# MINIMAL HEIGHTS AND DEFECT GROUPS WITH TWO CHARACTER DEGREES

GUNTER MALLE, ALEXANDER MORETÓ AND NOELIA RIZO

Abstract. Conjecture A of [3] predicts the equality between the smallest positive height of the irreducible characters in a $p$-block of a finite group and the smallest positive height of the irreducible characters in its defect group. Hence, it can be seen as a generalization of Brauer's famous height zero conjecture. One inequality was shown to be a consequence of Dade's Projective Conjecture. We prove the other, less well understood, inequality for principal blocks when the defect group has two character degrees.

## 1. Introduction

One of the main problems of the past decades in representation theory of finite groups is Brauer's Height Zero Conjecture. Posed by Richard Brauer in 1955, it asserts that all irreducible characters in a Brauer $p$-block $B$ of a finite group $G$ have height zero if and only if the defect groups of $B$ are abelian. The proof of this conjecture has recently been completed in [27] (building among others on earlier work in [33, 22, 36]). Now it seems appropriate to study possible extensions to blocks with non-abelian defect groups.

One such possibility was proposed in Conjecture A of [3], where it was conjectured that if $p$ is a prime, $B$ is a Brauer $p$-block with non-abelian defect group $D$ and $\mathrm{Irr}(B)$ is the set of irreducible characters in $B$, then the smallest positive height of the non-linear irreducible characters in $D$, i.e.,

$$\mathrm{mh}(D) = \min\{\log_p \varphi(1) \mid \varphi \in \mathrm{Irr}(D), \varphi(1) > 1\},$$

coincides with the smallest positive height of the non-linear characters in $\mathrm{Irr}(B)$, which we denote $\mathrm{mh}(B)$. In the following, we will refer to this conjecture as the EM-conjecture. If we adopt the convention that $\mathrm{mh}(B)$ or $\mathrm{mh}(D)$ are infinity if no such characters of positive height exist, the EM-conjecture extends Brauer's Height Zero Conjecture. The next natural case to consider is when the defect group has two character degrees.

In [3] it was proved that the inequality $\mathrm{mh}(D) \leq \mathrm{mh}(B)$ follows from Dade's Projective Conjecture, which has been reduced to a problem on simple groups. There is still less evidence for the inequality $\mathrm{mh}(D) \geq \mathrm{mh}(B)$. We refer the reader to [3, 1, 7, 37] for some

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cases where the conjecture is known to hold and to [21, Conj. 2.9] for a reformulation in terms of fusion systems.

Our investigations give strong evidence for the validity of the conjecture when $D$ has two character degrees. As usual, given a finite group $G$ and a fixed prime $p$, we write $B_0(G)$ to denote the principal $p$-block of $G$. Our main result is the following.

**Theorem 1.** *Let $p$ be a prime and let $G$ be a finite group. If a Sylow $p$-subgroup $P$ of $G$ has two character degrees, then* $\mathrm{mh}(B_0(G)) \leq \mathrm{mh}(P)$.

In other words, we prove the less well understood inequality of the EM-conjecture for principal $p$-blocks of groups whose Sylow $p$-subgroups have two character degrees (the case with just one degree being Brauer's Height Zero Conjecture). Our proof depends on the Classification of Finite Simple Groups and the Deligne–Lusztig theory of characters of finite reductive groups. Another fundamental tool that we use is the work in [10]. It may be worth remarking that the recent solution [27] of Brauer's Height Zero Conjecture (BHZ) was preceded by the easier proof in the case of principal blocks in [25]. While the proof of the general case of BHZ also depends heavily on [10] by means of [33], the proof of BHZ for principal blocks does not. We think that extending our main result to arbitrary blocks will be a very challenging problem.

Our paper is structured as follows: in Section 2 we collect some results on linear and permutation groups that are needed in the proof of Theorem 1. In Section 3 we prove some results on simple and almost simple groups, in particular we show a stronger version of Theorem 1 for almost simple groups that may be useful elsewhere (see Theorem 3.5). In Section 4 we prove Theorem 1.

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## 2. Linear and permutation groups

Let $p$ be a prime. In the following, $G$ will always be a finite group. Following [10], we say that a linear group $G \leq \mathrm{GL}_n(p)$ is *$p$-exceptional* if $p$ divides $|G|$ and all $G$-orbits on the natural module have $p'$-size, that is, their length is not divisible by $p$. The classification of solvable irreducible $p$-exceptional groups was already an impressive result (see Sections 9 and 10 of [29]), which lies at the heart of the proof of the $p$-solvable case of BHZ by D. Gluck and T. R. Wolf [12]. It took more than 30 years and a team of experts [10] to extend this result to arbitrary groups. Since it is fundamental for our work, we recall it here.

**Theorem 2.1.** *Let $G$ be an irreducible $p$-exceptional subgroup of $\mathrm{GL}_n(p) = \mathrm{GL}(V)$ and suppose that $G$ acts primitively on $V$. Then one of the following holds:*

(1) *$G$ is transitive on $V \setminus \{0\}$;*
(2) *$G \leq \Gamma\mathrm{L}_1(p^n)$;*
(3) *$G$ is one of the following:*
   (i) *$G = \mathfrak{A}_c, \mathfrak{S}_c$ where $c = 2^r - 2$ or $2^r - 1$, with $V$ the deleted permutation module over $\mathbb{F}_2$, of dimension $c-2$ or $c-1$ respectively;*
   (ii) *$\mathrm{SL}_2(5) \trianglelefteq G < \Gamma\mathrm{L}_2(9) < \mathrm{GL}_4(3)$, orbit sizes $1, 40, 40$;*
   (iii) *$\mathrm{PSL}_2(11) \trianglelefteq G < \mathrm{GL}_5(3)$, orbit sizes $1, 22, 110, 110$;*

(iv) $M_{11} \trianglelefteq G < \mathrm{GL}_5(3)$, *orbit sizes* $1, 22, 220$;
(v) $M_{23} = G < \mathrm{GL}_{11}(2)$, *orbit sizes* $1, 23, 253, 1771$.

*Proof.* This is Theorem 1 of [10]. □

The classification of the groups in case (1) was achieved by Hering in another deep theorem (see [14]). We will use the description of these groups that appears in Appendix 1 of [23]. For the imprimitive case of Theorem 2.1, a result on permutation groups is necessary. Given a prime $p$, a subgroup $G \leq \mathfrak{S}_n$ is *p-concealed* if it has order divisible by $p$ and all the orbits on the power set of $\{1, \dots, n\}$ have $p'$-size.

**Theorem 2.2.** *Let $H$ be a primitive subgroup of $\mathfrak{S}_n$ of order divisible by a prime $p$. Then $H$ is $p$-concealed if and only if one of the following holds:*
(1) $\mathfrak{A}_n \trianglelefteq H \leq \mathfrak{S}_n$, *and* $n = ap^s - 1$ *with* $s \geq 1$, $a \leq p-1$ *and* $(a, s) \neq (1, 1)$; *also* $H \neq \mathfrak{A}_3$ *if* $(n, p) = (3, 2)$;
(2) $(n, p) = (8, 3)$, *and* $H = \mathrm{AGL}_3(2) = 2^3 : \mathrm{SL}_3(2)$ *or* $H = \mathrm{A\Gamma L}_1(8) = 2^3 : 7 : 3$;
(3) $(n, p) = (5, 2)$ *and* $H = D_{10}$.
*In cases* (2) *and* (3) *there exists* $\chi \in \mathrm{Irr}(H)$ *such that* $\chi(1)_p = p$.

*Proof.* The first part is [10, Thm 2]. The final assertion can be easily checked. □

Given a finite group $G$, we write $\mathbf{O}^{p'}(G)$ to denote the smallest normal subgroup of $G$ with $p'$ factor group, similarly $\mathbf{O}^p(G)$. Analogously, we write $\mathbf{O}_{p'}(G)$ (resp. $\mathbf{O}_p(G)$) to denote the largest normal $p'$-subgroup (resp. $p$-subgroup) of $G$.

**Theorem 2.3.** *Suppose $G \leq \mathrm{GL}_n(p) = \mathrm{GL}(V)$ is irreducible and $p$-exceptional with $G = \mathbf{O}^{p'}(G)$. If $V = V_1 \oplus \cdots \oplus V_n$ $(n > 1)$ is an imprimitivity decomposition for $G$, then $G_{V_1}$ is transitive on $V_1 \setminus \{0\}$ and $G$ induces a primitive $p$-concealed subgroup of $\mathfrak{S}_\Omega$, where $\Omega = \{V_1, \dots, V_n\}$.*

*Proof.* This is [10, Thm 3]. □

We will use the following elementary result. In this lemma, the word block will have a different meaning than in the rest of the paper. Suppose that a group $G$ acts transitively on a set $\Omega$, and let $\Delta \subseteq \Omega$. Recall that $\Delta$ is a *block* if for all $g \in G$, either $\Delta^g \cap \Delta = \emptyset$ or $\Delta^g = \Delta$.

**Lemma 2.4.** *Suppose that $G$ acts transitively but not primitively on $\Omega = \{1, \dots, n\}$. Let $\Delta \subsetneq \Omega = \{1, \dots, n\}$ be a block of maximal order. Then:*
(a) $\Omega = \Delta_1 \cup \cdots \cup \Delta_r$, *where $\Delta_i$ are the translates of $\Delta$. This union is disjoint and $|\Delta_i| = |\Delta|$ for all $i$.*
(b) *Let $L = \bigcap_{i=1}^r \mathrm{Stab}_G(\Delta_i)$. Then the induced action of $G/L$ on $\{\Delta_1, \dots, \Delta_r\}$ is primitive.*
(c) *If the action of $G$ on $\Omega$ is $p$-concealed, the action of $G/L$ on $\{\Delta_1, \dots, \Delta_r\}$ is $p$-concealed.*

*Proof.* Part (a) and (b) are well-known.

For the last part, write $\bar{G} = G/L$. Suppose that $\Gamma = \{\Delta_1, \dots, \Delta_s\}$ is a block for the action of $\bar{G}$ and write $\Gamma' = \bigcup_{i=1}^s \Delta_i$. Notice that we have $\mathrm{Stab}_{\bar{G}}(\Gamma) = \mathrm{Stab}_G(\Gamma')/L$. Now,

$$|\bar{G} : \mathrm{Stab}_{\bar{G}}(\Gamma)| = |G/L : \mathrm{Stab}_G(\Gamma')/L| = |G : \mathrm{Stab}_G(\Gamma')|$$

is not divisible by $p$. □

For a set $\Omega$, we write $\mathfrak{P}(\Omega)$ to denote the power set of $\Omega$. More generally, following [2], for any integer $n > 1$ we set

$$\mathfrak{P}_n(\Omega) = \{(\Lambda_1, \dots, \Lambda_n) \mid \Omega = \bigcup_{i=1}^{n} \Lambda_i,\ \Lambda_i \cap \Lambda_j = \emptyset \text{ for } i \neq j\}.$$

By a theorem of Gluck, see [11] or [29, Thm 5.6], an odd order permutation group on $\Omega$ has a regular orbit on $\mathfrak{P}(\Omega)$ or, equivalently, on $\mathfrak{P}_2(\Omega)$ (see for instance [29, Cor. 5.7(b)]). As pointed out in [11, Prop. 1], this is false for 2-groups. In [2, Cor. 6] and independently in [38, Thm 1.2] it was proved that a solvable permutation group on $\Omega$ has a regular orbit on $\mathfrak{P}_4(\Omega)$. We will need a regular orbit on $\mathfrak{P}_3(\Omega)$ in the case when the group is a $p$-group. (Again, as examples in [2] and [38] show, this is not true for arbitrary solvable groups.) By the previous comments, it suffices to do it when $p = 2$.

**Lemma 2.5.** *Let $P \leq \mathfrak{S}_\Omega$ be a $p$-group. If $p$ is odd, then there exists a regular $P$-orbit on $\mathfrak{P}(\Omega)$. If $p = 2$ then there exists a regular $P$-orbit on $\mathfrak{P}_3(\Omega)$.*

*Proof.* The first part follows from [29, Cor. 5.7(b)]. The second part is a consequence of work in [2]. Indeed, by [2, Lemma 1(c)], every proper primitive group $H < \mathfrak{S}_\Omega$ has at least four regular orbits on $\mathfrak{P}_3(\Omega)$. It follows from [2, Thm 2] that $P$ has a regular orbit on $\mathfrak{P}_3(\Omega)$, as desired. □

## 3. Results on almost simple groups

In this section we collect some results on almost simple groups that will be used in the proof of Theorem 1.

3.1. **Miscellaneous results.** We start with an elementary result on the groups $\mathrm{PSL}_2$.

**Lemma 3.1.** *Let $S \cong \mathrm{SL}_2(p^n)$ or $S \cong \mathrm{PSL}_2(p^n)$, with $(p, n) \neq (2, 1), (3, 1)$. Let $R \in \mathrm{Syl}_p(S)$. Then:*
(a) *If $R \leq H < S$, then $H \leq \mathbf{N}_S(R)$.*
(b) *If $Q \in \mathrm{Syl}_p(S)$ with $Q \neq R$, then $Q \cap R = 1$.*

*Proof.* Suppose first that $S \cong \mathrm{PSL}_2(p^n)$. Part (a) follows from Dickson's classification of the subgroups of $\mathrm{PSL}_2(p^n)$ (see [17, Hauptsatz II.8.27]) and part (b) follows from [17, Satz II.8.2]. Now, suppose that $S = \mathrm{SL}_2(p^n)$ and let $Z = \mathbf{Z}(S)$ so that $S/Z$ is as in the previous paragraph. Let $H < S$ containing $R \in \mathrm{Syl}_p(S)$. Therefore, $HZ/Z \leq \mathbf{N}_{S/Z}(RZ/Z)$. Since $Z$ is a $p'$-group, $R$ is normal in $H$, as wanted. Claim (b) also follows from the previous case. □

Next, we obtain a result on the character degrees of alternating groups.

**Lemma 3.2.** *Let $p$ be an odd prime and $p \leq n < p^2$, $n > 4$ and $(n, p) \neq (6, 3)$. Then there exists $\chi \in \mathrm{Irr}(\mathfrak{A}_n)$ such that $\chi(1)_p = p$.*

*Proof.* If $p = 3$, the result can be easily checked. So we may assume that $p > 3$. Write $n = ap + b$ with $1 \leq a < p$ and $0 \leq b < p$. By the hook length formula [20, Thm 2.3.21], given a partition $\lambda$ of $n$, the character $\chi^\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ indexed by $\lambda$ has degree

$$\chi^\lambda(1) = \frac{n!}{\prod_{i,j} h_{ij}^\lambda},$$

where $h_{ij}^\lambda$ are the hooklengths of $\lambda$. Let $\chi \in \mathrm{Irr}(\mathfrak{A}_n)$ under $\chi^\lambda$. Since $p$ is odd, we have $\chi(1)_p = \chi^\lambda(1)_p$, so it is enough to find a character $\chi^\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ such that $\chi^\lambda(1)_p = p$. Since $(n!)_p = p^a$, we need to find a partition $\lambda$ such that $\prod_{i,j} h_{ij}^\lambda = p^{a-1}$. If $b \neq 0$, we take the partition $\lambda = (ap, 1, 1, \ldots, 1)$. If $b = 0$, we take the partition $\lambda = (ap - 2, 2)$. □

We need the following character extendability result in almost simple groups. As usual, given a group $G$ and a prime $p$, $\mathrm{Irr}_{p'}(B_0(G))$ is the set of $p'$-degree irreducible characters in the principal $p$-block of $G$.

**Lemma 3.3.** *Let $p$ be a prime and let $S$ be a non-abelian simple group. Then there exists $1_S \neq \alpha \in \mathrm{Irr}_{p'}(B_0(S))$ which is $X$-invariant for some Sylow $p$-subgroup $X \in \mathrm{Syl}_p(\mathrm{Aut}(S))$.*

*Proof.* This follows from [9, Prop. 2.1] when $p > 3$ and from [34, Thm C] when $p \leq 3$. □

Finally, we prove a technical lemma that will be used several times in the proof of Theorem 1.

**Lemma 3.4.** *Let $p$ be a prime. Let $H$ be a primitive subgroup of $\mathfrak{S}_n$ of order divisible by $p$ with abelian Sylow $p$-subgroups such that $\mathbf{O}^{p'}(H) = H$. If $H$ is $p$-concealed, then there exists $\chi \in \mathrm{Irr}(H)$ such that $\chi(1)_p = p$.*

*Proof.* By hypothesis, $H$ is one of the groups listed in Theorem 2.2. We have already seen in Theorem 2.2 that in cases (2) and (3) the result holds. Now, assume that we are in case (1) of that theorem, so $\mathfrak{A}_r \lhd H \leq \mathfrak{S}_r$, with $r = ap^s - 1$, $s \geq 1$ and $(a, s) \neq (1, 1)$. If $r = 2$, we have $p = 3$ and $(a, s) = (1, 1)$, a contradiction. If $r = 3$ then $p = 2$ and $H \neq \mathfrak{A}_3$ by hypothesis, so $H \cong \mathfrak{S}_3$, and there is $\chi \in \mathrm{Irr}(H)$ of degree 2. We are done in this case. If $r = 4$, we have $p = 5$ and $(a, s) = (1, 1)$, another contradiction. If $r = 6$, then $p = 7$ and $(a, s) = (1, 1)$, contradiction. Hence we have that $r \geq 5$, $r \neq 6$. Since $H$ has abelian Sylow $p$-subgroups, $r < p^2$. Suppose first that $H = \mathfrak{A}_r$. Since $p$ divides $|H|$, $p \leq r$ and we are done by Lemma 3.2. Similarly, if $H = \mathfrak{S}_r$, we deduce that $p = 2$. But then $H$ does not have abelian Sylow $p$-subgroups, which is a contradiction. □

3.2. **Almost simple groups of Lie type.** We now aim to prove the following, which is slightly stronger than what is needed later on:

**Theorem 3.5.** *Let $A$ be almost simple with socle a simple group $S$ of Lie type, and $p$ a prime such that $A = O^{p'}(A)$. If $S < A$ and the Sylow $p$-subgroups of $A$ are non-abelian then there exists $\chi$ in the principal $p$-block of $A$ of height 1.*

That is, the minimal non-zero height in the principal block is as small as possible. Note that the conclusion fails in general when $A = S$ and $p$ is the defining characteristic of $S$ (see [1] for examples). We will give the proof in several steps. We will use the following:

**Lemma 3.6.** *Let $A$ be a finite group and let $S \lhd A$ be a perfect group. Let $p$ be a prime and let $\chi \in \mathrm{Irr}(B_0(S))$ with stabiliser $A_\chi$ in $A$. Suppose that one of the following holds:*

(1) $\chi(1)$ *is not divisible by* $p$ *and* $|A : A_\chi|_p = p$*; or*
(2) $\chi(1)_p = p$, $|A : A_\chi|_p = 1$ *and* $\chi$ *extends to* $P$*, where* $P/S \in \mathrm{Syl}_p(A_\chi/S)$.
*Then there exists* $\psi \in \mathrm{Irr}(B_0(A))$ *of height 1.*

*Proof.* Suppose that (1) or (2) holds. Then $\chi$ extends to $P$ by [19, Cor. 6.28] or the hypothesis, where $P/S \in \mathrm{Syl}_p(A_\chi/S)$. By [31, Lemma 4.3] there exists $\varphi \in \mathrm{Irr}(B_0(A_\chi))$ with $\varphi(1)_p = \chi(1)_p$ lying over $\chi$. By the Clifford correspondence we know that $\psi := \varphi^A \in \mathrm{Irr}(A)$. By [32, Cors 6.2 and 6.7], $\psi$ lies in $B_0(A)$ and

$$\psi(1)_p = |A : A_\chi|_p \chi(1)_p = p. \qquad \square$$

Throughout, let $\mathbf{G}$ be a simple linear algebraic group over an algebraically closed field $k$ of characteristic $r > 0$. Let $\mathbf{T} \leq \mathbf{G}$ be a maximal torus contained in a Borel subgroup $\mathbf{B} \leq \mathbf{G}$ and let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. We denote by $\Phi$ the root system of $\mathbf{G}$ with respect to $\mathbf{T}$, and by $\Phi^+ \subset \Phi$ the set of positive roots with respect to $\mathbf{B}$. Write $\Delta \subset \Phi^+$ for the set of simple roots. For any $\alpha \in \Phi^+$, we denote by $\mathbf{U}_\alpha$ the corresponding root subgroup in $\mathbf{U}$ normalized by $\mathbf{T}$, and we choose an isomorphism $x_\alpha : k \to \mathbf{U}_\alpha$ (see e.g. [28, Sect. 8]).

For any graph automorphism $\gamma$ of (the Dynkin diagram of) $\Phi$ and for any power $q = r^f$, there is a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ with the property that $F(x_\alpha(c)) = x_{\gamma(\alpha)}(c^q)$ for all $\alpha \in \Phi$, $c \in k$ (see [13, Thm 1.15.2]). We set $F_r : \mathbf{G} \to \mathbf{G}$, $x_\alpha(c) \mapsto x_\alpha(c^r)$. Note that $F, F_r$ commute in their action on $\mathbf{G}$, and $\mathbf{T}, \mathbf{B}$ and $\mathbf{U}$ are $F$- and $F_r$-stable. Furthermore $F$ acts on the set of subgroups $\mathbf{U}_\alpha$, which induces an action of $F$ on the space $V = \mathbb{R}\Phi$. The resulting map is $q\phi$, where $\phi$ is an automorphism of $V$ of finite order. We recall the setup from [28, §23]. Write $\pi : V \to V^\phi$ for the projection (Reynolds operator) onto the fixed space $V^\phi$. We define an equivalence relation $\sim$ on $\pi(\Phi)$ by setting $\pi(\alpha) \sim \pi(\beta)$ if and only if there is some positive $c \in \mathbb{R}$ such that $\pi(\alpha) = c\pi(\beta)$, and we let $\widehat{\Phi}$ be the set of equivalence classes under this relation. For $\alpha \in \Phi^+$, we write $\widehat{\alpha}$ for the class of $\pi(\alpha)$ and set $\mathbf{U}_{\widehat{\alpha}} = \prod_{\beta:\pi(\beta)\sim\pi(\alpha)} \mathbf{U}_\beta$. Then $U_{\widehat{\alpha}} = \mathbf{U}_{\widehat{\alpha}}^F$ is the corresponding root subgroup of $G := \mathbf{G}^F$. The different possible root subgroups are described in [13, Table 2.4], for example. Now, by [13, Thm 2.3.7], $U = \mathbf{U}^F = \prod_{\{\widehat{\alpha}|\alpha\in\Phi^+\}} U_{\widehat{\alpha}}$ is a Sylow $r$-subgroup of $G$.

We first consider the case when $p$ is the defining prime for $S$. For this let $\mathbf{G}^*$ be dual to $\mathbf{G}$ with corresponding Frobenius endomorphisms again denoted $F, F_r$ and $G^* := \mathbf{G}^{*F}$.

**Lemma 3.7.** *In the notation introduced above, let* $A \leq \mathrm{Aut}(G)$ *and suppose* $r = p$*. Assume there is a semisimple element* $s \in [G^*, G^*]$ *such that the stabiliser of the* $G^*$*-conjugacy class of* $s$ *has index* $p$ *in* $A$*. Then* $B_0(A)$ *contains a character of height 1.*

*Proof.* Let $\chi \in \mathcal{E}(G, s)$ be a semisimple character in the Lusztig series labelled by $s$. Then $\chi(1)$ is prime to $p$ by the degree formula [8, Thm 2.6.11 with Cor. 2.6.18]. Furthermore, $\chi$ is trivial on $Z(G)$ since $s \in [G^*, G^*]$ (see [33, Lemma 4.4(ii)]). By the main result of [16], then $\chi \in \mathrm{Irr}(B_0(G))$ and thus $\chi$ can be considered as a character in $\mathrm{Irr}(B_0(S))$ by [32, Thms 9.9(c) and 9.10]. Finally, by [39, Prop. 7.2] our assumption on the class of $s$ implies that the stabiliser of $\chi$ in $A$ has index $p$, so by Clifford theory and Lemma 3.6 there is a character of height 1 above $\chi$ in $\mathrm{Irr}(B_0(A))$. $\square$

**Proposition 3.8.** *Let $S$ be a simple group of Lie type in characteristic $p$, $\sigma$ a non-trivial $p$-power order field automorphism of $S$ and $A := \langle S, \sigma\rangle \le \mathrm{Aut}(S)$. Let $P \in \mathrm{Syl}_p(A)$. Then:*

(a) *there exists $\chi \in \mathrm{Irr}(B_0(A))$ of height 1; and*

(b) *there exists $\theta \in \mathrm{Irr}(P)$ of height 1.*

*In particular, the EM-conjecture holds for the principal $p$-block of $A$.*

*Proof.* The assumptions imply that $S$ is not a Suzuki group nor a big Ree group (since these are defined in characteristic 2 but only have odd order field automorphisms). Let us postpone the treatment of the small Ree groups for the moment. Thus, now $S = G/Z(G)$ for $\mathbf{G}$ a simple algebraic group of simply connected type as introduced above, with $r = p$. By definition any field automorphism $\sigma$ of $S$ is induced, up to inner automorphisms, by a suitable power $F_1 := F_p^e$, and up to replacing $\sigma$ by a primitive power, we may even assume $e|f$. So let $e|f$ be such that $\sigma = F_1|_G$. In particular, $f/e$ is a $p$-power.

For part (a), as above let $\mathbf{G}^*$ be dual to $\mathbf{G}$ and $G^* := \mathbf{G}^{*F}$. Let $d$ be maximal such that the cyclotomic polynomial $\Phi_d$ divides the order polynomial of $\mathbf{G}^*$ (see [28, Tab. 24.1]) and let $l$ be a primitive prime divisor of $\Phi_{dp}(p^e)$. Note that $d \ge 2$ by loc. cit. and hence such a divisor always exists unless $p = 2$, $d = 3$, $e = 1$ (see [28, Thm 28.3]). In the latter case we instead choose $l = 5$. Then there exists a (semisimple) element $s \in [\mathbf{G}^{*F_1^p}, \mathbf{G}^{*F_1^p}]$ of order $l$. If the $\mathbf{G}^*$-conjugacy class $C$ of $s$ were $F_1$-stable, by the Lang–Steinberg theorem [28, Thm 21.11] there would exist $F_1$-stable elements in $C$, which is not the case since by construction $l$ does not divide the order of $\mathbf{G}^{*F_1}$. Now an application of Lemma 3.7(1) shows the existence of a character of $A$ above $\chi$ in $B_0(A)$ with height 1, proving (a).

For (b) let $U = \mathbf{U}^F$, considered as a Sylow $p$-subgroup of $S$, which is possible since $Z(G)$ has order prime to $p$. Now by [15, Lemma 7], we have $[U, U] = \prod_{\alpha\in\Phi^+\setminus\Delta} U_{\widehat{\alpha}}$ unless

$$G \in \{B_n(2),\, C_n(2),\, G_2(3),\, F_4(2),\, {}^2B_2(2),\, {}^2G_2(3),\, {}^2F_4(2)\}.$$

Since none of the excluded groups does possess non-trivial field automorphisms we may in fact assume $[U, U] = \prod_{\alpha\in\Phi^+\setminus\Delta} U_{\widehat{\alpha}}$. Note that $U$ is $\sigma$-stable. Now any $c \in k^{F_1^p} \setminus k^{F_1}$ lies in an $F_1$-orbit of length $p$. So for any $\alpha \in \Delta$, the element $\prod_{\alpha\in\widehat{\alpha}} x_\alpha(c)[U, U] \in U_{\widehat{\alpha}}[U, U]$ lies in a $\sigma$-orbit of length $p$ in $U/[U, U]$. Thus, by Brauer's permutation lemma, there also is a $\sigma$-orbit of length $p$ on $\mathrm{Irr}(U/[U, U])$, hence on $\mathrm{Irr}(U)$. That is, there is a linear character of $U$ whose inertia group in $\langle U, \sigma\rangle \in \mathrm{Syl}_p(A)$ has index $p$. So, by Clifford theory, $P$ has an irreducible character of degree $p$, showing (b).

Finally, if $S = G = {}^2G_2(q^2)$ is a small Ree group and $p = 3$, then for any semisimple element $s \in G^{\sigma^3} \setminus G^\sigma$ the corresponding semisimple character in $\mathcal{E}(G, s)$ is as in Lemma 3.7(1), giving (a). Further, using that the description in [28, §23] also applies in this very twisted case, as before we can construct a linear character $\chi$ of $P \cap S$ stable under $\sigma^3$ but not under $\sigma$, and then any character of $P$ above $\chi$ is as claimed in (b). ☐

**Proposition 3.9.** *Theorem 3.5 holds for $A$ with socle a simple group of Lie type in characteristic $p$.*

*Proof.* The outer automorphisms of $p$-power order of simple groups of Lie type $S$ in characteristic $p$ are field, graph and graph-field automorphisms (see e.g. [28, Thm 24.24]).

In particular, for $p \geq 5$ only field automorphisms occur, and these are central in $\mathrm{Out}(S)$. Hence in this case, our claim follows from Proposition 3.8.

If $p = 3$ the only groups $S$ with additional $p$-automorphisms are those of untwisted type $D_4$, for which $\mathrm{Out}(S) \cong \mathfrak{S}_4 \times C_f$. Thus $A/S \leq \mathfrak{A}_4 \times C_{3^k}$ with $3^k|f$. By Proposition 3.8 we may assume that $A$ induces some non-field automorphisms of 3-power order. Let $\mathbf{G}$ be simply connected and $\mathbf{G}^*$, $F$, $G = \mathbf{G}^F$ be as above, so that $S = G/Z(G)$. Let $3^e$ be the exponent of a Sylow 3-subgroup of $A/S$. Let $s \in \mathbf{G}^{*F}$ be a (semisimple) element whose order is a primitive prime divisor $l$ of $\Phi_4(3^{f/3^e})$ (so that in fact $s \in [\mathbf{G}^{*F}, \mathbf{G}^{*F}]$). Then $l$ does not divide the order of $G_2(3^f)$, the centraliser of a graph automorphism of $S$, and thus the $G^*$-conjugacy class of $s$ is not stabilised by the graph automorphism of $S$. On the other hand, by construction $s$ is fixed by the field automorphisms in $A/S$. Hence again we may conclude with Lemma 3.7.

Finally, assume $p = 2$. The only groups with graph automorphisms of order two are those of types $A_n$ $(n \geq 2)$, $D_n$ $(n \geq 4)$ and $E_6$. By the above we may assume $A$ induces some non-field automorphisms of 2-power order. Let $2^e$ be the exponent of a Sylow 2-subgroup of $A/S$ and $\mathbf{G}$, $F$ and $\mathbf{G}^*$ as before with $G := \mathbf{G}^F = E_6(2^f)$ and set $F_1 := F_p^{f/2^e}$. Let $s \in \mathbf{G}^{*F_1}$ be a (semisimple) element whose order is a primitive prime divisor of $\Phi_9(2^{f/2^e})$. Then $s$ is stable under all 2-power order field automorphisms in $A$ but inverted by a graph automorphism (while it is non-real in $G^*$). Thus the class of $s$ lies in an orbit of length 2 under $A$ and we may apply Lemma 3.7. Next, let $G = \mathrm{SL}_n(2^f)$ with $n \geq 3$ and $s \in \mathbf{G}^{*F_1}$ be a (semisimple) element whose order is a primitive prime divisor of $\Phi_n(2^{f/2^e})$ if $n$ is odd, respectively of $\Phi_{n-1}(2^{f/2^e})$ if $n$ is even. Then $s$ is non-real, but inverted by a graph automorphism and we may conclude as in the previous case.

Next, assume $G = D_n(q)$, with $q = 2^f$ and $n \geq 4$. Here, $\mathbf{G}^{*F} \cong \mathrm{SO}_{2n}^+(q)$. We now argue as in the proof of [26, Prop. 2.8]. Let $s \in \mathbf{G}^{*F_1}$ be a (semisimple) element in the stabiliser $\mathrm{GL}_n(q)$ of a maximal totally isotropic subspace of $\mathbf{G}^{*F}$ whose order is a primitive prime divisor of $\Phi_n(2^{f/2^e})$. If $n$ is odd, $s$ is non-real but the graph automorphism of $S$ induces the transpose-inverse automorphism of $\mathrm{GL}_n(q)$, so conjugates $s$ to its inverse. If $n$ is even, then $s$ is real but the graph automorphism interchanges the two conjugacy classes of stabilisers of totally singular subspaces, hence does not fix the class of $s$. So in either case the semisimple character $\chi$ of $S$ labelled by $s$ is invariant under $F_1$ but not under the graph automorphism, so gives rise to a character in $\mathrm{Irr}(B_0(A))$ of height 1.

Finally, let $S = \mathrm{Sp}_4(2^f)$ or $S = F_4(2^f)$ with $A/S$ inducing (and hence being generated by) a graph-field automorphism $\sigma$. Set $a = f_{2'}$ and let $s$ be a regular semisimple element in the first factor of an $A_1(2^a) \times \tilde{A}_1(2^a)$-, respectively $A_2(2^a) \times \tilde{A}_2(2^a)$-subsystem subfield subgroup. Then $s$ is centralised by 2-power order field automorphisms, but not by $\sigma$, since $s$ centralises a long root element, but no short root element, and these two are interchanged by $\sigma$. Then the semisimple character $\chi_s$ is again as desired. □

Next, we consider groups of Lie type in characteristic $r$ different from $p$.

**Proposition 3.10.** *Theorem* 3.5 *holds for $A$ with socle $S$ simple of Lie type in characteristic $r \neq p$ with non-abelian Sylow $p$-subgroups.*

*Proof.* Let $G = \mathbf{G}^F$ with $\mathbf{G}$ simple of simply connected type such that $S = G/Z(G)$. For $p = 2$, [1, Prop. 4.5] shows the existence of a unipotent character of $S$ of 2-height 1 in $B_0(G)$, except for $A_1(q)$ and $A_2(\epsilon q)$ with $q \equiv -\epsilon \pmod 4$. For $p \geq 3$ the existence of a unipotent character of height 1 in $B_0(G)$ is shown in [1, Prop. 4.3 and Thm 4.7], except for $G_2(q)$, ${}^3D_4(q)$, ${}^2F_4(q^2)$, $A_2(\epsilon q)$ and $A_5(\epsilon q)$ with $q \equiv \epsilon \pmod 3$ when $p = 3$. Since unipotent characters have $Z(G)$ in their kernel, all of the above can be regarded as characters in $B_0(S)$. Furthermore, using [8, Thm 4.5.11] one sees that these unipotent characters are invariant under $p$-automorphisms of $S$ and since by [24, Thm 2.4] they extend to their inertia group in $A$, this provides the desired character of $A$ by Clifford theory and Lemma 3.6.

It remains to discuss the types left open above. First assume $p = 3$. For $S = G = G_2(r^f)$ let $s \in H := \mathbf{G}^{*F_r} = G_2(r)$ be a 3-element in a maximal torus $T$ of $H$ of order $(r-\epsilon)^2$, where $\epsilon \in \{\pm 1\}$ and $r \equiv \epsilon \pmod 3$, whose centraliser in a Sylow 3-subgroup of $H$ has index 3 (since $N_H(T)$ contains a Sylow 3-subgroup of $H$ and the latter is non-abelian). Then $s$ is invariant under all field automorphisms of $S$, and any semisimple character in $\mathcal{E}(G, s)$ has height 1 (by the degree formula [8, Thm 2.6.11 with Cor. 2.6.18]), lies in the principal 3-block of $G$ by [4, Thm B] and is invariant under field automorphisms (since $s$ is), so extends to $A$ as $A/S$ is cyclic. Exactly the same construction allows one to deal with all of the other excluded types for $p = 3$ if $A$ induces only field automorphisms.

So next let $S = \mathrm{PSL}_3(q)$ with $q \equiv 1 \pmod 3$ and assume $A$ induces some non-field automorphisms. Let $3^e$ be the exponent of a Sylow 3-subgroup of $A/S$ and set $F_1 := F_r^{f/3^e}$. Let now $\mathbf{G}$ be simple of adjoint type such that $G := \mathbf{G}^F = \mathrm{PGL}_3(q)$, and so $S = [G, G]$. Let $s = \mathrm{diag}(1, \omega, \omega^2) \in \mathbf{G}^{*F} = \mathrm{SL}_3(q)$ with $\omega \in \mathbb{F}_q^\times$ a primitive third root of unity. Thus $s$ is a semisimple element of order three. Since $r^{f/3^e} \equiv r^f = q \equiv 1 \pmod 3$ this is $F_1$-stable. Let $\chi \in \mathcal{E}(G, s)$ be the semisimple character. Then $\chi$ lies in $B_0(G)$ since $G$ has a unique unipotent 3-block. Furthermore, $\chi$ has height 1, while its restriction to $[G, G] = S$ splits into three characters $\chi_1, \chi_2, \chi_3$ of height 0, since the image of $s$ in $\mathbf{G}^*/Z(\mathbf{G}^*) = \mathrm{PGL}_3$ has disconnected centraliser. Since $s$ is $F_1$-invariant, the inertia group of $\chi_1$ in $A$ has index 3, yielding the claimed character of $A$ by Lemma 3.7(1). For $S = \mathrm{PSL}_6(q)$ let $s = \mathrm{diag}(1, 1, 1, 1, \omega, \omega^2) \in \mathrm{SL}_6(q)$, an $F_1$-invariant semisimple element of order three. Let $\chi \in \mathcal{E}(\mathrm{PGL}_6(q), s)$ be the corresponding semisimple character. Then $\chi$ lies in the principal 3-block, since there is a unique unipotent block, has height 1 and is invariant under field automorphisms. So we may conclude as in the previous case. The unitary groups $\mathrm{PSU}_3(q)$ and $\mathrm{PSU}_6(q)$ are handled entirely similarly.

Finally let $p = 2$ and $S$ one of the groups excluded in the first paragraph. If $S = \mathrm{PSL}_3(\epsilon q)$ with $q \equiv -\epsilon \pmod 4$ let $s$ be an element of order 4 in a maximal torus of $\mathrm{PGL}_3(\epsilon q)$ of order $q^2 - 1$. The corresponding semisimple character of $S$ has degree $q^3 - \epsilon$, hence height 1, lies in the principal 2-block and is $A$-invariant as $s$ is centralised by the field and graph automorphisms. Now note that $A/S$ is cyclic since $q$ is not a square when $\epsilon = 1$, so $\chi$ extends to a character of $A$ as required.

Now let $S = \mathrm{PSL}_2(q)$. Since by assumption Sylow 2-subgroups of $S$ are non-abelian we have $q \equiv \epsilon \pmod 8$ with $\epsilon \in \{\pm 1\}$. Let $s \in \mathrm{SL}_2(q)$ be an element of order 4 whose image in $\mathrm{PGL}_2$ has disconnected centraliser. The (semisimple) character $\chi \in \mathcal{E}(\mathrm{PGL}_2(q), s)$ has degree $q + \epsilon$, hence height 1, lies in the principal 2-block (being labelled by a 2-element) and is invariant under the field automorphisms of $S$. Thus we are done whenever $A$ induces some non-field automorphism. If $A$ induces only field automorphisms, then $q$ is a square, so there exists a (rational) element of order 4 in a maximal torus of order $q - 1$ of $\mathrm{PGL}_2(q)$ which is a square, so lies in $\mathrm{PSL}_2(q)$. So the corresponding semisimple character of $\mathrm{SL}_2(q)$ of degree $q+1$ has $Z(\mathrm{SL}_2(q))$ in its kernel, lies in the principal 2-block, is of height 1 and is invariant under all field automorphisms. It thus extends to a character of $A$ as desired. □

**Proposition 3.11.** *Theorem* 3.5 *holds for $A$ with socle $S$ simple of Lie type in characteristic $r \neq p$ with abelian Sylow $p$-subgroups.*

*Proof.* Note that $S = {}^2G_2(q^2)$ has no even order outer automorphisms. Thus, if $p = 2$ then Sylow 2-subgroups of $S$ being abelian forces $S = \mathrm{PSL}_2(q)$ with $q \equiv \epsilon 3 \pmod 8$, $\epsilon \in \{\pm 1\}$. In particular, $S$ has no 2-power order field automorphisms and hence $A = \mathrm{PGL}_2(q)$. Here we take the irreducible character $\chi$ of $A$ labelled by an element of order 4, of degree $q+\epsilon$, which has height 1 and lies in the principal 2-block. So we have $p > 2$.

Let $p = 3$. For the groups $S = \mathrm{PSL}_3(\epsilon q)$ with abelian Sylow 3-subgroup and outer automorphisms of order 3 the arguments in the proof of Proposition 3.10 still do apply. The only other cases with abelian Sylow 3-subgroups are when $S = \mathrm{PSL}_2(q)$, with $q$ not a 3-power by Proposition 3.9. If $A$ has non-abelian Sylow 3-subgroups then $q = r^f$ with $3|f$. Write $|A/S| = e$ and let $s$ be a 3-element in $\mathrm{PGL}_2(r^{3f/e})$ not lying in $\mathrm{PGL}_2(r^{f/e})$. Then the semisimple character $\chi$ of $S$ labelled by $s$ lies in an orbit of length 3 under $A$, in the principal 3-block, and has height 0; hence there is a character of $A$ above $\chi$ as claimed.

For $p \geq 5$, outer automorphisms of order $p$ are either field automorphisms or $S = \mathrm{PSL}_n(\epsilon q)$ with $q \equiv \epsilon \pmod p$, but in the latter case, Sylow $p$-subgroups of $S$ are non-abelian. Thus we have that $A$ induces $p$-power field automorphisms of $S$, say generated by a Frobenius map $F_1$. Let $\mathbf{G}$ be of simply connected type such that $S = G/Z(G)$ for $G = \mathbf{G}^F$. Let $s \in G^*$ be a $p$-element whose class is invariant under $F_1^p$ but not under $F_1$ and $\chi \in \mathcal{E}(G,s)$ the corresponding semisimple character. Since Sylow $p$-subgroups of $S$ and hence of $G$ are abelian, this has height 0. Then $\chi \in \mathrm{Irr}(B_0(G))$ by [4, Thm B]. Note that $p$ does not divide $|G^* : [G^*, G^*]|$, so $\chi$ is a character of $S$. By construction its inertia group in $A$ has index $p$, and thus we are done by Lemma 3.6. □

This completes the proof of Theorem 3.5. Given a group $G$ we write $\mathrm{cd}(G)$ to denote the set of character degrees of $G$.

**Corollary 3.12.** *Let $p$ be a prime and let $A$ be an almost simple group with socle $S$ and $\mathbf{O}^{p'}(A) = A$. Let $P \in \mathrm{Syl}_p(A)$ and assume that $\mathrm{cd}(P) = \{1, p^a\}$ for some $a \geq 1$. Then there exists $\chi \in \mathrm{Irr}(B_0(A))$ such that $1 < \chi(1)_p \leq p^a$.*

*Proof.* The EM-conjecture for principal blocks of simple groups was proved in [1, Thm 4.7], so we may assume $A$ is not simple. If $S$ is sporadic then $|A : S| = 2$, so $p = 2$, and the result can be checked with [40]. If $S$ is alternating then the claim was shown in [1, Thm 2.1]. Finally, for $S$ simple of Lie type the result is contained in Theorem 3.5. □

## 4. Proof of Theorem 1

We will use several times the description of the structure of groups with an abelian Sylow $p$-subgroup.

**Theorem 4.1.** *Let $p$ be a prime. Let $G$ be a finite group with $G = \mathbf{O}^{p'}(G)$ and abelian Sylow $p$-subgroups. If $N = \mathbf{O}_{p'}(G)$, then $G/N = X/N \times Y/N$, where $X/N$ is an abelian $p$-group and $Y/N$ is trivial or a direct product of non-abelian simple groups of order divisible* *by $p$. In particular, any non-abelian chief factor of order divisible by $p$ of a group with abelian Sylow $p$-subgroups is a simple group.*

*Proof.* This follows from [33, Thm 2.1]. □

We need the following general lemma on principal blocks. It should be compared with [35, Lemmas 1.2 and 1.3].

**Lemma 4.2.** *Let $G$ be a finite group and $N \trianglelefteq G$. Let $Q \in \mathrm{Syl}_p(N)$ and suppose that $\mathbf{C}_G(Q) \subseteq N$. Then $B_0(G)$ is the unique block of $G$ that covers $B_0(N)$. In particular $\mathrm{Irr}(G/N) \subseteq \mathrm{Irr}(B_0(G))$.*

*Proof.* Let $B$ be a $p$-block of $G$ covering $B_0(N)$. By Knörr's theorem [32, Thm 9.26], there exists a defect group $P$ of $B$ containing $Q$. Since $\mathbf{C}_G(P) \subseteq \mathbf{C}_G(Q) \subseteq N$, [32, Lemma 9.20] implies that $B$ is regular with respect to $N$. Now, by [32, Thm 9.19], $B_0(N)^G$ is defined and $B_0(N)^G = B$.

Now, Brauer's Third Main Theorem [32, Thm 6.7] implies that $B = B_0(G)$, as desired. The second assertion follows from [32, Thm 9.2]. □

In Step 1 of the proof of Theorem 1 we handle the solvable case. When $p$ is odd, that case follows from [3, Thm 6.4 and Prop. 6.5]. We extend this result to the prime 2. We will need orbit theorems of Espuelas and Isaacs that we recall next. While Espuelas' result shows the existence of regular orbits in a certain situation, Isaacs' result provides short orbits in a similar situation.

**Theorem 4.3.** *Let $p$ be a prime and let $G$ be a solvable group with $\mathbf{O}_p(G) = 1$. Let $A$ be an abelian $p$-subgroup of $G$. Suppose that $V$ is a faithful $G$-module in characteristic $p$. Then there exists $v \in V$ in a regular $A$-orbit.*

*Proof.* This is a consequence of the main result of [5]. □

**Theorem 4.4.** *Let $p$ be a prime and let $G$ be a solvable group of order divisible by $p$ with $\mathbf{O}_p(G) = 1$. Let $V$ be a faithful completely reducible $G$-module in characteristic $p$. Suppose that $P \in \mathrm{Syl}_p(G)$ is abelian. Then there exists $v \in V$ in a $P$-orbit of size $p$.*

*Proof.* This is Isaacs' Lemma 5.1 of [3]. We take this opportunity to point out that in that statement it should say 'reducible' instead of 'irreducible'. □

Finally we recall a particular case of [19, Problem 6.18] that will be used several times in this paper. The result is well-known, but for the reader's convenience we sketch a proof.

**Lemma 4.5.** *Let $G = HN$, where $H \leq G$, $N \trianglelefteq G$ and $N \cap H = 1$. If $\lambda \in \mathrm{Irr}(N)$ is linear and $G$-invariant then $\lambda$ is extendible to $G$.*

*Proof.* It can easily be checked that $\tilde{\lambda} : G \longrightarrow \mathbb{C}^\times$ defined by $\tilde{\lambda}(hn) = \lambda(n)$ for $h \in H, n \in N$, is a group homomorphism, extending the character $\lambda$. □

Now, we proceed to prove Theorem 1 which we restate:

**Theorem 4.6.** *Let $G$ be a finite group and $p$ a prime number. Suppose that the set of character degrees of a Sylow $p$-subgroup of $G$ is $\{1, p^a\}$, for some $a \geq 1$. Then there exists $\chi \in \mathrm{Irr}(B_0(G))$ with $1 \neq \chi(1)_p \leq p^a$.*

*Proof.* We proceed by induction on $|G|$.

*Step* 0. *We may assume* $\mathbf{O}^{p'}(G) = G$ *and* $\mathbf{O}_{p'}(G) = 1$. *Moreover, if* $1 < N \lhd G$, $G/N$ *has abelian Sylow* $p$*-subgroups. Also, there is just one minimal normal subgroup of* $G$.

Let $P \in \mathrm{Syl}_p(G)$. Let $M = \mathbf{O}^{p'}(G)$, so that $P \subseteq M$. If $M < G$, by induction there exists $\psi \in \mathrm{Irr}(B_0(M))$ with $1 \neq \psi(1)_p \leq p^a$. Let $\chi \in \mathrm{Irr}(B_0(G))$ lying over $\psi$, which exists by [32, Thm 9.4]. Then $\chi(1)_p = \psi(1)_p$, and we are done.

Now let $N = \mathbf{O}_{p'}(G)$ and notice that $\mathrm{Irr}(B_0(G/N)) = \mathrm{Irr}(B_0(G))$ by [32, Thm 9.9(c)]. If $N > 1$ then, since $P \cong PN/N$, by induction there exists $\chi \in \mathrm{Irr}(B_0(G))$ with $1 \neq \chi(1)_p \leq p^a$, as wanted.

Next we show that $G/N$ has abelian Sylow $p$-subgroups for every non-trivial normal subgroup $N$ of $G$. We have that $\mathrm{cd}(PN/N) \subseteq \mathrm{cd}(P) = \{1, p^a\}$. Then, if $\mathrm{cd}(PN/N) = \mathrm{cd}(P)$, we are done by induction, since $\mathrm{Irr}(B_0(G/N)) \subseteq \mathrm{Irr}(B_0(G))$. Hence $\mathrm{cd}(PN/N) = \{1\}$ and $PN/N$ is abelian.

The last part of the statement follows from the fact that if $N$ and $M$ are minimal normal subgroups of $G$, then $G/N$ and $G/M$ have abelian Sylow $p$-subgroups and hence $G$ is isomorphic to a subgroup of $G/N \times G/M$ (consider the projection $G \to G/N \times G/M$), which has abelian Sylow $p$-subgroups, a contradiction since $\mathrm{cd}(P) = \{1, p^a\}$ with $a \geq 1$, by hypothesis.

*Step* 1. *We may assume that* $G$ *is not solvable.*

Suppose that $G$ is solvable and let $F = \mathbf{F}(G)$. By Step 0, $F = \mathbf{O}_p(G)$. Suppose that $F$ is not elementary abelian, then the Frattini subgroup $\Phi(G)$ is not trivial and by Step 0 we have $G/\Phi(G)$ has abelian Sylow $p$-subgroups. By Gaschütz's theorem (see [17, III.4.2 and III.4.5]) $G/F$ acts faithfully and completely reducibly on $F/\Phi(G)$. It follows that $F$ is a Sylow $p$-subgroup of $G$ and by Step 0 $G = F$ is a $p$-group and the claim follows trivially. Hence we may assume that $F$ is elementary abelian. Using again Gaschütz's theorem, we have that $G = FH$ is a semidirect product with $H$ acting faithfully and completely reducibly on $F$. Arguing as before we see that $H/\mathbf{O}_{p'}(H)$ is a $p$-group. By Schur–Zassenhaus we have that $H = RQ$, where $R = \mathbf{O}_{p'}(H)$ and $Q \in \mathrm{Syl}_p(H)$ is abelian.

By Theorem 4.3, there exists a regular $Q$-orbit on $\mathrm{Irr}(F)$. Now, by Theorem 4.4, there exists a $Q$-orbit of size $p$ on $\mathrm{Irr}(F)$. Let $P = FQ \in \mathrm{Syl}_p(G)$. Then $P$ has two character degrees and using Lemma 4.5, we conclude that $|Q| = p$. Since $P$ is not abelian, Itô's theorem [19, Cor. 12.34] implies that there exists $\chi \in \mathrm{Irr}(G)$ of degree divisible by $p$. Since $F$ is a normal abelian subgroup and $|G : F|_p = p$, it follows that $\chi(1)_p = p$, and we are done.

*Step* 2. *Let* $N$ *be the unique minimal normal subgroup of* $G$. *If* $N = S_1 \times \cdots \times S_t$ *for non-abelian simple groups* $S_i$, *then we are done.*

Since $N$ is the unique minimal normal subgroup of $G$ and it is non-abelian, we have that $\mathbf{C}_G(N) = 1$, so $G \leq \mathrm{Aut}(N)$. If $t = 1$, we are done by Corollary 3.12, hence we may assume that $t > 1$. Write $H = \bigcap_{i=1}^t \mathbf{N}_G(S_i)$. We claim that $|G : H|_p \leq p^a$.

Write $S = S_1$. Let $Q \in \mathrm{Syl}_p(S)$ so that $R = Q \times \cdots \times Q \in \mathrm{Syl}_p(N)$. Note that $Q > 1$ by Step 0. Let $P \in \mathrm{Syl}_p(G)$ containing $R$, so $P \cap N = R$ and let $T = P \cap H \in \mathrm{Syl}_p(H)$. Note that $\bar{G} = G/H$ transitively permutes the set $\Omega = \{S_1, S_2, \ldots, S_t\}$, so $P/T$ is a permutation group on $\Omega$. Assume first that $p$ is odd. By the first part of Lemma 2.5, there exists $\Delta$ a non-empty proper subset of $\Omega$ in a regular $P/T$-orbit. Let $\gamma, \delta \in \mathrm{Irr}(Q)$ be two different characters and let $\mu \in \mathrm{Irr}(R)$ be the character that is a product of copies of $\gamma$ in the positions corresponding to $\Delta$ and copies of $\delta$ elsewhere. It follows that $P_\mu \leq \mathrm{Stab}_P(\Delta) = T$. Thus if $\varphi \in \mathrm{Irr}(P)$ lies over $\mu$,

$$p^a \geq \varphi(1) \geq |P : T| = |G : H|_p,$$

as wanted.

Now, assume that $p = 2$. By the second part of Lemma 2.5, $P/T$ has a regular orbit on $\mathfrak{P}_3(\Omega)$. In other words, there exists a partition $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ of $\Omega$ (with at least two non-empty parts) such that no non-identity element in $P/T$ fixes $\Omega_i$ for every $i$. Since a 2-group of order greater than 2 has at least four conjugacy classes, we have that a Sylow 2-subgroup of $S$ has at least 4 conjugacy classes. Hence, we may choose four different characters $\delta_1, \delta_2, \delta_3, \delta_4 \in \mathrm{Irr}(Q)$. Let $\delta$ be the character of $R$ that is a product of copies of $\delta_i$ in the positions corresponding to $\Omega_i$ for $i = 1, 2, 3, 4$ (if $\Omega_i$ is empty then the character $\delta_i$ does not appear). As before, we see that $P_\delta \subseteq T$. Thus if $\varphi \in \mathrm{Irr}(P)$ lies over $\delta$,

$$2^a \geq \varphi(1) \geq |P : T| = |G : H|_2,$$

as wanted. Thus the claim is proven.

Next we claim that $\mathbf{C}_G(R) \subseteq H$. Let $g \in G \setminus H$. Since $G/H$ is a permutation group on $\{S_1, \ldots, S_t\}$, without loss of generality we may assume that $g$ moves $S_1$. Therefore, for any $x \in Q \setminus \{1\}$, $g$ does not fix $(x, 1, \ldots, 1) \in R$. It follows that $\mathbf{C}_G(R) \subseteq H$, as wanted.

By Lemma 3.3, there exists $1_S \neq \alpha \in \mathrm{Irr}_{p'}(B_0(S))$, $X$-invariant for a Sylow $p$-subgroup $X$ of $\mathrm{Aut}(S)$. Recall that $\bar{G}$ transitively permutes $\{S_1, \ldots, S_t\}$. Suppose that the action of $\bar{G}$ on $\Omega$ is not $p$-concealed. In this case, there exists $\Gamma \subseteq \Omega$ such that $p$ divides $|\bar{G} : \mathrm{Stab}_{\bar{G}}(\Gamma)|$. Consider now the character $\theta \in \mathrm{Irr}_{p'}(B_0(N))$ that is a product of copies of $\alpha$ in the positions corresponding to the copies of $S$ in $\Gamma$, and copies of $1_S$ in the positions corresponding to the copies of $S$ in $\Omega - \Gamma$. Then $\theta$ is invariant in a Sylow $p$-subgroup of $H$, so $|H : H_\theta|$ is prime to $p$. Then

$$|G : G_\theta|_p \leq |G : H|_p \leq p^a.$$

On the other hand, $G_\theta H/H \subseteq \mathrm{Stab}_{\bar{G}}(\Gamma)$, so $p$ divides $|G : G_\theta|$. It suffices to show that there exists $\chi \in \mathrm{Irr}(B_0(G))$ with $\chi(1)_p = |G : G_\theta|_p$. Now, $\theta$ is of $p'$-degree and $N$ is perfect, so if $YN/N \in \mathrm{Syl}_p(G_\theta/N)$, $\theta$ extends to $YN$ by [19, Cor. 6.28]. Then, by [31, Lemma 4.3], there is $\psi \in \mathrm{Irr}_{p'}(B_0(G_\theta))$ lying over $\theta$. Now $\chi = \psi^G \in \mathrm{Irr}(B_0(G))$ (by the Clifford correspondence and [32, Cor. 6.2 and Thm 6.7]) and $\chi(1)_p = |G : G_\theta|_p$, as wanted.

Hence we may assume that the action of $\bar{G}$ on $\Omega$ is $p$-concealed. Let $\Delta_1, \ldots, \Delta_r$ be a partition of $\Omega$ in blocks of maximal order (in the sense of Lemma 2.4). By Lemma 2.4

there exists a proper normal subgroup $L$ of $G$ such that $H \leq L \lhd G$ and the action of $G/L$ on $\{\Delta_1, \dots, \Delta_r\}$ is primitive and $p$-concealed. We claim that

$$\mathrm{Irr}(G/L) \subseteq \mathrm{Irr}(B_0(G)).$$

Recall that $R = Q \times \cdots \times Q$ is a Sylow $p$-subgroup of $N$, where $Q$ is a Sylow $p$-subgroup of $S$. Let $U \in \mathrm{Syl}_p(L)$ containing $R$, then

$$\mathbf{C}_G(U) \subseteq \mathbf{C}_G(R) \subseteq H \subseteq L.$$

The claim follows from Lemma 4.2.

Now, by Lemma 3.4, there exists $\chi \in \mathrm{Irr}(G/L)$ with $\chi(1)_p = p$. Since this character belongs to the principal $p$-block of $G$, the result follows.

*Step 3. Let $N$ be the unique minimal normal subgroup of $G$. If $N \subseteq \mathbf{Z}(G)$, then we are done.*

By Step 0 and Step 2, $N$ is an elementary abelian $p$-group. Let $H = P\mathbf{C}_G(P)$, where $P \in \mathrm{Syl}_p(G)$. If $H = G$, $P$ is normal in $G$, a contradiction to Step 0. Hence we may assume $H < G$, and by induction there is $\psi \in \mathrm{Irr}(B_0(H))$ with $1 \neq \psi(1)_p \leq p^a$. Now, by [32, Thm 4.14] we have that $B_0(H)^G$ is defined and by the Third Main Theorem [32, Thm 6.2], $B_0(H)^G = B_0(G)$. Now,

$$\psi^G = \sum_{\chi \in \mathrm{Irr}(B_0(G))} a_\chi \chi + \sum_{\chi \in \mathrm{Irr}(B_1(G))} a_\chi \chi + \cdots + \sum_{\chi \in \mathrm{Irr}(B_k(G))} a_\chi \chi$$

where $B_i(G)$ are $p$-blocks of $G$. By [32, Cor. 6.4] we have

$$\left( \sum_{\chi \in \mathrm{Irr}(B_0(G))} a_\chi \chi(1) \right)_p = \psi^G(1)_p \leq p^a.$$

Therefore there exists an irreducible constituent $\chi \in \mathrm{Irr}(B_0(G))$ of $\psi^G$ with $\chi(1)_p \leq p^a$. We need to show that $\chi(1)_p \neq 1$. Let $\lambda \in \mathrm{Irr}(N)$ be an irreducible constituent of $\psi_N$, so $\lambda$ lies under $\chi$. Notice that $\lambda$ is $G$-invariant, since $N \subseteq \mathbf{Z}(G)$. Now, if $1 = \chi(1)_p$, then $\chi_P$ has a linear constituent, which is an extension of $\lambda$. Since $P/N$ is abelian by Step 0, all characters in $\mathrm{Irr}(P|\lambda)$ are extensions of $\lambda$ by Gallagher's theorem. Let $\tau \in \mathrm{Irr}(P|\lambda)$ under $\psi$, then $\psi(1)_p = \tau(1) = 1$, a contradiction. Hence $1 \neq \chi(1)_p$ and the step is proved.

*Step 4. We may assume that $\mathbf{F}(G) = \mathbf{F}^*(G)$.*

Let $N$ be the unique minimal normal subgroup of $G$. By Step 0 and Step 2, $N$ is an elementary abelian $p$-group.

By Step 0, $F = \mathbf{F}(G) = \mathbf{O}_p(G) > 1$. Suppose that $E = \mathbf{E}(G) > 1$ and let $Z = \mathbf{Z}(E)$. Since $N$ is the unique minimal normal subgroup of $G$, $N \subseteq Z$ (notice that $Z > 1$ since otherwise $\mathbf{F}^*(G) = \mathbf{F}(G) \times E$ in contradiction to Step 0). We claim that $E/Z = S_1/Z \times \cdots \times S_n/Z$, where $S_i \trianglelefteq G$ for every $i$. Let $W/Z$ be a (non-abelian) minimal normal subgroup of $G/Z$ contained in $E/Z$. By the Schur–Zassenhaus theorem and Step 0, we know that $|W/Z|$ is divisible by $p$. Now, by Theorem 4.1 applied to $G/Z$, we have that $W/Z$ is simple and the claim follows. Write $S = S_1$, so that $S'$ is a quasi-simple normal subgroup of $G$. Using again that $N$ is the unique minimal normal subgroup of $G$, we have that $N \subseteq \mathbf{Z}(S) \cap S' \subseteq \mathbf{Z}(S')$. Looking at the Schur multipliers of the simple groups [13], if $p \geq 5$, we deduce that $\mathbf{Z}(S')$ has cyclic Sylow $p$-subgroups. We claim that $\mathbf{Z}(S')$ has cyclic Sylow $p$-subgroups for $p = 2, 3$ as well.

Suppose first that $p = 2$. Recall that $G/N$ has abelian Sylow $p$-subgroups, whence the simple group $S'/N$ has abelian Sylow $p$-subgroups. The simple groups with abelian Sylow 2-subgroups were classified by J. H. Walter [41]. They are $\mathrm{PSL}_2(2^n)$ for $n > 1$, $\mathrm{PSL}_2(q)$ where $q \equiv 3$ or $5 \pmod 8$ and $q > 3$, the Ree groups ${}^2G_2(3^{2n+1})$ and the Janko group $J_1$. As before, we can check in [40] that these groups have cyclic multiplier. Therefore $\mathbf{Z}(S')$ has cyclic Sylow 2-subgroups. It remains to consider the case $p = 3$. It can be checked in [40] that the unique simple group $S$ whose Schur multiplier has a non-cyclic Sylow 3-subgroup is $\mathrm{PSU}_4(3)$, but this group does not have abelian Sylow 3-subgroups.

In all cases, we conclude that $N$ is cyclic and hence $|N| = p$. Now, the order of $G/\mathbf{C}_G(N)$ divides $p-1$. Using Step 0 we conclude that $N$ is central in $G$ and we are done by Step 3. Therefore, we may assume that $E = 1$, so that $\mathbf{F}(G) = \mathbf{F}^*(G)$, as desired.

*Step* 5. *There is just one $p$-block in $G$.*

By Step 0 and Step 4, we have that $G$ is $p$-constrained, so there is just one $p$-block (see [6, Cor. V.3.11], for instance).

Thus, from now on, we want to find $\chi \in \mathrm{Irr}(G)$ of degree divisible by $p$ such that $\chi(1)_p \leq p^a$. Let $N$ be the unique minimal normal subgroup of $G$, which is an elementary abelian $p$-group. Recall that by Step 1 we have that $G$ is not solvable.

*Step* 6. *Let $\lambda \in \mathrm{Irr}(N)$. Then there exists $Q \in \mathrm{Syl}_p(G)$ such that $\lambda$ extends to $Q$. In particular $|G : G_\lambda|$ is not divisible by $p$.*

Let $P \in \mathrm{Syl}_p(G)$ and let $\psi \in \mathrm{Irr}(P|\lambda)$. If $\psi(1) = 1$, then $\psi_N = \lambda$, as wanted. Hence we may assume that $\psi(1) = p^a$. Since $\psi^G(1)_p = \psi(1) = p^a$, there exists an irreducible constituent $\chi \in \mathrm{Irr}(G)$ of $\psi^G$ with $\chi(1)_p \leq p^a$. If $1 < \chi(1)_p$, we are done. Hence we may assume that $\chi(1)_p = 1$. In this case, there exists $\xi \in \mathrm{Irr}(P)$ with $[\chi_P, \xi] \neq 0$ and $\xi(1) = 1$. Let $\mu \in \mathrm{Irr}(N)$ under $\xi$, so $\xi_N = \mu$ and $\mu$ lies under $\chi$. Hence $\mu^g = \lambda$ for some $g \in G$. Since $\mu$ extends to $P$, $\mu^g = \lambda$ extends to $P^g$, as wanted.

*Step* 7. *Let $N$ be the unique minimal normal subgroup of $G$ (which is an elementary abelian $p$-group). Suppose there exists $V \lhd G$ with $V/N \cong \mathrm{SL}_2(p^n)$ or $V/N \cong \mathrm{PSL}_2(p^n)$ with $p^n > 3$. Then $p^a \geq |V/N|_p$.*

Notice that since $V/N$ is perfect and $N$ is the unique minimal normal subgroup of $G$, we have that $V$ is perfect. We claim that if $1_N \neq \lambda \in \mathrm{Irr}(N)$ then $V_\lambda < V$. Indeed, let $\lambda \in \mathrm{Irr}(N)$ be $V$-invariant. By Step 6, $\lambda$ extends to some $R \in \mathrm{Syl}_p(V)$. Since $N$ is a $p$-group, $\lambda$ also extends to every $q$-Sylow subgroup of $V$ for $q \neq p$ by [19, Cor. 6.28]. Then $\lambda$ extends to $V$ by [19, Cor. 11.31]. But this is not possible since $V$ is perfect. The claim follows.

Let $1_N \neq \lambda \in \mathrm{Irr}(N)$, so $\lambda$ extends to some Sylow $p$-subgroup of $G$ by Step 6. Then $\lambda$ extends to some $R \in \mathrm{Syl}_p(V)$, in particular $R \subseteq V_\lambda$. Since $V_\lambda < V$, by Lemma 3.1, we have $V_\lambda \leq \mathbf{N}_V(R)$. Let $Q \in \mathrm{Syl}_p(V)$ with $Q \neq R$. Again by Lemma 3.1 we have that $Q \cap R = N$. Now,

$$Q_\lambda \subseteq V_\lambda \leq \mathbf{N}_V(R),$$

whence $Q_\lambda$ is a subgroup of a Sylow $p$-subgroup of $\mathbf{N}_V(R)$. It follows that $Q_\lambda \subseteq R$, so $Q_\lambda = N$. Therefore $\lambda^Q \in \mathrm{Irr}(Q)$, so

$$|Q : N| = \lambda^Q(1) \in \mathrm{cd}(Q).$$

Since $\lambda^Q$ lies under some irreducible character of some Sylow $p$-subgroup of $G$, we have that $|Q : N| \leq p^a$.

*Step* 8. *We may assume that* $\mathbf{O}_{p'}(G/N) > 1$.

Recall that by Step 0, $G/N$ has abelian Sylow $p$-subgroups. Suppose $\mathbf{O}_{p'}(G/N) = 1$. Then, applying Theorem 4.1 to $G/N$, we deduce that there exist $X, Y \trianglelefteq G$ containing $N$ such that $X/N$ is an abelian $p$-group, $Y/N$ is a direct product of non-abelian simple groups $V_i/N$ of order divisible by $p$ and $G/N = X/N \times Y/N$. Furthermore $V_i/N$ is normal in $G/N$. Notice that $X$ is a $p$-group, so $1 < \mathbf{Z}(X) \triangleleft G$. Therefore, $N \subseteq \mathbf{Z}(X)$. Then $X \subseteq \mathbf{C}_G(N)$. Moreover, since $V_i/N$ is simple, $\mathbf{C}_{V_i}(N) = V_i$ or $\mathbf{C}_{V_i}(N) = N$. If the first happens for some $i$, then $N \subseteq \mathbf{Z}(V_i)$ and hence, as in Step 4, it is cyclic. Then $|N| = p$, and we conclude arguing as in Step 4. Therefore $\mathbf{C}_G(N) = X$.

Using Step 6, we now apply Theorem 2.1 and Theorem 2.3 to the action of $G/X \cong Y/N$ on $\mathrm{Irr}(N)$. Suppose first that this action is primitive, so that Theorem 2.1 applies. Since $G/X$ is not solvable, we are not in case (2). Suppose now that we are in case (3). In subcase (i), $p = 2$. Since $G/X$ has abelian Sylow $p$-subgroups, $c = 5$ and $c = 2^r - 1$ or $2^r - 2$ for some $r$. This is a contradiction. Since $G/X \cong Y/N$ is a direct product of simple groups, subcase (ii) does not occur. In subcases (iii), (iv) and (v), we may assume that $G/X = \mathrm{PSL}_2(11), M_{11}$ or $M_{23}$ and $p = 3, 3$ or 2, respectively. In subcase (iii), there exists $\chi \in \mathrm{Irr}(G/X)$ such that $\chi(1) = 12$, so we are done by Step 5. Since $M_{23}$ does not have abelian Sylow 2-subgroups, subcase (v) does not occur. Suppose that $G/X \cong Y/N = M_{11}$ and $p = 3$. Furthermore, $|N| = 3^5$, so $Y$ is a perfect group of order $|M_{11}| \cdot 3^5 = 2^4 \cdot 3^7 \cdot 5 \cdot 11$. The group $M_{11}$ has an irreducible character of degree 45, so its 3-part is 9. It suffices to see that the largest character degree of a Sylow 3-subgroup of $Y$ is at least 9. We can check in [40] that there are two perfect groups of order $|Y|$. In both of them, the largest character degree of a Sylow 3-subgroup is 9. The result follows in this case too.

Finally, we may assume that we are in case (1). As mentioned before, these groups were classified by Hering. We will use the description in [23, App. 1]. We remark that, as was observed in the proof of [33, Lemma 6.4], for instance, these groups have at most one non-abelian composition factor. Since $Y/N$ is a direct product of non-abelian simple groups, it suffices to study the simple groups with abelian Sylow $p$-subgroups that appear in Hering's theorem. First we consider case (A) of that theorem. Since $Y/N$ is a direct product of non-abelian simple groups of order divisible by $p$, the only possibility is $p = 2$ and $Y/N = \mathrm{SL}_2(2^n)$, where $|N| = 2^{2n}$ and $n \geq 2$. By Step 7 we have that $1 < |Y/N|_p \leq p^a$. Moreover if $\chi$ is the Steinberg character of $G/X \cong Y/N$ then

$$\chi(1) = |Y/N|_p \in \mathrm{cd}(G/X)$$

and we are done.

We do not need to consider the extra-special classes described in (B) because they are not simple. Finally, we consider the exceptional classes described in (C) (see [23, Tab. 11]). If $Y/N \cong \mathfrak{A}_6$ or $Y/N \cong \mathfrak{A}_7$ and $p = 2$, $Y/N$ does not have abelian Sylow $p$-subgroups. The remaining groups are not simple

Now, we may assume that the action of $L = Y/N$ on $N$ is imprimitive. We apply Theorem 2.3 (recall that $L = \mathbf{O}^{p'}(L)$ by Step 0). Let $N = N_1 \oplus \cdots \oplus N_n$ be an imprimitivity decomposition for $N$. Therefore, $\mathbf{N}_L(N_i)$ is transitive on $N_i \setminus \{0\}$ and $M := L/\bigcap \mathbf{N}_L(N_i)$ induces a primitive $p$-concealed subgroup of $\mathfrak{S}_n$. Note that $M$ is a factor group of $G$. By Lemma 3.4, this group has an irreducible character $\chi$ such that $\chi(1)_p = p$ and the result follows in this case.

*Step* 9. *If $N$ is the unique minimal normal subgroup, we have that $N = \mathbf{F}(G)$. In particular, $\mathbf{C}_G(N) = N$.*

Let $K/N = \mathbf{O}_{p'}(G/N)$. By Step 8, we may assume that $K > N$. By the Schur–Zassenhaus theorem, there exists a $p$-complement $H$ of $K$, so $K = HN$ and $H \cap N = 1$. Then by the Frattini argument we have $G = N\mathbf{N}_G(H)$. Write $L = \mathbf{N}_G(H)$. Now, since $N$ is abelian and normal in $G$, $\mathbf{N}_N(H)$ is normal in $G = \mathbf{N}_G(H) = NL$, and so $\mathbf{N}_N(H) = 1$ or $\mathbf{N}_N(H) = N$. If $\mathbf{N}_N(H) = N$, then $H \lhd G$, and we get a contradiction since $\mathbf{O}_{p'}(G) = 1$ by Step 0. Thus, $L \cap N = \mathbf{N}_N(H) = 1$ and $L$ is a complement of $N$ in $G$.

Let $F = \mathbf{F}(G) = \mathbf{O}_p(G)$. We claim that $F = N$. Notice that $N \subseteq \mathbf{Z}(F)$ since $\mathbf{Z}(F) > 1$. Let $F_1 = F \cap L$. Then $F_1 \lhd L$ and since $G = NL$, we have that $F_1 \lhd G$. Since $N$ is the unique minimal normal subgroup, this forces $F_1 = 1$, so $F = N$ as claimed. Since $N = \mathbf{F}(G) = \mathbf{F}^*(G)$ by Step 4, we have $\mathbf{C}_G(N) \subseteq N$, as wanted.

*Step* 10. *If $G$ is $p$-solvable, then we are done.*

We already know that $G$ is not solvable by Step 1, so $p > 2$. Let $P \in \mathrm{Syl}_p(G)$. By Step 9, $G/N$ acts faithfully on $N$. Hence, $G/N$ acts faithfully on $\mathrm{Irr}(N)$ and by [30, Thm 3.1] we conclude that there exists $\lambda \in \mathrm{Irr}(N)$ such that $P_\lambda = N$. Therefore, by Clifford's correspondence, $\lambda^P \in \mathrm{Irr}(P)$ and $p^a = |P : N|$ since $P \neq N$. On the other hand, since $N$ is an abelian normal subgroup of $G$, it follows from Itô's theorem ([19, Thm 6.15], for instance) that for any $\chi \in \mathrm{Irr}(G)$, $\chi(1)$ divides $|G : N|$. In particular,

$$\chi(1)_p \leq |G : N|_p = |P : N|.$$

Hence, it suffices to see that there exists $\chi \in \mathrm{Irr}(G)$ of degree divisible by $p$. Since $G$ does not have a normal abelian Sylow $p$-subgroup, this follows from [19, Thm 12.33].

*Step* 11. *Completion of the proof.*

Now, we assume that $G$ is not $p$-solvable. Let $K/N = \mathbf{O}_{p'}(G/N)$. By Step 8, we may assume that $K > N$. Again, applying Theorem 4.1 to $G/N$, we deduce that there exist $X, Y \trianglelefteq G$ containing $N$ such that $X/K$ is an abelian $p$-group, $Y/K$ is a direct product of non-abelian simple groups $V_i/N \trianglelefteq G/N$ of order divisible by $p$ and $G/K = X/K \times Y/K$.

As in Step 9, let $H$ be a $p$-complement of $K$ and let $L = \mathbf{N}_G(H)$. Then $G = NL$ and $N \cap L = 1$. By Step 9, we have that $L$ acts faithfully and irreducibly on $\mathrm{Irr}(N)$. By Step 6, this action is $p$-exceptional.

Suppose first that the action of $L$ on $\mathrm{Irr}(N)$ is primitive. We apply Theorem 2.1. In case (2), $G$ is solvable so this case does not occur. Assume that we are in case (3). In subcases (i) and (v) we can argue as in Step 8. In subcases (ii), (iii) and (iv) $G/N$ has a normal subgroup $V/N = \mathrm{SL}_2(5), \mathrm{PSL}_2(11)$ or $M_{11}$ respectively and in all cases $p = 3$. Note that one of the simple direct factors of $Y/K$ is then $\mathfrak{A}_5, \mathrm{PSL}_2(11)$ or $M_{11}$, respectively. Since the first two groups have irreducible characters whose degree has 3-part 3, the result follows in these cases. Suppose now that $V/N = M_{11}$. In this case, note that $V$ is perfect and $|N| = 3^5$, so we can complete the proof in this case as in Step 8.

Now, we may assume that we are in case (1). So $L$ is transitive on $\mathrm{Irr}(N) \setminus \{1_N\}$ and $L$ is one of the groups from Hering's theorem. Again, we use the description in [23, App. 1]. We start with the infinite classes described in (A). Since $G/N$ is not solvable and has abelian Sylow $p$-subgroups, it follows that $G/N$ has a normal subgroup $V/N \cong \mathrm{SL}_2(p^n)$, where $|N| = p^{2n} > 3$. By Step 7, we have that $|V/N|_p \leq p^a$. Now, $\mathrm{PSL}_2(p^n)$ is a composition factor of $G/K$. Write $M/K$ for this composition factor. If $\chi$ is the Steinberg character of $M/K$ then

$$\chi(1) = |M/K|_p \in \mathrm{cd}(M/K) \subseteq \mathrm{cd}(G/K) \subseteq \mathrm{cd}(G)$$

and we are done since $|M/K|_p = |V/N|_p$.

Next, we consider the extra-special classes described in (B). The first four groups in [23, Tab. 10] are solvable (in fact they all have a normal subgroup $R/N \cong \mathrm{Q}_8$ such that $G/R$ is isomorphic to a subgroup of $\mathrm{SL}_2(3)$, see [18, Rem. XII.7.5]), so they do not occur. The last case of this table (where $|N| = 3^4$) can be handled with [40].

Finally, we consider the exceptional classes described in (C). In the cases where $p^d \in \{11^2, 19^2, 29^2, 59^2\}$ in [23, Tab. 11], we have that $G/N$ (and hence, $G$) is $p$-solvable, so we are done. Since $G/N$ has abelian Sylow $p$-subgroups, we are left with the first and the last cases of Table 11. In the last case, $p = 3$ and $G/N \cong \mathrm{SL}_2(13)$. Here, there is $\chi \in \mathrm{Irr}(G/N)$ of degree 6, and we are done. The first case can be handled with [40].

Now, assume that the action of $L$ on $N$ is imprimitive. Arguing as in the last paragraph of Step 8, we find $\chi \in \mathrm{Irr}(L)$ such that $\chi(1)_p = p$. This completes the proof. $\square$

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FB Mathematik, TU Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany.
*Email address*: malle@mathematik.uni-kl.de

Departament de Matemàtiques, Universitat de València, 46100 Burjassot, València, Spain
*Email address*: alexander.moreto@uv.es
*Email address*: Noelia.Rizo@uv.es