathbf{G}(\chi_\mathbf{H})$. By Proposition 2.4(c), $\tilde{\chi}_\mathbf{H} = (\chi_\mathbf{H})_1 \otimes (\chi_\mathbf{H})_2$ for some $(\chi_\mathbf{H})_i \in \mathcal{E}(\mathbf{H}_i^F, ((\mathbf{L}_H)_i, (\lambda_\mathbf{H})_i))$, $i = 1, 2$, and similarly $\tilde{\chi} = \chi_1 \otimes \chi_2$ with $\chi_i \in \mathcal{E}(\mathbf{G}^F, (\mathbf{H}_i, (\chi_\mathbf{H})_i))$. Since $\mathbf{H}_2 = \mathbf{G}_2$, we have $\chi_2 = (\chi_\mathbf{H})_2$ and as observed above, $(\mathbf{L}_2, \lambda_2) = ((\mathbf{L}_\mathbf{H})_2, (\lambda_\mathbf{H})_2)$. Hence, $\chi_2 \in \mathcal{E}(\mathbf{G}_2^F, (\mathbf{L}_2, \lambda_2)) \subseteq \mathcal{E}(\mathbf{G}_2^F, s_2) \cap \mathrm{Irr}(b_2)$. On the other hand, $\chi_1 \in \mathcal{E}(\mathbf{G}_1^F, s_1)$ and by Lemma 2.9, $\mathcal{E}(\mathbf{G}_1^F, s_1) = \mathcal{E}(\mathbf{G}_1^F, (\mathbf{L}_1, \lambda_1))$. It follows by Proposition 2.4(c) that $\chi \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) \subseteq \mathrm{Irr}(b)$. Since $\chi_\mathbf{H} \in \mathrm{Irr}(b_{\mathbf{H}^F}(\mathbf{H}, \lambda_\mathbf{H})$, the result follows by Proposition 4.13. □

Proposition 4.15. *Suppose that ℓ is odd and good for $\mathbf{G}$, and $\ell > 3$ if $\mathbf{G}^F$ has a component of type ${}^3D_4$. Suppose also that Theorem* 1 *holds for all proper F-stable Levi subgroups of $\mathbf{G}$. Then Theorem* 1 *holds for $\mathbf{G}$ and ℓ.*

Proof. By hypothesis and Proposition 4.6, we may assume $\mathbf{G}(t)$ is not contained in any proper e-split Levi subgroup of $\mathbf{G}$. Thus by [5, Prop. 2.5] we may assume that $\mathbf{G}_\mathbf{b} \leq \mathbf{G}(t)$. By Propositions 4.13 and 4.14 and the paragraph preceding Proposition 4.13 it suffices to prove the result for the case $\mathbf{G} = \mathbf{G}(t)$ and $t = 1$. This case was settled in Proposition 4.11. □

4.6. **Proof of Theorem 1.** We are now ready to complete the proof of our main theorem by checking individually the various isolated blocks at bad primes in simple groups of exceptional type.

Proof of Theorem 1. Let $\mathbf{G}$ be as in the statement and assume that Theorem 1 holds for every proper F-stable Levi subgroup of $\mathbf{G}$. If ℓ is odd and good for $\mathbf{G}$ and $\ell > 3$ if $\mathbf{G}^F$ has a component of type ${}^3D_4$, the result follows by Proposition 4.15. If $\mathbf{G}$ does not have any factors of exceptional type and $\ell = 2$, then $\mathcal{E}_\ell(\mathbf{G}^F, s)$ is a single ℓ-block (see [7, Thm 21.14]) and the claim holds trivially. So we are reduced to the case that $\mathbf{G}^F$ has a factor of exceptional type and that ℓ is bad for $\mathbf{G}$, or $\ell = 3$ and $\mathbf{G}^F$ has a component of type ${}^3D_4$. By Proposition 4.7 we may assume that s is isolated in $\mathbf{G}^*$ and by Proposition 4.8 that s is not central in $\mathbf{G}^*$.

Suppose that $\ell = 3$ and $\mathbf{G}^F$ has a component of type ${}^3D_4$. By Proposition 2.10 this implies $[\mathbf{X}, \mathbf{X}]$ is of exceptional type. Then, $\mathbf{G}$ is of type D_4, D_4A_1 or D_4A_2. Since groups of type A have no non-central isolated elements, the D_4-component of s in $\mathbf{G}^*$ is non-central and isolated. Thus, by [21, Tab. 9], all components of $C_{\mathbf{G}^*}(s)$ are of type A and unless $\mathbf{G}$ has type D_4A_2 they are all of type A_1. By Corollary 4.12, we may inductively assume that $\mathbf{G}$ is of type D_4A_2 and hence again by [21, Tab. 9] the rational type of $C_{\mathbf{G}^*}(s)$ is $A_1(q^3)A_1(q).A_2(\epsilon q)$ for some $\epsilon \in \{\pm 1\}$. If $3 \not| (q - \epsilon)$ then we conclude by Corollary 4.12, while in the opposite case, by [7, Prop. 3.3], $C_{\mathbf{G}^*}(s)$ has only one conjugacy class of unipotent e-cuspidal pairs. It follows that $\mathcal{E}_3(\mathbf{G}^*, s)$ is a single 3-block and the claim holds trivially.

Thus, we may assume that ℓ is bad for $\mathbf{G}$, s is isolated and non-central, and either $[\mathbf{G},\mathbf{G}]$ is simple of exceptional type or of type E_6A_1.

We now discuss these remaining possibilities according to the structure of $[\mathbf{G},\mathbf{G}]^F$.

Groups $G_2(q)$

Here the centralisers of isolated elements $s\neq 1$ are of rational types $A_1(q)^2$, $A_2(q)$ and ${}^2A_2(q)$, and in each case, $\mathcal{E}_\ell(\mathbf{G}^F,s)$ is a single ℓ-block for the appropriate primes ℓ, see [21, Tab. 9]. Thus Theorem 1 is trivially satisfied.

Groups $F_4(q)$

For $\ell=3$ let $1\neq s\in\mathbf{G}^{*F}$ be an isolated 2-element. Then $\mathbf{C}^*=C_{\mathbf{G}^*}(s)$ has only factors of classical type, so 3 is a good prime for $\mathbf{C}^*$, and moreover the assumptions of Corollary 4.12 are satisfied, whence we are done. For $\ell=2$ there is nothing to prove as by [21, Tab. 2] for all isolated 3-elements $s\neq 1$, $\mathcal{E}_2(G,s)$ is a single 2-block.

Groups $E_6(q)$ **and** ${}^2E_6(q)$

We give the arguments for $E_6(q)$, the case of ${}^2E_6(q)$ being entirely similar. For $\ell=3$ the centralisers of isolated 2-elements $1\neq s\in\mathbf{G}^*$ are of rational type $A_5(q)A_1(q)$. We discuss the various possibilities for 3-elements $t\in\mathbf{C}^{*F}$. If t is central, so $\ell=3$ divides $|Z(\mathbf{G}^*)^F|$ and hence $e=1$, we conclude by Proposition 4.11. Otherwise, by inspection $C_{\mathbf{G}^*}(st)$ lies in a proper e-split Levi subgroup of $\mathbf{G}^*$ and we can apply Proposition 4.6. For $\ell=2$ again by Table 3 for all isolated 3-elements $s\neq 1$, $\mathcal{E}_2(G,s)$ is a single 2-block.

Groups $E_6(q)A_1(q)$ **and** ${}^2E_6(q)A_1(q)$

Here we can argue in a completely similar fashion as in the previous case since the A_1-factor has no non-central isolated elements.

Groups $E_7(q)$

For $\ell=3$ by [21, Tab. 4] we only need to consider the blocks in Table 4 below. Here all centralisers of isolated 2-elements satisfy the assumptions of Corollary 4.12 and we are done. Note that $E_7(2)$ does not need to be considered as it has no (isolated) semisimple 2-elements. For $\ell=2$ again $\mathcal{E}_2(G,s)$ is a single 2-block for all isolated 3-elements $s\neq 1$ by [21, Tab. 4].

Groups $E_8(q)$

First assume that $\ell=5$. Let $1\neq s\in\mathbf{G}^{*F}$ be an isolated $5'$-element and $\mathbf{C}^*=C_{\mathbf{G}^*}(s)$. Then 5 is good for $\mathbf{C}^*$ and the assumptions of Corollary 4.12 are satisfied for $\mathbf{C}^*$ (see [21, Tab. 7 and 8] and Tables 5 and 6 below). This completes the argument when $\ell=5$.

Now assume that $\ell=3$ and let $e=e_3(q)$. Let $1\neq s\in\mathbf{G}^{*F}$ be an isolated $3'$-element and $\mathbf{C}^*=C_{\mathbf{G}^*}(s)$. If $\mathbf{C}^*$ has only classical factors not of rational type ${}^3D_4$, then 3 is a good prime for $\mathbf{C}^*$ and we can apply exactly the same argument as in the case $\ell=5$ to conclude. The only centraliser for which this approach fails is when s is an involution with $\mathbf{C}^*$ of rational type $E_7(q)A_1(q)$.

The Harish-Chandra series in this case are listed in Table 8 below (copied from [21, Tab. 6]) for $e=1$; for $e=2$ we have the Ennola dual situation which can be treated in exactly the same manner. We discuss the various 3-elements $t\in\mathbf{C}^{*F}$. Assume that t has a non-central component in the A_1-factor. Then $C_{\mathbf{G}^*}(st)$ is contained in a 1-split Levi subgroup of $\mathbf{G}^*$ of rational type $E_7(q).\Phi_1$, and we may conclude by Proposition 4.6. Thus, the centraliser of t does not contain the whole E_7-factor. But now by inspection the centraliser of any element of order 3 in $E_7(q)$ is either of type A_5A_2, or it is contained in a proper 1-split Levi subgroup of $E_7(q)$. Thus, any non-trivial 3-element of $E_7(q)$ none of whose powers has centraliser A_5A_2 has its centraliser in a proper 1-split Levi subgroup and Proposition 4.6 applies.

So finally assume t is such that $C_{\mathbf{G}^*}(st^k)^F = A_5(q)A_2(q)A_1(q)$ for some $k \geq 1$. It can be checked using Chevie and the known block distribution for $\mathcal{E}(\mathbf{G}^F, s)$ from [21, Prop. 6.7] that all but four classes of maximal tori $\mathbf{T}^*$ of $A_5(q)A_2(q)A_1(q)$ have the property that all constituents of $R_{\mathbf{T}}^{\mathbf{G}}(\hat{s})$ lie in the same 3-block of $\mathbf{G}^F$, namely the semisimple block in $\mathcal{E}_3(\mathbf{G}^F, s)$. Now let $\mathbf{T}^*$ be the maximally split torus of $C_{\mathbf{G}^*}(st)$; note that all factors of this centraliser have untwisted type A. So $R_{\mathbf{T}^*}^{C_{\mathbf{G}^*}(st)}(1)$ contains all unipotent characters of $C_{\mathbf{G}^*}(st)$. Note that the maximally split torus is uniquely determined up to conjugacy inside $C_{\mathbf{G}^*}(st)$ by its order. Thus, if $\mathbf{T}^*$ is not one of the four exceptions mentioned before, then by Lemma 2.8 all $\chi \in \mathcal{E}(\mathbf{G}^F, st)$ lie in the semisimple block in $\mathcal{E}_3(\mathbf{G}^F, s)$, and we may conclude with Proposition 4.3 that Theorem 1 holds in this case. The excluded maximal tori $\mathbf{T}^*$, intersected with the $A_5(q)A_2(q)$-factor of $C_{\mathbf{G}^*}(st^k)^F$, have orders $\Phi_3^3\Phi_1$ or $\Phi_2\Phi_3^2\Phi_6$, and they are maximally split in a centraliser in $A_5(q)A_2(q)$ only when this centraliser is contained in a subgroup $\Phi_3^3A_1(q)$, respectively in $\mathbf{T}^*$. But the second does not occur as the centraliser of a 3-element in $A_5(q)A_2(q)$, while for elements with centraliser the first type, no power has centraliser $A_5(q)A_2(q)A_1(q)$.

For $\ell = 2$, by [21, Tab. 5] there are only two types of non-central isolated elements $s \in \mathbf{G}^{*F}$ of order 3 to consider with centralisers as listed in Table 2, which is taken from loc. cit., for $q \equiv 1 \pmod 4$. The case $q \equiv 3 \pmod 4$ is Ennola dual to this one and analogous arguments apply to it.

TABLE 2. Harish-Chandra series in some isolated 2-blocks of $E_8(q)$, $q \equiv 1 \pmod 4$

No.	$C_{\mathbf{G}^*}(s)^F$	$\mathbf{L}^F$	$C_{\mathbf{L}^*}(s)^F$	λ	$W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$
3	$E_6(q).A_2(q)$	$\emptyset$	$\mathbf{L}^{*F}$	1	$E_6 \times A_2$
		D_4	$\mathbf{L}^{*F}$	$D_4[1]$	$G_2 \times A_2$
4		E_6	$\mathbf{L}^{*F}$	$E_6[\theta^{\pm 1}]$	A_2
5	${}^2E_6(q).{}^2A_2(q)$	A_1^3	$\Phi_1^5\Phi_2^3$	1	$F_4 \times A_1$
		D_4	$\Phi_1^4\Phi_2^2.{}^2A_2(q)$	ϕ_{21}	F_4
		D_6	$\Phi_1^2\Phi_2.{}^2A_5(q)$	ϕ_{321}	$A_1 \times A_1$
		E_7	$\Phi_1.{}^2A_5(q){}^2A_2(q)$	$\phi_{321} \otimes \phi_{21}$	A_1
		E_7	$\Phi_1\Phi_2.{}^2E_6(q)$	${}^2E_6[1]$	A_1
		E_8	$C_{\mathbf{G}^*}(s)^F$	${}^2E_6[1] \otimes \phi_{21}$	1
6		E_7	$\Phi_1\Phi_2.{}^2E_6(q)$	${}^2E_6[\theta^{\pm 1}]$	A_1
		E_8	$C_{\mathbf{G}^*}(s)^F$	${}^2E_6[\theta^{\pm 1}] \otimes \phi_{21}$	1

The involution centralisers in $C_{\mathbf{G}^*}(s)^F = E_6(q).A_2(q)$ either lie in a proper 1-split Levi subgroup, or equal $A_5(q)A_2(q)A_1(q)$. Thus, again, we only need to worry about 2-elements $t \neq 1$ such that $C_{\mathbf{G}^*}(st^k) = A_5(q)A_2(q)A_1(q)$ for some $k \geq 1$. In this case all constituents of the Deligne–Lusztig characters for maximal tori of $C_{\mathbf{G}^*}(st^k)$ lie in the semisimple 2-block in $\mathcal{E}_2(\mathbf{G}^F, s)$ as described in [21, Prop. 6.4] and we conclude with Lemma 2.8.

Finally assume $C_{\mathbf{G}^*}(s)^F = {}^2E_6(q).{}^2A_2(q)$. Here, centralisers of involutions are either contained in a proper 1-split Levi subgroup and we are done by induction, or are of rational type ${}^2A_5(q)A_1(q).{}^2A_2(q)$ or ${}^2D_5(q)\Phi_2.{}^2A_2(q)$. By a Chevie-computation, all maximal tori $\mathbf{T}^*$ of either of the latter two subgroups have the property that all constituents of $R_{\mathbf{T}}^{\mathbf{G}}(\hat{s})$ lie in the semisimple block of $\mathcal{E}_2(\mathbf{G}^F, s)$ as described in [21, Prop. 6.4], and so we may conclude as before. □

Proof of Corollary 2. By [11, Thm A, Thm A.bis], the Jordan correspondents of χ and χ' lie in the same unipotent ℓ-block of $C_{\mathbf{G}^*}(r)^F$. Now the result is immediate from Theorem 1. □

5. Descent to quasi-simple groups of types E_6 and E_7

Let $\mathbf{X}$ be as in Theorem 1 such that $\mathbf{G} := [\mathbf{X}, \mathbf{X}]$ is simple of simply connected type E_6 or E_7; so $\mathbf{G} \hookrightarrow \mathbf{X}$ is a regular embedding. Now $\mathbf{G}_{\text{ad}} := \mathbf{X}/Z(\mathbf{X})$ is simple of adjoint type, and $\mathbf{G}_{\text{ad}}^F = (\mathbf{X}/Z(\mathbf{X}))^F \cong \mathbf{X}^F/Z(\mathbf{X}^F)$ since $Z(\mathbf{X})$ is connected, so Theorem 1 immediately gives a description of the ℓ-blocks of $\mathbf{G}_{\text{ad}}^F$, by only considering those characters that are trivial on $Z(\mathbf{X}^F)$.

We also obtain strong information on the ℓ-blocks of groups of simply connected type. Namely, with $m := |Z(\mathbf{G}^F)|$ we have: if $m = 1$ then $\mathbf{G}^F = (\mathbf{G}/Z(\mathbf{G}))^F = \mathbf{G}_{\text{ad}}^F$ which was discussed above, while if $m \neq 1$ then $\mathbf{G}^F$ is quasi-simple with $m = 3$ for type E_6 and $m = 2$ for type E_7. For general facts on covering blocks see [24, Sec. 6.8].

Proposition 5.1. *In the setting introduced above, let ℓ be a non-defining prime for $\mathbf{G}$, let $b = b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ be an ℓ-block of $\mathbf{G}^F$ in $\mathcal{E}_\ell(\mathbf{G}^F, s)$ for $s \in \mathbf{G}^{*F}$ a semisimple ℓ'-element, and let B be an ℓ-block of $\mathbf{X}^F$ covering b.*

(a) *If b is $\mathbf{X}^F$-invariant, the members of* $\operatorname{Irr}(b)$ *are the constituents of the restriction to $\mathbf{G}^F$ of the members of* $\operatorname{Irr}(B)$*. If moreover $C_{\mathbf{G}^*}(s)^F = C^\circ_{\mathbf{G}^*}(s)^F$ and $\ell \neq m$, restriction defines a height preserving bijection from* $\operatorname{Irr}(B)$ *to* $\operatorname{Irr}(b)$.

(b) *If b is not $\mathbf{X}^F$-invariant, then $C^\circ_{\mathbf{G}^*}(s)^F < C_{\mathbf{G}^*}(s)^F$, in particular $C_{\mathbf{G}^*}(s)$ is disconnected, $\ell \neq m$, the block b is $\mathbf{X}^F$-conjugate to m distinct blocks of $\mathbf{G}^F$, all $\tilde{\chi} \in \operatorname{Irr}(B)$ have reducible restriction to $\mathbf{G}^F$ and the restriction of every member of* $\operatorname{Irr}(B)$ *to $\mathbf{G}^F$ contains one constituent in each of these blocks. This defines a height preserving bijection from* $\operatorname{Irr}(B)$ *to* $\operatorname{Irr}(b)$.

In particular, the ℓ-block distribution of $\mathbf{G}^F$ is determined, up to the labelling of characters in $\mathbf{X}^F$-orbits in case (b)*, by Theorem* 1.

Proof. The first claim in (a) is a standard fact about covering blocks. Assume $C_{\mathbf{G}^*}(s)^F = C^\circ_{\mathbf{G}^*}(s)^F$ and $m \neq \ell$. We claim that $C_{\mathbf{G}^*}(st)^F = C^\circ_{\mathbf{G}^*}(st)^F$ for all ℓ-elements $t \in C_{\mathbf{G}^*}(s)^F$. If not, then $|C_{\mathbf{G}^*}(st)^F : C^\circ_{\mathbf{G}^*}(st)^F| = m > 1$. But $C_{\mathbf{G}^*}(st) = C_{C_{\mathbf{G}^*}(s)}(t)$, and m is prime to ℓ, hence to the order of t, contradicting [26, Prop. 14.20]. Thus all characters in $\operatorname{Irr}(B)$ restrict irreducibly to $\mathbf{G}^F$. This gives the last claim in (a).

So now assume that b is not $\mathbf{X}^F$-invariant. Then no $\chi \in \operatorname{Irr}(b)$ is $\mathbf{X}^F$-invariant, so necessarily $C_{\mathbf{G}^*}(s)^F > C^\circ_{\mathbf{G}^*}(s)^F$ (and hence $C_{\mathbf{G}^*}(s)$ is disconnected), which again by [26, Prop. 14.20] implies that $o(s)$ is divisible by m and so $\ell \neq m$. Further, B covers the $m = |\mathbf{X}^F : \mathbf{G}^F Z(\mathbf{X}^F)|$ distinct $\mathbf{X}^F$-conjugates of b and the restriction to $\mathbf{G}^F$ of every character

$\tilde{\chi} \in \mathrm{Irr}(B)$ has constituents in each of them. Since $\mathbf{X}^F/\mathbf{G}^F$ is cyclic, all restrictions are multiplicity-free and thus have m constituents. The last claim of (b) then follows. □

Remark 5.2. (1) If $\mathbf{G}^F$ is of adjoint type then $m = 1$ and we are in the situation of case (a).

(2) If $\ell \neq 2$ then case (a) occurs precisely when the $\mathbf{G}^F$-class of $(\mathbf{L}, \lambda)$ is $\mathbf{X}^F$-invariant, by [22, Thm A] (but see the counter-example in [22, Exmp. 3.16] when $\ell = 2$.)

(3) In (a), if either $C^\circ_{\mathbf{G}^*}(s)^F < C_{\mathbf{G}^*}(s)^F$ or $\ell = m$, there will in general not exist a height preserving bijection.

(4) The proof shows that in case (b) the centralisers $C_{\mathbf{G}^*}(st)$ are disconnected for all ℓ-elements $t \in C_{\mathbf{G}^*}(s)$.

Example 5.3. In case (a) of Proposition 5.1, $C_{\mathbf{G}^*}(s)$ could be connected or disconnected: an example for the first case is the block 13 in Table 3 below, an example for the second is the block 1 in Table 3. Even if $C_{\mathbf{G}^*}(s)$ is connected but $\ell = m$ there may exist ℓ-elements $t \in C_{\mathbf{G}^*}(s)^F$ with $C_{\mathbf{G}^*}(st)$ disconnected (e.g., taking $s = 1$).

6. On e-Harish-Chandra series in exceptional groups

In this section we complement our results from [21] in several ways. First we parametrise the isolated blocks for bad primes in exceptional groups of type E_6 and E_7 with connected centre and verify the validity of an e-Harish-Chandra theory in this situation. Second, we give the block distribution for some isolated 5-blocks in groups of type E_8 inadvertently omitted in [21]. Finally, we give the proofs of the extensions of results from [21] and [22] to the situation considered in the present paper. The e-cuspidal pairs in the subsequent tables can be determined, for example, as explained in [33].

6.1. **Isolated blocks in exceptional groups of adjoint type.**

In this section $\mathbf{G}$ is a group with connected centre such that $[\mathbf{G}, \mathbf{G}]$ is simple of simply connected type E_6 or E_7 and $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius endomorphism with respect to an $\mathbb{F}_q$-structure.

It follows by looking at the root datum that $\mathbf{G}$ is isomorphic to its dual $\mathbf{G}^*$. Since $\mathbf{G}$ has connected centre, all centralisers of semisimple elements in $\mathbf{G}^*$ are connected and thus the notions of isolated and quasi-isolated elements do coincide. If $s \in \mathbf{G}^*$ is isolated, then so is sz for any $z \in Z(\mathbf{G}^*)$, with the same centraliser, and this defines an equivalence relation on the set of conjugacy classes of isolated semisimple elements. We describe the ℓ-block subdivision of $\mathcal{E}(\mathbf{G}, s)$ for $s \in \mathbf{G}^{*F}$ an isolated ℓ'-element and $\ell \in \{2, 3\}$ not divding q. Note that the blocks corresponding to two isolated elements s and sz, for $z \in Z(\mathbf{G}^{*F})$, are obtained from one another by tensoring with a linear character $\hat{z}$ of $\mathbf{G}^F$ (see [17, Prop. 2.5.21]). Also note that the case of $s = 1$, that is, of unipotent blocks has been dealt with by Enguehard [11], so we may assume s is non-central. As before, let $e = e_\ell(q)$ be the order of q modulo ℓ when ℓ is odd, respectively the order of q modulo 4 when $\ell = 2$.

We first determine the decomposition of Lusztig induction for e-Harish-Chandra series of $\mathbf{G}$ in $\mathcal{E}(\mathbf{G}^F, s)$.

Proposition 6.1. *Let $\mathbf{G}$ be as above, $\ell \in \{2, 3\}$ not dividing q, $s \in \mathbf{G}^{*F}$ a non-central isolated ℓ'-element, and set $e = e_\ell(q)$. Then we have:*

(a) *If* $[\mathbf{G},\mathbf{G}]^F = E_6(q)_{\rm sc}$ *then* $\mathcal{E}(\mathbf{G}^F,s)$ *is the disjoint union of the e-Harish-Chandra series listed in Table* 3*.*
(b) *If* $[\mathbf{G},\mathbf{G}]^F = {}^2E_6(q)_{\rm sc}$ *then the e-Harish-Chandra series in* $\mathcal{E}(\mathbf{G}^F,s)$ *are the Ennola duals of those in Table* 3*.*
(c) *If* $[\mathbf{G},\mathbf{G}]^F = E_7(q)_{\rm sc}$ *then* $\mathcal{E}(\mathbf{G}^F,s)$ *is the disjoint union of the e-Harish-Chandra series listed in Table* 4 *when* $e=1$*, respectively their Ennola duals when* $e=2$*.*
(d) *The assertion of* [21, Thm 1.4] *continues to hold for* $\mathbf{G}$*.*

In Tables 3 and 4, we give the data relative to $\overline{\mathbf{G}} := \mathbf{G}/Z(\mathbf{G})$, a simple group of adjoint type. The numbering of blocks follows [21, Tab. 3 and 4].

TABLE 3. e-Harish-Chandra series in $\overline{\mathbf{G}}^F = E_6(q)_{\rm ad}$

No.	$C_{\overline{\mathbf{G}}^*}(s)^F$	(ℓ,e)	$\overline{\mathbf{L}}^F$	$C_{\overline{\mathbf{L}}^*}(s)^F$	λ	$W_{\overline{\mathbf{G}}^F}(\overline{\mathbf{L}},\lambda)$
1	$A_2(q)^3$	$(2,1)$	Φ_1^6	$\overline{\mathbf{L}}^{*F}$	1	A_2^3
2	$A_2(q^3)$	$(2,1)$	$\Phi_1^2.A_2(q)^2$	$\Phi_1^2\Phi_3^2$	1	A_2
6	$A_2(q^2).{}^2A_2(q)$	$(2,1)$	$\Phi_1^3.A_1(q)^3$	$\Phi_1^3\Phi_2^3$	1	$A_2\times A_1$
			$\Phi_1^2.D_4(q)$	$\Phi_1^2\Phi_2^2.{}^2A_2(q)$	ϕ_{21}	A_2
7	$A_2(q)^3$	$(2,2)$	$\Phi_1^2\Phi_2^3.A_1(q)$	$\Phi_1^3\Phi_2^3$	1	A_1^3
			$\Phi_1\Phi_2^2.A_3(q)$	$\Phi_1^2\Phi_2^2.A_2(q)$	ϕ_{21} (3×)	$A_1\times A_1$
			$\Phi_2.A_5(q)$	$\Phi_1\Phi_2.A_2(q)^2$	$\phi_{21}^{\otimes 2}$ (3×)	A_1
			$\overline{\mathbf{G}}^F$	$C_{\overline{\mathbf{G}}^*}(s)^F$	$\phi_{21}^{\otimes 3}$	1
8	$A_2(q^3)$	$(2,2)$	$\Phi_2.A_2(q^2)A_1(q)$	$\Phi_1\Phi_2\Phi_3\Phi_6$	1	A_1
			$\overline{\mathbf{G}}^F$	$C_{\overline{\mathbf{G}}^*}(s)^F$	ϕ_{21}	1
12	$A_2(q^2).{}^2A_2(q)$	$(2,2)$	$\Phi_1^2\Phi_2^4$	$\overline{\mathbf{L}}^{*F}$	1	$A_2\times A_2$
13	$A_5(q)A_1(q)$	$(3,1)$	Φ_1^6	$\overline{\mathbf{L}}^{*F}$	1	$A_5\times A_1$
14	$A_5(q)A_1(q)$	$(3,2)$	$\Phi_1^2\Phi_2^4$	$\overline{\mathbf{L}}^{*F}$	1	$C_3\times A_1$
15			$\Phi_2.A_5(q)$	$\overline{\mathbf{L}}^{*F}$	ϕ_{321}	A_1

Proof. Let $\mathbf{G}_0 := [\mathbf{G},\mathbf{G}]$, a simple group of simply connected type, and consider the regular embedding $\mathbf{G}_0 \hookrightarrow \mathbf{G}$. Then for any F-stable Levi subgroup $\mathbf{L}_0 \le \mathbf{G}_0$ we have

$$\mathrm{Ind}_{\mathbf{G}_0^F}^{\mathbf{G}^F} \circ R_{\mathbf{L}_0}^{\mathbf{G}_0} = R_{\mathbf{L}}^{\mathbf{G}} \circ \mathrm{Ind}_{\mathbf{L}_0^F}^{\mathbf{L}^F}$$

(see [17, Prop. 3.2.9]), where $\mathbf{L} = Z(\mathbf{G})\mathbf{L}_0$ is the corresponding Levi subgroup of $\mathbf{G}$. Now above every character in $\mathcal{E}(\mathbf{G}_0^F,s)$ there are $|C_{\mathbf{G}^*}(s)^F : C_{\mathbf{G}_0^*}(s)^F|$ characters of $\mathbf{G}^F$, lying in distinct Lusztig series, and similarly, above every character in $\mathcal{E}(\mathbf{L}_0^F,s)$ there are $|C_{\mathbf{L}^*}(s)^F : C_{\mathbf{L}_0^*}(s)^F|$ characters of $\mathbf{L}^F$, lying in distinct Lusztig series. Write $\mathrm{Ind}_{\mathbf{L}_0^F}^{\mathbf{L}^F}(\lambda) = \sum_i \lambda_i$ with $\lambda_i \in \mathrm{Irr}(\mathbf{L}^F)$. Thus all constituents λ_i lie in distinct Lusztig series of $\mathbf{L}^F$, so any summand $R_{\mathbf{L}}^{\mathbf{G}}(\lambda_i)$ of $(R_{\mathbf{L}}^{\mathbf{G}} \circ \mathrm{Ind}_{\mathbf{L}_0^F}^{\mathbf{L}^F})(\lambda) = \sum_i R_{\mathbf{L}}^{\mathbf{G}}(\lambda_i)$ lies inside a fixed Lusztig series

Table 4. e-Harish-Chandra series in $\overline{\mathbf{G}}^F = E_7(q)_{\rm ad}$

No.	$C_{\overline{\mathbf{G}}^*}(s)^F$	(ℓ, e)	$\overline{\mathbf{L}}^F$	$C_{\overline{\mathbf{L}}^*}(s)^F$	λ	$W_{\overline{\mathbf{G}}^F}(\overline{\mathbf{L}}, \lambda)$
1	$A_5(q)A_2(q)$	$(2,1)$	Φ_1^7	$\overline{\mathbf{L}}^{*F}$	1	$A_5 \times A_2$
2	${}^2A_5(q){}^2A_2(q)$	$(2,1)$	$\Phi_1^4.(A_1(q)^3)'$	$\Phi_1^4\Phi_2^3$	1	$C_3 \times A_1$
			$\Phi_1^3.D_4(q)$	$\Phi_1^3\Phi_2^2.{}^2A_2(q)$	ϕ_{21}	C_3
			$\Phi_1.D_6(q)$	$\Phi_1\Phi_2.{}^2A_5(q)$	ϕ_{321}	A_1
			$E_7(q)$	$C_{\overline{\mathbf{G}}^*}(s)^F$	$\phi_{321} \otimes \phi_{21}$	1
3	$D_6(q)A_1(q)$	$(3,1)$	Φ_1^7	$\overline{\mathbf{L}}^{*F}$	1	$D_6 \times A_1$
4			$\Phi_1^3.D_4(q)$	$\overline{\mathbf{L}}^{*F}$	$D_4[1]$	$B_2 \times A_1$
5	$A_7(q)$	$(3,1)$	Φ_1^7	$\overline{\mathbf{L}}^{*F}$	1	A_7
6	${}^2A_7(q)$	$(3,1)$	$\Phi_1^4.(A_1(q)^3)'$	$\Phi_1^4\Phi_2^3$	1	C_4
7			$\Phi_1.D_6(q)$	$\Phi_1\Phi_2.{}^2A_5(q)$	ϕ_{321}	A_1
12	$A_3(q){}^2A_1(q)$	$(3,1)$	Φ_1^7	$\overline{\mathbf{L}}^{*F}$	1	$A_3^2 \times A_1$
13	${}^2A_3(q){}^2A_1(q)$	$(3,1)$	$\Phi_1^5.A_1(q)^2$	$\Phi_1^5\Phi_2^2$	1	$B_2^2 \times A_1$
14	$A_3(q^2)A_1(q)$	$(3,1)$	$\Phi_1^4.(A_1(q)^3)'$	$\Phi_1^4\Phi_2^3$	1	$A_3 \times A_1$

of $\mathbf{G}^F$ and those lying inside a fixed Lusztig series are equal. In particular, knowing $R_{\mathbf{L}_0}^{\mathbf{G}_0}$ and the decomposition of $\mathrm{Ind}_{\mathbf{G}_0^F}^{\mathbf{G}^F}$ along Lusztig series, we can recover the $R_{\mathbf{L}}^{\mathbf{G}}(\lambda_i)$ uniquely.

Thus we may recover the stated decomposition of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ from the one for $\mathbf{G}_0^F$ that was determined in [21, Prop. 4.1 and 5.1] (with the correction given in the proof of [22, Thm 3.14], see Remark 6.8 below), from where our numbering of cases is taken. This also shows that Harish-Chandra series for the case when $\mathbf{G}_0^F = {}^2E_6(q)_{\rm sc}$ are Ennola dual to those for the untwisted case, and (as in [21]) we do not print them here. The existence of an e-Harish-Chandra theory as in [21, Thm 1.4] follows (see [21, Def. 2.9]). □

Lemma 6.2. *Let* $\mathbf{G}$, $\mathbf{L}$ *and* ℓ *be as in Proposition* 6.1, *with* $e = e_\ell(q)$. *Then:*

(a) $\mathbf{L} = C_{\mathbf{G}}(Z(\mathbf{L})_\ell^F)$ *and* $\mathbf{L}$ *is* (e,ℓ)*-adapted; and*

(b) *in Tables* 3 *and* 4, λ *is of quasi-central* ℓ*-defect precisely in the numbered lines.*

In fact, in all numbered lines except 6, 7 and 8 in Table 3, λ is even of central ℓ-defect. We had shown in [22, Lemma 2.7(b)] that regular embeddings do preserve the property of having quasi-central ℓ-defect.

By Proposition 6.1 and Lemma 6.2 the assumptions of [21, Prop. 2.17] are satisfied, so each e-Harish-Chandra series in the two tables is contained in a unique ℓ-block of $\mathbf{G}^F$.

Proposition 6.3. *Let* $[\mathbf{G},\mathbf{G}]^F = E_6(q)_{\rm sc}$ *(resp.* $[\mathbf{G},\mathbf{G}]^F = E_7(q)_{\rm sc}$*). Then for any isolated non-central* ℓ'*-element* $s \in \mathbf{G}^{*F}$ *the* ℓ*-block subdivision of* $\mathcal{E}(\mathbf{G}^F, s)$ *is as indicated by the horizontal lines in Tables* 3 *and* 4.

For each ℓ*-block corresponding to one of the cases in the tables there is a defect group* $P \leq N_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ *with the structure described in* [21, Thm 1.2].

The analogous, Ennola dual statement holds for $[\mathbf{G},\mathbf{G}]^F = {}^2E_6(q)_{\rm sc}$.

Proof. This is entirely analogous to the proofs of [21, Prop. 4.3 and 5.3]. □

Remark 6.4. The group $\mathbf{G}/Z(\mathbf{G})$ is simple of adjoint type, and as $Z(\mathbf{G})$ is connected, $(\mathbf{G}/Z(\mathbf{G}))^F = \mathbf{G}^F/Z(\mathbf{G})^F$, so the above result provides also a parametrisation of the isolated ℓ-blocks of the groups $E_6(q)_{\mathrm{ad}}$, ${}^2E_6(q)_{\mathrm{ad}}$ and $E_7(q)_{\mathrm{ad}}$ for $\ell = 2, 3$.

6.2. **Some 5-blocks in $E_8(q)$.** We deal with a situation missed in our paper [21]. We thank Niamh Farrell for pointing this omission out to us. Let $\mathbf{G}$ be of type E_8 with a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ such that $G = \mathbf{G}^F = E_8(q)$. If $q \equiv \pm 1 \pmod 6$ there exists an isolated element $s \in G^* \cong G$ of order six with centraliser $C_{\mathbf{G}^*}(s)$ of type $A_5A_2A_1$. It was inadvertently left out of [21, Tab. 1] (probably as its order is divisible by two distinct bad primes; but type E_8 has three bad primes). The centraliser of s in $\mathbf{G}^*$ has rational type $A_5(q)A_2(q)A_1(q)$ if $q \equiv 1 \pmod 6$, and ${}^2A_5(q).{}^2A_2(q)A_1(q)$ if $q \equiv 5 \pmod 6$. We parametrise the ℓ-blocks in $\mathcal{E}_\ell(G, s)$ by e-cuspidal pairs for the only relevant bad prime $\ell = 5$.

Theorem 6.5. *Theorems* 1.2, 1.4 *and* 1.5 *of* [21] *continue to hold for all quasi-isolated* 5*-blocks of* $E_8(q)$ *described above.*

Proof. The method is completely analogous to that employed in [21]. There are three cases to distinguish, depending on whether $e := e_5(q)$ is 1,2 or 4. First we determine the decomposition of the Lusztig functor $R_{\mathbf{L}}^{\mathbf{G}}$ for the relevant e-cuspidal pairs $(\mathbf{L}, \lambda)$ below $(\mathbf{G}^F, s)$. Since the unipotent characters of groups of type A are uniform, and Jordan decomposition commutes with Deligne–Lusztig induction, this decomposition follows from the known corresponding one for unipotent characters. This also shows that $\mathbf{G}^F$ satisfies an e-Harish-Chandra theory above each e-cuspidal pair below $(\mathbf{G}^F, s)$, thus showing the assertion of [21, Thm. 1.4] in this case. The decomposition is given in Tables 5 and 6. The notation is as in the analogous tables in [21]. Here, the case $e = 2$ can be obtained by Ennola duality from the one for $e = 1$, and the case of centraliser ${}^2A_5(q).{}^2A_2(q)A_1(q)$ when $q \equiv \pm 2 \pmod 5$ from the one with centraliser $A_5(q)A_2(q)A_1(q)$ when $q \equiv \mp 2 \pmod 5$. (Note that the e-Harish-Chandra series in Lines 1–5 are exactly as the Lines (1) and (2) in [21, Tab. 4].)

TABLE 5. Quasi-isolated 5-blocks in $E_8(q)$, $q \equiv 1 \pmod 5$

No.	$C_{\mathbf{G}^*}(s)^F$	$\mathbf{L}^F$	$C_{\mathbf{L}^*}(s)^F$	λ	$W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$
1	$A_5(q)A_2(q)A_1(q)$	Φ_1^8	$\mathbf{L}^{*F}$	1	$A_5 \times A_2 \times A_1$
2	${}^2A_5(q).{}^2A_2(q)A_1(q)$	$\Phi_1^5.A_1(q)^3$	$\Phi_1^5\Phi_2^3$	1	$C_3 \times A_1 \times A_1$
3		$\Phi_1^4.D_4(q)$	$\Phi_1^4\Phi_2^2.{}^2A_2(q)$	ϕ_{21}	$C_3 \times A_1$
4		$\Phi_1^2.D_6(q)$	$\Phi_1^2\Phi_2.{}^2A_5(q)$	ϕ_{321}	$A_1 \times A_1$
5		$\Phi_1.E_7(q)$	$\Phi_1.{}^2A_5(q).{}^2A_2(q)$	$\phi_{321} \otimes \phi_{21}$	A_1

It has been checked in [21, Lemma 6.9] that all relevant e-split Levi subgroups of $\mathbf{G}$ satisfy $C_{\mathbf{G}}(Z(\mathbf{L})_\ell^F) = \mathbf{L}$. (In fact they all already occur in Lines 19–23 of Table 7 respectively in Line 43 of Table 8 in [21].) Furthermore, all relevant e-cuspidal characters λ are readily seen to be of central ℓ-defect. But then by [21, Prop. 2.13 and 2.15] the two

Table 6. Quasi-isolated 5-blocks in $E_8(q)$, $q \equiv \pm 2 \pmod 5$

No.	$C_{\mathbf{G}^*}(s)^F$	$\mathbf{L}^F$	$C_{\mathbf{L}^*}(s)^F$	λ	$W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$
6	$A_5(q)A_2(q)A_1(q)$	$\Phi_4.{}^2D_6(q)$	$\Phi_1^3\Phi_4.A_2(q)A_1(q)$	6 chars	$Z_4 \times A_1$
7		$\mathbf{G}^F$	$C_{\mathbf{G}^*}(s)^F$	18 chars	1

conditions in [21, Prop. 2.12] are satisfied and so for all relevant e-cuspidal pairs $(\mathbf{L}, \lambda)$ all constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ lie in a single 5-block $b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$. Moreover, $Z^\circ(\mathbf{L})^F \cap [\mathbf{L}, \mathbf{L}]^F$ is a $5'$-group, hence by [21, Prop. 2.7(g)], in each case $(Z(\mathbf{L}^F)_\ell, b_{\mathbf{L}^F}(\lambda))$ is a centric $b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$-Brauer pair. If $(\mathbf{L}, \lambda)$ corresponds to Line 1 of Table 5, then by [21, Prop. 2.7(c)], a defect group of $b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ is an extension of $Z(\mathbf{L}^F)_\ell$ by a Sylow 5-subgroup of $W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$. In all other cases, the relative Weyl group is a $5'$-group, hence by [21, Prop. 2.7], $(Z(\mathbf{L})_\ell^F, \lambda)$ is a maximal $b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$-Brauer pair, and in particular $Z(\mathbf{L})_\ell^F$ is a defect group of $b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$. Thus the defect groups of the various blocks are as described in [21, Thm 1.2].

Since the orders of the Sylow 5-subgroups of the various $Z(\mathbf{L})^F$ in Lines 2–5 are all distinct, these lines correspond to different blocks. To see that the blocks represented by the six characters of Line 6 are distinct, note that since $\mathbf{L} = C_{\mathbf{G}}(Z(\mathbf{L})_\ell^F)$ and since the pairs $(\mathbf{L}, \lambda)$ are not $\mathbf{G}^F$-conjugate, neither are the corresponding maximal Brauer pairs $(Z(\mathbf{L})_\ell^F, \lambda)$. The blocks corresponding to Line 7 are all of defect zero, hence are distinct. □

Remark 6.6. Let us point out that in Table 6, as in Table 8 of [21] we suppressed the Ennola dual situations (obtained by changing q to $-q$) for which all Harish-Chandra series look completely similar since the congruence of q^2 modulo 5 remains unchanged.

The above results show that [22, Thm. A, Thm. B] remain unchanged (note that Remark 2.2(4) of [23] applies also to the isolated element s of order 6 above). We also obtain the following consequences, completing the gap in the proofs of Theorem [21, Thm.1.1] and [23, Main Thm] caused by the missing case.

Corollary 6.7. *For s as above, the 5-blocks in $\mathcal{E}_5(\mathbf{G}^F, s)$ with non-abelian defect group have characters of positive height. Further, $(\mathbf{G}^F, \chi)$ is not a minimal counter-example to (HZC1) for any semisimple 5-element $t \in \mathbf{G}^{*F}$ commuting with s and any $\chi \in \mathcal{E}(\mathbf{G}^F, st)$.*

Proof. As already discussed in the proof of Theorem 6.5, the block in Line 1 has non-abelian defect groups (as the relative Weyl group has order divisible by 5) and the blocks in all other lines have abelian defect groups. For the block in Line 1, the character in $\mathcal{E}(\mathbf{G}^F, s)$ corresponding to the unipotent character of $A_5(q)A_2(q)A_1(q)$ labelled by $41 \otimes 2 \otimes 1$ has positive height. This proves the first assertion.

Suppose that s corresponds to Lines 6–7. The blocks in Line 7 are all of defect 0, and all remaining characters in $\mathcal{E}_5(\mathbf{G}^F, s)$ have the same 5-part in their degree, so are all of height 0 in their respective blocks. In particular, the second assertion holds. Now suppose that s corresponds to Lines 2–5. Here we may apply [21, Lemma 8.5(3)(b)] in conjunction with [21, Prop. 8.6(1)] to conclude that the second assertion holds (see the argument in the last part of the proof of [21, Prop. 8.8]). □

Remark 6.8. We take the opportunity to repeat what we already pointed out in the proof of Theorem 3.14 of [22]: in Table 4 of [21] each of the lines 6, 7, 10, 11, 14 and 20 give rise to two e-cuspidal pairs and so to two distinct e-Harish-Chandra series, but the two pairs also give rise to different blocks, and similarly lines 2, 5, 8 and 11 in Table 3 of [21] give rise to three e-Harish-Chandra series and three different blocks. We thank Ruwen Hollenbach for bringing this misprint to our attention.

6.3. **Further correction to the block distribution for $\ell = 3$.** We discuss one more issue connected to the tables of block distributions printed in [21]. In the accompanying statements, we say that the block distribution is 'indicated by the horizontal lines' in the tables, but in fact, as we prove, the block distribution is related to the numbered lines, in the sense that all unnumbered lines below a numbered line fall into the block for the numbered line. This amended formulation fails in two places, though: a misinterpretation of the statement of [11] on unipotent blocks of groups of type E_6 led to a wrong assignment of certain e-Harish-Chandra series to 3-blocks. More concretely, we have the following:

Proposition 6.9. *In each of the following two cases and their Ennola duals, the e-Harish-Chandra series in the unnumbered line in the corresponding box of the table lies in the semisimple (first) block of the box:*

(1) $\mathbf{G}^F = E_7(q)_{\mathrm{sc}}$, $\ell = 3$, $C_{\mathbf{G}^*}(s)^F = \Phi_1.E_6(q).2$, *see* [21, Tab. 4]*; and*
(2) $\mathbf{G}^F = E_8(q)$, $\ell = 3$, $C_{\mathbf{G}^*}(s)^F = E_7(q).A_1(q)$, *see* [21, Tab. 6].

The corrected parts of the tables are thus as shown in Table 7 *and* 8.

Table 7. Harish-Chandra series in some isolated 3-blocks of $E_7(q)_{\mathrm{sc}}$, $q \equiv 1 \pmod 3$

No.	$C_{\mathbf{G}^*}(s)^F$	$\mathbf{L}^F$	$C_{\mathbf{L}^*}(s)^F$	λ	$W_{\mathbf{G}^F}(\mathbf{L},\lambda)$
8	$\Phi_1.E_6(q).2$	Φ_1^7	$\mathbf{L}^{*F}$	1	$E_6.2$
		$\Phi_1.E_6(q)$	$\mathbf{L}^{*F}$	$E_6[\theta^{\pm 1}]$	2
9		$\Phi_1^3.D_4(q)$	$\mathbf{L}^{*F}$	$D_4[1]$	$A_2.2$

Table 8. Harish-Chandra series in some isolated 3-blocks of $E_8(q)$, $q \equiv 1 \pmod 3$

No.	$C_{\mathbf{G}^*}(s)^F$	$\mathbf{L}^F$	$C_{\mathbf{L}^*}(s)^F$	λ	$W_{\mathbf{G}^F}(\mathbf{L},\lambda)$
3	$E_7(q)A_1(q)$	Φ_1^8	$\mathbf{L}^{*F}$	1	$E_7 \times A_1$
		$\Phi_1^2.E_6(q)$	$\mathbf{L}^{*F}$	$E_6[\theta^{\pm 1}]$	$A_1 \times A_1$
4		$\Phi_1^4.D_4(q)$	$\mathbf{L}^{*F}$	$D_4[1]$	$C_3 \times A_1$
5		$\Phi_1.E_7(q)$	$\mathbf{L}^{*F}$	$E_7[\pm\xi]$	A_1

Proof. The arguments given in the proofs of [21] apply verbatim, when using the correct interpretation of the block distribution for $E_6(q)$ and ${}^2E_6(q)$ from [11]. □

The 3-block b labelled by Line 9 in Table 7 has non-abelian defect groups, and the argument for b containing characters of different heights given in [23] is no longer valid in view of the correction above since we cannot deduce the existence of a $3'$-character of positive height coming from a 1-Harish-Chandra series of non-central 3-defect. We remedy this as follows. Let $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding. Then $Z(\tilde{\mathbf{G}}^F)\mathbf{G}^F$ is a normal subgroup of $\tilde{\mathbf{G}}^F$ of $3'$-index and $Z(\tilde{\mathbf{G}}^F) \cap \mathbf{G}^F$ is a $3'$-group. Hence it suffices to prove that a block of $\tilde{\mathbf{G}}^F$ covering b has characters of different 3-defect. By Bonnafé–Rouquier it then suffices to prove that the corresponding unipotent 3-block, say B, of a group $\mathbf{C}^F$, with $\mathbf{C}$ in duality with $C_{\tilde{\mathbf{G}}^*}(\tilde{s})$ ($\tilde{s}$ a preimage of s in $\tilde{\mathbf{G}}^{*F}$) has characters of different 3-defect. Let $(\mathbf{L}, \lambda)$ be a unipotent 1-cuspidal pair of $\mathbf{C}^F$ defining B and note that $\mathbf{C}$ (a group of type E_6) is a Levi subgroup of $\tilde{\mathbf{G}}$, hence has connected centre. Let $t \in C_{\tilde{\mathbf{G}}^*}(\tilde{s})$ be a 3-element with centraliser of type D_5 and $(\mathbf{L}_t, \lambda_t)$ be a 1-cuspidal pair of $C_{\tilde{\mathbf{G}}^*}(\tilde{s}t)$ such that $(\mathbf{L}_t, \lambda_t) \to_t (\mathbf{L}, \lambda)$ as in [11, Thm B]. Since $C_{\tilde{\mathbf{G}}^*}(\tilde{s}t)$ does not contain a Sylow 3-subgroup of $C_{\tilde{\mathbf{G}}^*}(\tilde{s})$, the characters in the 1-Harish-Chandra-series of $C_{\tilde{\mathbf{G}}^*}(\tilde{s}t)^F$ defined by $(\mathbf{L}_t, \lambda_t)$ have 3-defect different from that of the characters in $\mathcal{E}(\mathbf{C}^F, (\mathbf{L}, \lambda))$. Then we are done by [11, Thm B].

6.4. **Proof of Theorems 3.1 and 3.2.**

Proof of Theorem 3.1. Suppose first that $s = 1$. Then part (c) follows from [3] and parts (a) and (b) follow from Theorems A and A.bis of [11] (see also Lemma 2.5). Parts (a), (b) and (c) in case s is central follow easily from the $s = 1$ case. Now suppose that s is non-central. If $[\mathbf{G}, \mathbf{G}]$ is of type G_2, F_4 or E_8, then there is an F-stable decomposition $\mathbf{G} = [\mathbf{G}, \mathbf{G}] \times Z^\circ(\mathbf{G})$. Hence parts (a), (b) and (c) follow from the analogous results in [21] and Section 6.1, respectively.

The first assertion of part (f) follows as in Remark 2.2 and Section 4 of [22]. Note that if ℓ is good for $\mathbf{G}$, then all unipotent e-cuspidal pairs are of (quasi-)central ℓ-defect. The second assertion of part (f) follows from this and by inspection of the tables of [11] and [21] and of Tables 3–8.

Parts (d) and (e) are implicit in the construction of the tables cited above. □

Proof of Theorem 3.2. The proof of part (a) follows along the same lines as that of Theorem 3.4 of [22], with some simplifications coming from the fact that $Z(\mathbf{X})$ is connected. The only additional input required is the existence of an e-Harish-Chandra theory at isolated elements and bad ℓ in the case $[\mathbf{X}, \mathbf{X}]$ is of exceptional type which is provided in this section. Here note that Lemma 3.1 of [22] is stated for all connected reductive groups. Part (b) is immediate from part (a). Part (c) follows from (a) and the proof of Theorem 3.6 of [22]. Part (d) follows from part (b) and [22, Lemmas 2.3 and 3.7, Thm A(c)] and the remarks following Definition 2.12 of [22]. □

6.5. **Decomposition of $R_{\mathbf{L}}^{\mathbf{G}}$.** We take the opportunity to resolve the last ambiguities left in the determination of the decomposition of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ for certain unipotent characters λ in exceptional groups of Lie type, viz. the cases denoted “15+16”, “40+41” and “42+43” in [3, Tab. 2] (note that since the statements only concern unipotent characters, the precise isogeny type of $\mathbf{G}$ is not relevant):

Lemma 6.10. (a) *Let* $\mathbf{L}$ *be a Levi subgroup of rational type* ${}^2E_6(q).\Phi_2$ *in* $\mathbf{G}$ *of type* E_7. *Then*

$$R_{\mathbf{L}}^{\mathbf{G}}({}^2E_6[\theta^j]) = (E_6[\theta^j],1) - (E_6[\theta^j],\epsilon) \quad \textit{for } j=1,2.$$

(b) *Let* $\mathbf{L}$ *be a Levi subgroup of rational type* ${}^2E_6(q).\Phi_2^2$ *in* $\mathbf{G}$ *of type* E_8. *Then*

$$R_{\mathbf{L}}^{\mathbf{G}}({}^2E_6[\theta]) = (E_6[\theta],\phi_{1,0}) - (E_6[\theta],\phi_{1,3}') - (E_6[\theta],\phi_{1,3}'') + (E_6[\theta],\phi_{1,6}) - 2E_8[-\theta] - 2E_8[\theta].$$

(c) *Let* $\mathbf{L}$ *be a Levi subgroup of rational type* $E_7(q).\Phi_2$ *in* $\mathbf{G}$ *of type* E_8. *Then*

$$R_{\mathbf{L}}^{\mathbf{G}}(\phi_{512,11}) = \phi_{4096,11} - \phi_{4096,26} \quad \textit{and} \quad R_{\mathbf{L}}^{\mathbf{G}}(\phi_{512,12}) = \phi_{4096,12} - \phi_{4096,27}.$$

Proof. In all three cases this follows from Shoji's explicit description of Lusztig induction of unipotent characters in terms of the Fourier transform [32]. In cases (a) and (c) we can also provide a block theoretic argument. It will be sufficient to show the claim for one value of q since by [3, Thm 1.33] the decomposition of Lusztig induction of unipotent characters is generic. We choose q such that $q+1$ is divisible by a prime ℓ larger than 7, say $q=37$, and take $\ell=19$, so $e_\ell(q)=2$. The four unipotent characters $\lambda = {}^2E_6[\theta], {}^2E_6[\theta^2], \phi_{512,11}, \phi_{512,12}$ considered are 2-cuspidal in their respective Levi subgroups and thus of ℓ-defect zero. It then follows by [5, Thm] that all constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ lie in the same ℓ-block of $\mathbf{G}^F$. On the other hand, the two characters λ given in either case are not conjugate under any automorphism of $\mathbf{L}^F$ [17, Thm 4.5.11] and thus by [5] they label distinct blocks of $\mathbf{G}^F$. It was already shown in [3] that in all cases $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ has norm 2, and that the constituents are among the ones listed in the statement. Thus we are done if we can show that the stated decompositions agree with the ℓ-block distribution.

For (a) consider the unipotent character $\rho := E_6[\theta]$ of the split Levi subgroup $\mathbf{L}_1$ of $\mathbf{G}=E_7$ of rational type $E_6(q).\Phi_1$. It is 1-cuspidal, and its relative Hecke algebra in $\mathbf{G}^F$ is of type A_1 with parameter q^9 [17, Tab. 4.8]. Thus its Harish-Chandra induction $R_{\mathbf{L}_1}^{\mathbf{G}}(\rho)$ is indecomposable modulo all primes ℓ dividing q^9+1 by [17, Prop. 3.1.29], whence its two constituents $E_6[\theta],1$ and $E_6[\theta],\epsilon$ lie in the same ℓ-block of $\mathbf{G}^F$. In case (c) it follows from [18, Table F.6] that the block distribution is as claimed. □

7. Robinson's conjecture on defects

Robinson's conjecture on character defects [28] presented in the introduction was reduced to the case of minimal counter-examples in quasi-simple groups in [14, Thm 2.3] based on work of Murai, and it was shown to hold for all odd primes ℓ and for the 2-blocks of any quasi-simple group not of exceptional Lie type in odd characteristic in [14, 15]. Here, we investigate its validity in the following situation. Let $\mathbf{G}$ be a simple algebraic group of simply connected type with a Frobenius map F such that $G=\mathbf{G}^F$ is of type $G_2(q)$, ${}^3D_4(q)$, $F_4(q)$, $E_6(\pm q)$, $E_7(q)$ or $E_8(q)$, for some odd prime power q. Let S be a central quotient of G. Note that $G=S$ unless possibly when G is of type E_6 or E_7. Let $\bar{B}$ be a block of S dominated by an isolated 2-block B of G.

Lemma 7.1. *Suppose that* $\mathbf{G}$ *as above is of type* E_6 *and let* $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ *be a regular embedding. Let* S, B *and* $\bar{B}$ *be as above and let* $\tilde{B}$ *be a block of* $\tilde{\mathbf{G}}^F$ *covering* B. *Then any one of* B, $\bar{B}$ *and* $\tilde{B}$ *satisfies Robinson's conjecture if and only if any of the other two does.*

Proof. Since $Z(G)$ is a $2'$-group, B and $\bar{B}$ are isomorphic blocks, and in particular B and $\bar{B}$ have isomorphic defect groups and the same set of character degrees. Thus B satisfies

Robinson's conjecture if and only if $\bar{B}$ does. Again, since $Z(G)$ is a $2'$-group and since $[\tilde{\mathbf{G}}, \tilde{\mathbf{G}}] = \mathbf{G}$, $Z(\tilde{\mathbf{G}})^F\mathbf{G}^F$ is a normal subgroup of odd index in $\tilde{\mathbf{G}}^F$, and thus $\tilde{B}$ satisfies Robinson's conjecture if and only if some (hence any) block, say C, of $Z(\tilde{\mathbf{G}})^F\mathbf{G}^F$ covering B satisfies the conjecture. Since $Z(\tilde{\mathbf{G}})^F$ is central in $Z(\tilde{\mathbf{G}})^F\mathbf{G}^F$, B and C have the same set of character degrees. Moreover, since $Z(\mathbf{G}^F) = Z(\tilde{\mathbf{G}})^F \cap \mathbf{G}^F$ is a $2'$-group, if D is a defect group of B, then $Z(\tilde{\mathbf{G}})^F_2 D \cong Z(\tilde{\mathbf{G}})^F_2 \times D$ is a defect group of C. So C satisfies Robinson's conjecture if and only if B does. □

Proposition 7.2. *Let B be a unipotent 2-block of G as above. Then the block $\bar{B}$ of S is not a minimal counter-examples to Robinson's conjecture.*

Proof. First assume B is the principal block of G. Then its defect groups are the Sylow 2-subgroups of G. Their centres are given in Table 9; here T_ϵ denotes a torus of order $q - \epsilon 1$, and $\overline{E_7(q)} := E_7(q)/Z(E_7(q))$.

TABLE 9. Centres of Sylow 2-subgroups $P \in \mathrm{Syl}_2(S)$

S	$C_G(t)$	$Z(P)$	S	$C_G(t)$	$Z(P)$
$G_2(q)$	$A_1(q)^2$	C_2	${}^2E_6(q)$	${}^2D_5(q).T_-$	$C_{\vert q+1\vert_2}$
${}^3D_4(q)$	$A_1(q^3)A_1(q)$	C_2	$E_7(q)$	$D_6(q)A_1(q)$	C_2^2
$F_4(q)$	$B_4(q)$	C_2	$\overline{E_7(q)}$	$D_6(q)A_1(q)$	C_2
$E_6(q)$	$D_5(q).T_+$	$C_{\vert q-1\vert_2}$	$E_8(q)$	$D_8(q)$	C_2

This is obtained as follows. If $P \in \mathrm{Syl}_2(G)$ and $t \in Z(P)$ then $C_G(t)$ has odd index in G and $Z(P) \leq P \leq C_G(t)$. The centralisers of semisimple elements in G can be enumerated with the algorithm of Borel–de Siebenthal (see e.g. [26, 13.2]). It turns out that the only centralisers $C_G(t)$ of 2-elements $t \in G \setminus Z(G)$ of odd index in G are as listed in Table 9. (In fact, these can also be found on the website [25].) Then $|Z(P)|$ can be read off from the structure of $C_G(t)$. If $S = \overline{E_7(q)}$ then we may consider S as the derived subgroup of an adjoint type group, and with the same argument as before we find that $|Z(P)| = 2$ in this case.

By [14, Lemma 3.1] Robinson's conjecture holds when $|Z(P)| = 2$. Thus, by Table 9 we only need to concern ourselves with S of type $E_6(q)$, ${}^2E_6(q)$, or $S = E_7(q)$. Let's first assume that $\mathbf{G}$ is of type E_6. Then by Lemma 7.1 we may instead argue for the principal block $\tilde{B}$ of $\tilde{\mathbf{G}}^F$. Now, by [4, Thm 12] for every $\chi \in \mathrm{Irr}(\tilde{B})$ there is a character of the same height in the principal block of $G_{\mathrm{ad}} := \tilde{G}/Z(\tilde{G})$, a group of adjoint type. Note that G and $\tilde{G}/Z(\tilde{G})$ have isomorphic Sylow 2-subgroups as G is a central extension of degree dividing 3 of the derived subgroup of G_{ad}. We may hence argue for the principal 2-block of G_{ad}; here $G^*_{\mathrm{ad}} \cong G$.

Now, according to Enguehard's description in [11, Thm B] a character $\chi \in \mathcal{E}(G_{\mathrm{ad}}, t)$ lies in the principal 2-block if and only if $t \in G \cong G^*_{\mathrm{ad}}$ is a 2-element and moreover the Jordan correspondent of χ in $\mathcal{E}(C_G(t), 1)$ lies in a Harish-Chandra series with Harish-Chandra vertex either a torus or a Levi subgroup of type D_4. First assume that χ has Harish-Chandra vertex D_4. Then $\mathbf{H} := C_{\mathbf{G}}(t)$ has a Levi subgroup of type D_4 and hence is either of type D_4, D_5 or E_6. In either case, [14, Prop. 6.2] shows that

$(|S|/\chi(1))_2 \geq (q-1)_2^2 > |Z(P)|$ and we are done. If χ has trivial Harish-Chandra vertex then by the same argument we are done when $\mathbf{H}$ has $\mathbb{F}_q$-rank at least 2. Note that $\mathbf{H}$ has $\mathbb{F}_q$-rank at least 1 as by Table 9 every 2-element of $\mathbf{G}^*_{\mathrm{ad}}$ centralises a split torus of rank 1. Now if $\mathbf{H}$ has $\mathbb{F}_q$-rank 1, then all of its unipotent characters have odd degree, and it is easy to see that $(|S|/\chi(1))_2 \geq 2(q-1)_2 > |Z(P)|$.

The same discussion applies to $G = {}^2E_6(q)$ by interchanging the cases corresponding to the two possible congruences of q modulo 4.

Now assume $S = G = E_7(q)$. Let χ lie in the principal 2-block B of G, so in $\mathcal{E}(G,t)$ for some 2-element $t \in G^*$. If $t = 1$, so χ is unipotent, then by inspection $\mathrm{def}(\chi) \geq 3$ unless χ lies in an ordinary Harish-Chandra series above a Levi subgroup of type E_6, but by [11, p. 354] these are not in the principal 2-block of G. Now assume $t \neq 1$, and let t_1 be the involution in $\langle t\rangle$. Then $C_{G^*}(t_1)$ has one of the structures $D_6(q)A_1(q)$, $A_7(\pm q).2$ or $E_6(\pm q)(q \mp 1).2$. Assume $C_{G^*}(t)$ involves an E_6-factor. Let $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding, $\tilde{t} \in \tilde{\mathbf{G}}^{*F}$ a preimage of t of 2-power order, and let $\tilde{B}$ be the principal block of $\tilde{\mathbf{G}}^F$. By [11, Thm B] the characters in $\tilde{B}$ in series $\tilde{t}$ are those in Jordan correspondence with unipotent characters of $C_{\tilde{\mathbf{G}}^{*F}}(\tilde{t})$ with a rational Frobenius eigenvalue. Thus, the characters in B in series t are again among those in Jordan correspondence with unipotent characters with a rational Frobenius eigenvalue, and by inspection all of these have defect at least 3. So we may assume that $C_{G^*}(t)$ has only factors of classical type. Again by using the lists of unipotent character degrees and arguing as before, one sees that all characters in $\mathcal{E}(G,t) \cap \mathrm{Irr}(B)$ are of defect at least 3, so $B = \bar{B}$ is not a counter-example.

The non-principal unipotent 2-blocks of groups of exceptional type were determined in [11]. In Table 10 we list those blocks having non-abelian defect groups, as well as properties of their defect groups D which have also been taken from [11]. We label the blocks by their Harish-Chandra vertex $(\mathbf{L},\lambda)$ of quasi-central defect, in the notation of loc. cit.

Table 10. Non-principal unipotent 2-blocks of non-abelian defect

G	cond.	$([\mathbf{L},\mathbf{L}],\lambda)$	D	$\|Z(D)\|$
$E_7(q)$	$q \equiv 1\,(4)$	$(E_6, E_6[\theta]), (E_6, E_6[\theta^2])$	$Z_{(q-1)_2}.2$	2
	$q \equiv 3\,(4)$	$({}^2E_6, {}^2E_6[\theta]), ({}^2E_6, {}^2E_6[\theta^2])$	$Z_{(q+1)_2}.2$	2
$E_8(q)$	$q \equiv 1\,(4)$	$(E_6, E_6[\theta]), (E_6, E_6[\theta^2])$	$Z^2_{(q-1)_2}.2^2$	≤ 4
	$q \equiv 3\,(4)$	$({}^2E_6, {}^2E_6[\theta]), ({}^2E_6, {}^2E_6[\theta^2])$	$Z^2_{(q+1)_2}.2^2$	≤ 4

Since the conjecture holds for blocks with $|Z(D)| = 2$ by [14, Lemma 3.1], we only need to consider the 2-blocks B in $E_8(q)$. First from the list of unipotent degrees it is easy to check that all unipotent characters χ in these blocks, as described in [11, Thm A], have $\mathrm{def}(\chi) \geq 3$. By [11, Thm B] all characters in $\mathcal{E}_2(G,1)\cap\mathrm{Irr}(B)$ have Harish-Chandra vertex of type E_6. Thus the centralisers of the relevant 2-elements $1 \neq t \in \mathbf{G}^{*F}$ contain either an E_6- or an E_7-factor. Again, it is straightforward to verify that all such characters χ have $\mathrm{def}(\chi) \geq 3$, whence our claim holds. ∎

We suspect that in fact the defect groups of the blocks B for $E_8(q)$ as in Table 10 are isomorphic to Sylow 2-subgroups of $G_2(q)$, in which case their centres would have order 2 and the proof would be even easier.

Proposition 7.3. *Let B be an isolated, non-unipotent 2-block of G in $\mathcal{E}_2(G,s)$, with $\mathbf{G}$ not of type E_7. Assume $\mathbf{C}^* := C_{\mathbf{G}^*}(s)$ has only factors of classical type. Then the block $\bar{B}$ of S is not a minimal counter-example to Robinson's conjecture.*

Proof. Let B be as in the statement. First assume $\mathbf{G}$ is not of type E_6. Then $S = G$, $\mathbf{C}^*$ is connected, and Jordan decomposition on the level of G and of $C = \mathbf{C}^F$ together yield a defect preserving bijection $\mathcal{E}_2(G,s) \to \mathcal{E}_2(C,1)$. Since $\mathbf{C}^*$ (and hence also $\mathbf{C}$) has only factors of classical type, $\mathcal{E}_2(C,1)$ is a single 2-block, namely the principal block b_0 of C, and by Proposition 3.3 so is $\mathcal{E}_2(G,s)$. In particular, B is a *minimal block*, that is, B contains a semisimple character in $\mathcal{E}(G,s)$. Then by [29, Prop. E] a defect group of B is isomorphic to a Sylow 2-subgroup of C. Thus, if B is a counter-example then so is b_0. Since C is strictly smaller than G, this shows that B cannot be minimal.

Finally, if $\mathbf{G}$ is of type E_6, then by Lemma 7.1 we may replace B by a block $\tilde{B}$ of $\tilde{G}$ instead and the same arguments go through. □

Lemma 7.4. *Suppose that $\mathbf{G}$ is of type E_8. Let $s \in \mathbf{G}^*$ be isolated such that $C_{\mathbf{G}^*}(s)$ is of type E_6A_2 and let $\mathbf{C}$ be dual to $\mathbf{C}^* := C_{\mathbf{G}^*}(s)$. There is a 2-defect preserving bijection $\psi : \mathcal{E}_2(\mathbf{G}^F, s) \to \mathcal{E}_2(\mathbf{C}^F, 1)$ which preserves 2-blocks.*

Proof. Since the centre of a simply-connected covering of $\mathbf{C}$ is a 3-group, by [4, Thm 12], we may replace $\mathbf{C}$ by its adjoint quotient $\bar{\mathbf{C}} = \mathbf{C}/Z(\mathbf{C})$. By inspection of [21] and [11], both $\mathcal{E}_2(\mathbf{G}^F, s)$ and $\mathcal{E}_2(\bar{\mathbf{C}}^F, 1)$ contain exactly two 2-blocks. Let B be the minimal 2-block in $\mathcal{E}_2(\mathbf{G}^F, s)$. Now by Theorem 1, for any 2-element $t \in C_{\mathbf{G}^*}(s)^F$, the characters of $\mathcal{E}(\mathbf{G}^F, st)$ which lie in B are precisely those whose Jordan correspondents in $\mathcal{E}(C_{\mathbf{G}^*}(st)^F, 1)$ lie in e-Harish-Chandra series above e-cuspidal characters with rational Frobenius eigenvalue. The same description applies to the characters in the principal block of $\bar{\mathbf{C}}^F$ by [11, Thm B]. The result follows by noting that the natural surjection $\bar{\mathbf{C}}^* \to \mathbf{C}^*$ induces a bijection from the set of conjugacy classes of 2-elements of $\bar{\mathbf{C}}^{*F}$ to those of $\mathbf{C}^{*F}$ and that this surjection induces isogenies between centralisers of corresponding elements, so in particular preserves their orders. □

Proposition 7.5. *Let B be an isolated, non-unipotent 2-block of G. Then the block $\bar{B}$ of S is not a minimal counter-example to Robinson's conjecture.*

Proof. Let B be isolated but not unipotent, labelled by a semisimple $2'$-element $1 \neq s \in G^*$. By inspection of the tables in [21] the only cases where $\mathbf{C}^* = C_{\mathbf{G}^*}(s)$ has factors of exceptional type are for $\mathbf{G}$ of type E_8 and $\mathbf{C}^*$ of type E_6A_2. Thus by Proposition 7.3 we only need to discuss these blocks, and the case when $\mathbf{G}$ has type E_7.

First assume $\mathbf{G}$ is of type E_7. According to [21, Tab. 4], the relevant 2-blocks are those with $\mathbf{C}^* := C_{\mathbf{G}^*}(s)$ of rational type $A_5(q)A_2(q)$ or ${}^2A_5(q).{}^2A_2(q)$ (depending on the congruence of q modulo 3). In both cases, $\mathcal{E}_2(G,s)$ is a single 2-block by [21], with defect group D isomorphic to a Sylow 2-subgroup of $C_{\mathbf{G}^{*F}}(s)^*$ by [29, Prop. E]. By Ennola duality, it suffices to consider the case $C^* := \mathbf{C}^{*F} \cong A_5(q)A_2(q)$. Assume $q \equiv 1 \pmod 4$. Then, $|Z(D)| = 2(q-1)_2^2$. By the description of centralisers of elements t of 2-power order in linear groups, any character in Lusztig series $\mathcal{E}(C^*, t)$ has defect at least $\log_2(8(q-1)_2^2)$. Since the characters in $\mathcal{E}_2(G, s)$ are in height preserving bijection to the characters of $\mathcal{E}_2(C, 1)$ via Jordan decomposition, where $C := \mathbf{C}^F$, B is not a counter-example. Now consider the corresponding block $\bar{B}$ of $S = G/Z(G)$, with defect group $\bar{D} = D/Z(G)$. By inspection, $|Z(\bar{D})| \leq (q-1)_2^2$. Since any character of $\bar{B}$ is a character of B, and defects decrease by 1, neither is the block $\bar{B}$ a counter-example. Now assume $q \equiv 3 \pmod 4$. Then $|Z(D)| = 2^3$. Here any character in $\mathcal{E}(C^*, t)$ has defect at least 5. Furthermore, $\bar{D} = D/Z(G)$ has centre of order 2^2. Thus we may conclude as in the previous case.

It remains to consider the isolated blocks of E_8 in series s with $\mathbf{C}^*$ of type E_6A_2. The defect groups of the minimal block B in $\mathcal{E}_2(G, s)$ are isomorphic to Sylow 2-subgroups of C, by [29, Prop. E]. Hence by Lemma 7.4 and its proof, if B is a counter-example, then so is the principal block of $\mathbf{C}^F$, contradicting the minimality of B.

Finally, let B be the minimal block in $\mathcal{E}_2(G, s)$ as above, see Table 11 (taken from [21, Tab. 5]) and their Ennola duals. According to [21, Thm 1.2], the defect groups for B the block No. 6 are meta-cyclic and hence Robinson's conjecture holds by [30, Cor. 8.2].

TABLE 11. Isolated non-unipotent non-maximal 2-blocks in $E_8(q)$, $q \equiv 1 \pmod 4$

No.	$C_{\mathbf{G}^*}(s)^F$	$\mathbf{L}^F$	$C_{\mathbf{L}^*}(s)^F$	λ	$W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$
4	$E_6(q).A_2(q)$	E_6	$\mathbf{L}^{*F}$	$E_6[\theta^{\pm 1}]$	A_2
6	${}^2E_6(q).{}^2A_2(q)$	E_7	$\Phi_1\Phi_2.{}^2E_6(q)$	${}^2E_6[\theta^{\pm 1}]$	A_1
		E_8	$C_{\mathbf{G}^*}(s)^F$	${}^2E_6[\theta^{\pm 1}] \otimes \phi_{21}$	1

Now let B be the No. 4 block. As before, let $\mathbf{C} \leq \mathbf{G}$ be F-stable of type E_6A_2, corresponding to $\mathbf{C}^*$ under an identification of $\mathbf{G}$ with $\mathbf{G}^*$. Let $b_{\mathbf{L}^F}(\lambda)$ be the 2-block of $\mathbf{L}^F$ containing λ. As explained in the proof of [21, Prop. 6.4], $(Z(\mathbf{L})_2^F, b_{\mathbf{L}^F}(\lambda))$ is a B-Brauer pair. Moreover, by [21, Lemma 6.2], $C_{\mathbf{G}^F}(Z(\mathbf{L})_2^F) = \mathbf{L}^F \leq \mathbf{C}^F$. So, by general block-theoretic considerations, letting d be the unique 2-block of $\mathbf{C}^F$ such that $(Z(\mathbf{L})_2^F, b_{\mathbf{L}^F}(\lambda))$ is a d-Brauer pair, there is a defect group R of d and a defect group P of B such that $Z(\mathbf{L})_2^F \leq R \leq P$. Also, note from the table entry for B that $Z(\mathbf{L})_2^F$ is of index 2 in P. Thus, either $R = Z(\mathbf{L})_2^F$ or $R = P$. Since $W_{\mathbf{G}^F}(\mathbf{L}, \lambda) \cong W_{\mathbf{C}^F}(\mathbf{L}, \lambda)$ is not a $2'$-group, R is not abelian by [21, Prop. 2.7(e)] and hence $R = P$.

Let $\mathbf{C}_1$ be the E_6-subgroup of $\mathbf{C}$ and $\mathbf{C}_2$ the A_2-subgroup of $\mathbf{C}$. Then $\mathbf{C}_1^F\mathbf{C}_2^F$ is normal of index 3 in $\mathbf{C}^F$, $\mathbf{C}_1^F$ and $\mathbf{C}_2^F$ commute and $\mathbf{C}_1^F \cap \mathbf{C}_2^F$ is of order 3. It follows that $R = R_1R_2 \cong R_1 \times R_2$ with R_i a defect group of a 2-block of $\mathbf{C}_i^F$, $i = 1, 2$. Since $Z(\mathbf{L})_2^F \leq R \cap \mathbf{C}_2^F = R_2$ is of index 2 in R and $Z(\mathbf{L})_2^F$ is self-centralising in R, it follows that R_1 is trivial and hence $Z(\mathbf{L})_2^F$ is of index 2 in $P = R = R_2$. Since $|\mathbf{C}_2^F|_2 = 2|Z(\mathbf{L})_2^F|$, $P = R_2$ is a Sylow 2-subgroup of $\mathbf{C}_2^F$. Now the result follows by Lemma 7.4 and its proof. □

Complementing earlier investigations in [14, 15] this proves Robinson's conjecture:

Proof of Theorem 3. According to [14, 15] a minimal counter-example would have to occur as a quasi-isolated 2-block of an exceptional type quasi-simple group S in odd characteristic. Note that the exceptional covering group $3.G_2(3)$ was handled in [14, Thm. 3.6].

By the defect group preserving Morita equivalences from [2], we can further restrict to isolated blocks. Note that here the centre of $\mathbf{G}^F$ is always cyclic, so the known gap in [2] is not relevant. By the main result of [21] Robinson's conjecture holds for blocks with abelian defect groups, so we need not consider the Ree groups ${}^2G_2(q^2)$. But now Propositions 7.2 and 7.5 give the claim. □

References

[1] C. Bonnafé, Quasi-isolated elements in reductive groups. *Comm. Algebra* **33** (2005), 2315–2337.

[2] C. Bonnafé, J.-F. Dat, R. Rouquier, Derived categories and Deligne–Lusztig varieties II. *Ann. of Math. (2)* **185** (2017), 609–670.

[3] M. Broué, G. Malle, J. Michel, Generic blocks of finite reductive groups. *Astérisque* No. 212 (1993), 7–92.

[4] M. Cabanes, M. Enguehard, Unipotent blocks of finite reductive groups of a given type. *Math. Z.* **213** (1993), 479–490.

[5] M. Cabanes, M. Enguehard, On unipotent blocks and their ordinary characters. *Invent. Math.* **117** (1994), 149–164.

[6] M. Cabanes, M. Enguehard, On blocks of finite reductive groups and twisted induction. *Adv. Math.* **145** (1999), 189–229.

[7] M. Cabanes, M. Enguehard, *Representation Theory of Finite Reductive Groups.* Cambridge University Press, Cambridge, 2004.

[8] D. Deriziotis, G. O. Michler, Character table and blocks of finite simple triality groups ${}^3D_4(q)$. *Trans. Amer. Math. Soc.* **303** (1987), 39–70.

[9] F. Digne, J. Michel, On Lusztig's parametrization of characters of finite groups of Lie type. *Astérisque* No. 181-182 (1990), 113–156.

[10] F. Digne, J. Michel, *Representations of Finite Groups of Lie Type.* Second edition. London Mathematical Society Student Texts, 95. Cambridge University Press, Cambridge, 2020.

[11] M. Enguehard, Sur les l-blocs unipotents des groupes réductifs finis quand l est mauvais. *J. Algebra* **230** (2000), 334–377.

[12] M. Enguehard, Vers une décomposition de Jordan des blocs des groupes réductifs finis. *J. Algebra* **319** (2008), 1035–1115.

[13] M. Enguehard, Towards a Jordan decomposition of blocks of finite reductive groups. ArXiv: 1312.0106.

[14] Z. Feng, C. Li, Y. Liu, G. Malle, J. Zhang, Robinson's conjecture on heights of characters. *Compos. Math.* **155** (2019), 1098–1117.

[15] Z. Feng, C. Li, Y. Liu, G. Malle, J. Zhang, Robinson's conjecture for classical groups. *J. Group Theory* **22** (2019), 555–578.

[16] P. Fong, B. Srinivasan, The blocks of finite classical groups. *J. reine angew. Math.* **396** (1989), 122–191.

[17] M. Geck, G. Malle, *The Character Theory of Finite Groups of Lie Type. A Guided Tour.* Cambridge University Press, Cambridge, 2020.

[18] M. Geck and G. Pfeiffer, *Characters of Finite Coxeter Groups and Iwahori–Hecke Algebras.* The Clarendon Press, Oxford University Press, New York, 2000.

[19] R. Hollenbach, On e-cuspidal pairs of finite groups of exceptional Lie type. *J. Pure Appl. Algebra* **226** (2022), 106781, 22 pp.

[20] R. Howlett, Normalizers of parabolic subgroups of reflection groups. *J. London Math. Soc.* **21** (1980), 62–80.

[21] R. Kessar, G. Malle, Quasi-isolated blocks and Brauer's height zero conjecture. *Ann. of Math. (2)* **178** (2013), 321–384.

[22] R. Kessar, G. Malle, Lusztig induction and ℓ-blocks of finite reductive groups. *Pacific J. Math.* **279** (2015), 269–298.

[23] R. Kessar, G. Malle, Brauer's height zero conjecture for quasi-simple groups. *J. Algebra* **475** (2017), 43–60.
[24] M. Linckelmann, *The Block Theory of Finite Group Algebras. Vol. 2.* London Math. Soc. Students Texts, **91**, Cambridge University Press, Cambridge, 2018.
[25] F. Lübeck, *Data for Finite Groups of Lie Type and Related Algebraic Groups.* https://www.math.rwth-aachen.de/ Frank.Luebeck/chev/index.html
[26] G. Malle, D. Testerman, *Linear Algebraic Groups and Finite Groups of Lie Type*. Cambridge Studies in Advanced Mathematics, 133, Cambridge University Press, Cambridge, 2011.
[27] J. Michel, The development version of the CHEVIE package of GAP3. *J. Algebra* **435** (2015), 308–336.
[28] G. Robinson, Local structure, vertices and Alperin's conjecture. *Proc. London Math. Soc.* **72** (1996), 312–330.
[29] L. Ruhstorfer, The Alperin–McKay and Brauer's height zero conjecture for the prime 2. *Ann. of Math. (2)*, to appear. ArXiv:2204.06373.
[30] B. Sambale, *Blocks of Finite Groups and their Invariants.* Lecture Notes in Mathematics **2127**, Springer, Cham, 2014.
[31] A.A. Schaeffer-Fry, Jay Taylor, C. R. Vinroot, Galois automorphisms and a unique Jordan decomposition in the case of a connected centralizer. *J. Algebra* **664** (2025), 123–149.
[32] T. Shoji, Shintani descent for exceptional groups over a finite field. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **34** (1987), 599–653.
[33] J. Taylor, Harish-Chandra cuspidal pairs. *Indag. Math. (N.S.)* **33** (2022), 24–38.

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