A BRAUER–GALOIS HEIGHT ZERO CONJECTURE

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Abstract. Recently, Malle and Navarro provided a Galois-theoretic enhancement of Brauer's height zero conjecture for principal p-blocks, limited to the case $p = 2$, utilizing a specific Galois automorphism of order 2. While they left open the question of whether a similar result could hold for odd primes, in this paper, we significantly advance their work by formulating a broader Galois version of the conjecture for any prime p, using an elementary abelian p-subgroup of the absolute Galois group. We not only strengthen their result for $p = 2$, but also prove the conjecture for arbitrary primes p, except when G contains certain small-rank Lie-type groups as composition factors. Moreover, we establish the conjecture for almost simple groups and for p-solvable groups.

1. Introduction

Some of the fundamental local/global counting conjectures in representation theory of finite groups were strengthened by Navarro [35] using the action of Frobenius elements of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. This strengthening has had a large impact on research in the last two decades. It was asked in [29] whether there is also a strengthening of Brauer's height zero conjecture (which was very recently solved [30]) to include Galois automorphisms. Brauer's conjecture, posed in 1955, asserts that if G is a finite group, and B is a Brauer p-block of G with defect group D, then D is abelian if and only if all the complex irreducible characters in B have height zero. (Recall that $\chi \in \mathrm{Irr}(B)$ has height zero if $\chi(1)_p = |G|_p/|D|$ is minimal possible.) The main result of [29] provides such a strengthening for the principal 2-block. It was proved there that if σ denotes the Galois automorphism that fixes 2-power order roots of unity and complex-conjugates odd-order roots of unity, then all σ-invariant irreducible characters in the principal 2-block $B_0(G)$ of a finite group G have $2'$-degree if and only if G has an abelian Sylow 2-subgroup. As pointed out in [29], this result does not extend to arbitrary 2-blocks.

For p a fixed prime number, let $\mathcal{J}$ be the subgroup of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ consisting of the automorphisms of order p that fix all p-power order roots of unity. We write $\mathrm{Irr}_{\mathcal{J}}(G)$

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to denote the set of $\mathcal{J}$-invariant complex irreducible characters of a finite group G and $\mathrm{Irr}_{\mathcal{J}}(B_0(G))$ for the $\mathcal{J}$-invariant irreducible characters in the principal p-block $B_0(G)$.

We propose the following conjecture.

Conjecture 1. *Let G be a finite group, let p be a prime number and let $P \in \mathrm{Syl}_p(G)$. Then all characters in $\mathrm{Irr}_{\mathcal{J}}(B_0(G))$ have height zero if and only if P is abelian.*

This provides the desired Galois strengthening of the height zero conjecture for principal p-blocks. We prove this conjecture when $p = 2$ and also when p is odd and certain simple groups are not composition factors of G. Since $\sigma \in \mathcal{J}$, this result also strengthens the main result of [29] when $p = 2$. We remark that the original height zero conjecture had remained open even for principal p-blocks until recently [28].

As pointed out in [29], the "only if" implication of Conjecture 1 fails for non-principal blocks when $p = 2$. It is also easy to find examples in GAP [43] showing that this fails for non-principal p-blocks also when p is odd; for instance `SmallGroup`(1701, 119). It is worth remarking, however, that we have not found counterexamples among blocks of maximal defect that contain some $\mathcal{J}$-invariant character. If true, proving this for p-solvable groups would most likely require a version for $\mathcal{J}$-invariant characters of the very deep Gluck-Wolf theorem [12]. The proof of the general case would probably require a $\mathcal{J}$-version of the fundamental Navarro–Tiep theorem [37]. This theorem asserts that if G is a finite group, $N \trianglelefteq G$, $\theta \in \mathrm{Irr}(N)$ and a prime p does not divide $\chi(1)/\theta(1)$ for every $\chi \in \mathrm{Irr}(G)$ lying over θ then G/N has abelian Sylow p-subgroups. We plan to address this daunting task elsewhere.

On the other hand, the conclusion of Conjecture 1 is definitely false for $p = 2$ if we define $\mathcal{J}$ to be the set of elements of order p of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. For instance, the non-linear characters of the non-abelian 2-group `SmallGroup`(16, 6) in [43] are not fixed by complex conjugation. As we will see in Remark 3.7, there are also counterexamples for odd primes. Conjecture 1 is also false if we replace $\mathcal{J}$ by the set of elements of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ that fix p-power roots of unity and have order a power of p. `SmallGroup`(160, 234) is a counterexample. Therefore it does not seem possible to replace $\mathcal{J}$ by a natural larger subgroup.

The following are our main results.

Theorem 2. *Conjecture 1 holds for $p = 2$.*

Theorem 3. *Let G be a finite group, let $p > 2$ be a prime number and let $P \in \mathrm{Syl}_p(G)$. Suppose that G does not have composition factors isomorphic to S with (S, p) a pair as listed in Proposition 2.5. Then Conjecture 1 holds for G.*

Our proof of Theorem 2 and Theorem 3 relies on a partial strengthening of another celebrated theorem in character theory of finite groups: the Itô–Michler theorem (see e.g. [36, Thm 7.1]).

Theorem 4. *Let G be a finite group and let p be a prime. Suppose that G does not have composition factors isomorphic to S with (S, p) a pair as listed in Proposition 2.5. Then all characters in $\mathrm{Irr}_{\mathcal{J}}(G)$ have p'-degree if and only if G has a normal abelian Sylow p-subgroup. In particular, if $p = 2$ or G is p-solvable then all characters in $\mathrm{Irr}_{\mathcal{J}}(G)$ have p'-degree if and only if G has a normal abelian Sylow p-subgroup.*

The conclusion of Theorem 4 is definitely false when $G = S$ for some of the pairs (S, p) listed in Proposition 2.5, see Example 2.6. Notice that when $p = 2$ or G is p-solvable, we have a complete $\mathcal{J}$-version of the Itô–Michler theorem. This could be compared with Theorem A of [14].

Note that the Itô–Michler theorem is a fundamental tool to prove Brauer's height zero conjecture for principal blocks. The failure of the $\mathcal{J}$-version of it suggests that completely new ideas may be necessary to prove Conjecture 1. Our main goal in this paper has been to give evidence for Conjecture 1. Hopefully this conjecture (and the yet much tougher possible version for blocks of maximal defect) will stimulate interesting new ideas in this field.

We remark that we had reduced the proof of Theorem 3 to two sets of questions on simple groups, addressed in Section 2, when the preprint [15] appeared. In that paper, a different extension of the Itô–Michler theorem is proved. Those of our questions relating to Theorem 4 are now handled by a careful analysis and extension of the result of Theorem C from [15], while those relevant for the proof of Theorem 3, involving statements related to principal blocks, have to be solved differently. As one step, in particular, we show Conjecture 1 for almost simple groups. We then prove Theorem 4 and the p-solvable case of Conjecture 1 in Section 3. We conclude in Section 4 with the proof of Theorem 2 and Theorem 3.

2. Almost simple and quasi-simple groups

In this section we collect several results on characters of simple and almost simple groups needed for the proofs of our main theorems.

Let n be an integer. We write $\mathbb{Q}_n := \mathbb{Q}(\zeta)$ where ζ is a primitive nth root of unity. In the proofs we will work with the following subgroup of $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$. Let $\Omega = \Omega(n)$ be the set of elements of order p of $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}_{n_p})$. Note that this definition depends on the integer n and of course on p. If G is a finite group, we often write Ω for $\Omega(|G|)$. We let $\mathrm{Irr}_\Omega(G)$ denote the set of Ω-invariant complex irreducible characters of G and $\mathrm{Irr}_\Omega(B_0(G))$ to denote the Ω-invariant complex irreducible characters in the principal p-block $B_0(G)$. This relates to $\mathrm{Irr}_{\mathcal{J}}(B_0(G))$ thanks to the following lemma.

Lemma 2.1. *Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$. Then χ is $\mathcal{J}$-invariant if and only if χ is $\Omega(|G|)$-invariant.*

Proof. Clearly all elements of $\mathcal{J}$ restrict to elements of $\Omega := \Omega(|G|)$. Conversely, we show that elements of Ω lift to elements of the same order in $\mathcal{J}$. It suffices to do this for $\sigma \in \mathrm{Gal}(\mathbb{Q}_{r^n}/\mathbb{Q})$ of order p, where r runs over primes different from p. The hypotheses imply that $r > 2$. Therefore, for all $m \geq n$, $\mathrm{Gal}(\mathbb{Q}_{r^m}/\mathbb{Q})$ and $\mathrm{Gal}(\mathbb{Q}_{r^n}/\mathbb{Q})$ are cyclic groups with the same p-part, with the second group being a factor group of the first one. In particular, any p-element of $\mathrm{Gal}(\mathbb{Q}_{r^n}/\mathbb{Q})$ lifts to a *unique* element of the same order of $\mathrm{Gal}(\mathbb{Q}_{r^m}/\mathbb{Q})$. Hence σ lifts to an automorphism of order p of $\cup_{m\geq 1}\mathbb{Q}_{r^m}$, which moreover we may choose to act trivially on the linearly disjoint extensions $\mathbb{Q}_k/\mathbb{Q}$ with k prime to r, so to an element of $\mathcal{J}$. □

In particular, we have the following.

Corollary 2.2. *Let H be a group of order n and let $\theta \in \mathrm{Irr}(H)$. Suppose that m is a multiple of n. Then θ is $\Omega(n)$-invariant if and only if it is $\Omega(m)$-invariant.*

We also record the following observation about $\Omega(n)$.

Lemma 2.3. *Assume that r is a prime such that $r \not\equiv 0, 1 \pmod p$. Then $\Omega(n)$ acts trivially on $\mathbb{Q}_r$.*

Proof. Let $\xi \in \mathbb{Q}_r$ be a primitive rth root of unity and let $\sigma \in \Omega(n)$. Then $\xi^\sigma = \xi^k$ for some k relatively prime to r, so that $\xi^{k^p} = \xi^{\sigma^p} = \xi$. But then r divides $k^p - 1$, so either $\xi^k = \xi$ or k has order p modulo r. In the latter case, p divides $r - 1$, contradicting that $r \not\equiv 1 \pmod p$. It follows that σ fixes ξ, so $\Omega(n)$ acts trivially on $\mathbb{Q}_r$. □

2.1. **Extensions of Itô–Michler for almost simple groups.** We consider the following setting: $\mathbf{G}$ is a simple linear algebraic group of simply connected type and $F : \mathbf{G} \to \mathbf{G}$ is a Steinberg endomorphism with (finite) group of fixed points $G = \mathbf{G}^F$. We let $(\mathbf{G}^*, F)$ be in duality with $(\mathbf{G}, F)$ and $G^* := \mathbf{G}^{*F}$ (see e.g. [10, Def. 1.5.17]). For q a prime power we denote by $e_p(q)$ the order of q in $\mathbb{F}_p^\times$.

Recall that a character χ of a finite group is *p-rational* if its character field $\mathbb{Q}(\chi)$ is contained in $\mathbb{Q}_n$ for some n prime to p.

Lemma 2.4. *Let $G = \mathbf{G}^F$, $G^* = \mathbf{G}^{*F}$ be as above with p a prime different from the underlying characteristic of $\mathbf{G}$. Assume there is a prime $r \not\equiv 0, 1 \pmod p$ with:*
(1) *r does not divide $|\mathbf{Z}(G)|$; and*
(2) *there is an r-element $s \in G^*$ which does not centralise a Sylow p-subgroup of G^*.*
Then $S = G/\mathbf{Z}(G)$ has an irreducible p-rational and $\Omega(|S|)$-invariant character of degree divisible by p.

Proof. Since r does not divide $|\mathbf{Z}(G)| = |G^* : [G^*, G^*]|$ (see [31, Prop. 24.21]) we have $s \in [G^*, G^*]$. Further, we have $(\mathbf{C}_{\mathbf{G}^*}(s)/\mathbf{C}_{\mathbf{G}^*}(s)^\circ)^F = 1$ using [31, Prop. 14.20]. But then applying [10, Cor. 2.6.18, Lemma 2.6.20, Rem. 2.6.15], we see there exists a unique semisimple character $\chi \in \mathrm{Irr}(G)$ in the Lusztig series of s (see [10, Def. 2.6.9, 2.6.10]) and this has $\mathbf{Z}(G)$ in its kernel by [37, Lemma 4.4], hence descends to a character of S. By Jordan decomposition [10, Thm 2.6.11] the degree of χ is divisible by the p-part of $|G^* : \mathbf{C}_{G^*}(s)|$, hence by p. From the explicit definition of χ as a linear combination of Deligne–Lusztig characters, the character field $\mathbb{Q}(\chi)$ is contained in $\mathbb{Q}_r$; since $r \neq p$ this shows that χ is p-rational and, as $r \not\equiv 1 \pmod p$, also that χ is $\Omega(|G|)$-invariant using Lemma 2.3. □

Proposition 2.5. *Let S be a non-abelian simple group and $p > 2$ a prime divisor of $|S|$. If all p-rational irreducible characters of S which are $\Omega(|S|)$-invariant have p'-degree, then one of the following holds:*
(1) *$S = {}^2\mathrm{B}_2(q^2)$ with $q^2 = 2^{2f+1}$ and $p|(q^2 + \epsilon\sqrt{2}q + 1)$, $\epsilon \in \{\pm 1\}$, and all prime divisors of $(q^2 - 1)(q^2 - \epsilon\sqrt{2}q + 1)$ are congruent to $1 \pmod p$;*
(2) *$S = \mathrm{PSL}_2(q)$ with $p|(q-\epsilon)$, and all prime divisors of $(q+\epsilon)/\gcd(2, q+\epsilon)$ are congruent to $1 \pmod p$;*
(3) *$S = \mathrm{PSU}_3(q)$ with $q = 2^{2f+1}$ and $p = 3$, and all prime divisors of $(q-1)(q^2-q+1)/3$ are congruent to $1 \pmod 3$;*

(4) *$S=\mathrm{PSL}_4(q)$ with $q=2^{2f+1}$ and $p=3$, and all prime divisors of $(q^3-1)(q^2+1)$ are congruent to $1 \pmod 3$;*
(5) *$S=\mathrm{PSU}_4(q)$ with $p|(q-1)$, and all prime divisors $r>2$ of $(q^3+1)(q^2+1)$ are congruent to $1 \pmod p$.*

Proof. For the sporadic simple groups, the claim is easily checked from the Atlas [4] or using [43]. Now assume $S=\mathfrak{A}_n$ with $n\geq 5$. If $p\geq 5$ there exists a p-core of size n [13], that is, an irreducible (and rational) character of $\mathfrak{S}_n$ of p-defect zero. Its restriction to $\mathfrak{A}_n$ has at most two constituents which hence must be Ω-invariant and still of p-defect zero, so p-rational. If $p=3$ then we take for χ the character of $\mathfrak{S}_n$ labelled by the partition $(n-2,2),(n-2,1^2)$ of degree $n(n-3)/2$, $(n-1)(n-2)/2$ respectively, at least one of which is divisible by 3. Again, its restriction to $\mathfrak{A}_n$ has at most two constituents and thus is $\Omega(|S|)$-invariant.

Thus, finally S is of Lie type and $p>2$. If (S,p) is an exception to the claim, then in particular, all rational irreducible characters of S must have p'-degree. Thus (S,p) appears in the conclusion of [15, Thm C]. We discuss these cases, numbered (1)–(7), in turn, striving to exhibit a suitable prime r such that Lemma 2.4 applies. For this, except for case (1), we view S as $G/\mathbf{Z}(G)$ for $G=\mathbf{G}^F$ for a simple simply connected group $\mathbf{G}$ with a Frobenius map $F:\mathbf{G}\to\mathbf{G}$. Note that in all cases p is not the defining characteristic of S, as otherwise the Steinberg character is as desired (see the proof of [15, Thm C]).

(1) Here $S={}^2\mathrm{B}_2(q^2)$ with $q^2=2^{2f+1}$ and $p|(q^2+\epsilon\sqrt{2}q+1)$. If $r\neq p$ is a prime divisor of $(q^2-1)(q^2-\epsilon\sqrt{2}q+1)$ with $r\not\equiv 1 \pmod p$ then r is as required in Lemma 2.4.
(2) Here $S=\mathrm{PSL}_n(q)$ and $n<p|(q-1)$. We may assume $n>2$ since the case $n=2$ is treated in (4) below. Then we have
$$(q^{n-1}-1)/(q-1)\equiv n-1 \pmod{q-1}\equiv n-1 \pmod p\not\equiv 0,1 \pmod p,$$
hence $(q^{n-1}-1)/(q-1)$ has a prime divisor $r\not\equiv 0,1 \pmod p$. Since r divides $(q^{n-1}-1)/(q-1)$ we have r is prime to $|\mathbf{Z}(G)|=\gcd(n,q-1)$, with $G=\mathrm{SL}_n(q)$. Let s be a (semisimple) element of $G^*=\mathrm{PGL}_n(q)$ generating the Sylow r-subgroup of a (cyclic) maximal torus of order $q^{n-1}-1$. As s is not contained in a maximally split torus of $\mathrm{PGL}_n(q)$ (whose exponent is $q-1$, while by the choice of r, the order of s is larger than $(q-1)_r$), s does not centralise a maximally split torus of $\mathrm{PGL}_n(q)$, hence does not centralise a Sylow p-subgroup of G^* by [31, Thm 25.19]. Thus all assumptions of Lemma 2.4 are satisfied and this case is not an exception.
(3) Here $S=\mathrm{PSU}_n(q)$ and $2<n<p|(q+1)$. Now
$$(q^{n-1}-(-1)^{n-1})/(q+1)\equiv n-1 \pmod{q+1}\equiv n-1 \pmod p\not\equiv 0,1 \pmod p,$$
hence $(q^{n-1}-(-1)^{n-1})/(q+1)$ has a prime divisor $r\not\equiv 0,1 \pmod p$, and as before, r is prime to $|\mathbf{Z}(G)|=\gcd(n,q+1)$, with $G=\mathrm{SU}_n(q)$. The rest of the argument is now entirely analogous to the previous case to show that Lemma 2.4 applies.
(4) Here $S=\mathrm{PSL}_2(q)$, $S=\mathrm{PSL}_4(q)$ with $e_p(q)=2$, or $S=\mathrm{PSL}_3(q)$ with $p=3|(q-1)$. In the latter case, $q+1\equiv 2 \pmod 3$, so $q+1$ has a prime divisor $r\equiv 2 \pmod 3$. Let s be an element of order r in the dual group $G^*=\mathrm{PGL}_3(q)$. Since r divides $q+1$, the centraliser of s in G^* does not contain a Sylow 3-subgroup of G^*, and r does not divide $|\mathbf{Z}(\mathrm{SL}_3(q))|$. Thus, Lemma 2.4 applies. Now assume $S=\mathrm{PSL}_2(q)$. If $p|(q-\epsilon)$ and $q+\epsilon$ has a prime divisor $r>2$ with $r\not\equiv 1 \pmod p$ then we are done

by Lemma 2.4. Similarly, if $2|(q+\epsilon)/2$ then the characters of S of degree $(q-\epsilon)/2$ have values in $\mathbb{Q}(\sqrt{\pm q})$ (see [10, Tab. 2.6]), so are p-rational and Ω-invariant. Finally, let $S = \mathrm{PSL}_4(q)$ with $2 < p|(q+1)$. If $p > 3$ then let r be any prime divisor of $q-1 \equiv -2 \not\equiv 1 \pmod p$ which is not congruent to 1 (mod p) and s an element of maximal r-power order in G^*. If $p = 3$ and q is odd, let $s \neq 1$ be a 2-element in $\mathrm{PSL}_4(q) = [G^*, G^*]$ not centralising a Sylow 3-subgroup of G^*, and we may again conclude. Finally, when $p = 3$ and q is even (and hence an odd 2-power), then whenever r is any prime divisor of $(q^3-1)(q^2+1)$ not congruent to 1 (mod 3), we can again apply Lemma 2.4.

(5) Here $S = \mathrm{PSU}_4(q)$ with $p|(q-1)$ or $S = \mathrm{PSU}_3(q)$ with $p = 3|(q+1)$. In the latter case, if q is odd then take for s a 2-element of G^* of maximal possible order (and hence not centralising a Sylow 3-subgroup), and apply Lemma 2.4. If q is even, and there is a prime r not congruent to 1 (mod 3) dividing $(q-1)(q^2-q+1)/3$, this will do. Finally, if $S = \mathrm{PSU}_4(q)$ with $p|(q-1)$, we are done by Lemma 2.4 if there is a prime divisor $r > 2$ of $(q^3+1)(q^2+1)$ not congruent to 1 (mod p).

(6) Here $S = \mathrm{PSO}_{2n}^+(q)$ with $n \in \{5, 7\}$ and $p|(q+1)$. If $p = 3$ any (rational) unipotent character of degree divisible by q^2-q+1, hence by p, is as desired. So assume $p > 3$. Then there exists a prime divisor $r \neq p$ of $q^2-q+1 \equiv 3 \pmod p$ not congruent to 1 modulo p. Let s be an element generating a Sylow r-subgroup of a torus of order q^3+1 in G^*, where $G = \mathrm{Spin}_{2n}^+(q)$. Since $r \neq 2$ this satisfies the assumptions of Lemma 2.4.

(7) Here $S = \mathrm{PSO}_{2n}^-(q)$ with

$$(n, e_p(q)) \in \{(4,1), (4,2), (4,4), (5,1), (6,1), (6,2), (7,1), (8,1), (8,2)\}.$$

First assume $e_p(q) = 1$, so $p|(q-1)$. If $p = 3$ we use a (rational) unipotent character of degree divisible by q^2+q+1, so by 3. If $p \neq 3$ there exists a prime divisor $r \neq p$ of $q^2+q+1 \equiv 3 \pmod p$ not congruent to 1 modulo p, and any r-element generating a torus of order q^3-1 in the dual group G^* of $G = \mathrm{Spin}_{2n}^-(q)$ satisfies the assumptions of Lemma 2.4. Next assume $e_p(q) = 2$. Again, when $p = 3$ there is a suitable unipotent character of degree divisible by q^2-q+1. Otherwise, let r be a prime divisor of $q^2-q+1 \equiv 3 \pmod p$ not congruent to 1 modulo p, and argue as before. So finally $S = \mathrm{PSO}_8^-(q)$ and $2 \neq p|(q^2+1)$. With r a prime divisor of $q^2+q+1 \equiv q \not\equiv 1 \pmod p$ not congruent to 1 modulo p, we are done as before. □

Example 2.6. The second case listed in Proposition 2.5 is a true exception, for example the groups $S = \mathrm{PSL}_2(q)$ with

$$(p, q) \in \{(3,8), (5,61), (7,421), (11,397), (13,157), (17,613), (19,457), (23,277)\},$$

and similarly the first case is a true exception for example when $S = {}^2\mathrm{B}_2(8)$ with $p = 5$. We have not tried to determine whether there are cases with arbitrarily large p.

Example 2.7. The sporadic simple group J_1 has only three irreducible characters of degree divisible by 3; they are cyclically permuted by the Galois automorphism of order 9 of $\mathbb{Q}_{19}/\mathbb{Q}$ (see [4]), but fixed by the elements of order 3 in $\Omega(19)$.

Theorem 2.8. *Let A be an almost simple group with socle S such that A/S is a p-group for a prime p dividing $|A|$. If A has no $\Omega(|A|)$-invariant irreducible character of degree divisible by p, then $p \neq 2$ and (S, p) is as in (1)–(5) of Proposition 2.5.*

Proof. First assume p is odd. Assume there is a p-rational and Ω-invariant character $\chi \in \mathrm{Irr}(S)$ of degree divisible by p. Then by [19, Thm 6.30], χ has a unique p-rational extension $\tilde{\chi}$ to its inertia group $I_A(\chi)$, which then, by uniqueness, must also be Ω-invariant. Then $\tilde{\chi}^A$ is an Ω-invariant character of A of degree divisible by p, contradicting our assumption. Hence, we arrive at (S, p) being one of the exceptions in Proposition 2.5.

This leaves the case when p does not divide $|S|$. Then according to the classification, $p \geq 3$, and $p \geq 5$ unless S is a Suzuki group. Thus, necessarily S is of Lie type and A/S is generated by field automorphisms (see e.g. [4, Tab. 5]), say $A/S = \langle \gamma \rangle$ with $|\gamma| = p^a$. Now as before let $G = \mathbf{G}^F$ be such that $S = G/\mathbf{Z}(G)$, for $\mathbf{G}$ simple of simply connected type, and let $G^* = \mathbf{G}^{*F}$ be a dual group. Then γ is induced by a field automorphism of G. Assume for the moment that S is not a Suzuki or Ree group, and defined over the finite field $\mathbb{F}_q$. Then the centraliser $\mathbf{C}_G(\gamma^{p^{a-1}})$ is of the same Lie type over the subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$ with $q = q_0^p$. Let d be maximal with respect to Φ_d dividing the order polynomial of G. Then $|G|$ is divisible by $\Phi_d(q_0^p)$ and hence by $\Phi_{dp}(q_0)$, while $|\mathbf{C}_G(\gamma^{p^{a-1}})|$ is not. Let r be a Zsigmondy prime divisor of $\Phi_{dp}(q_0)$ (note that here $p \geq 5$), so r does not divide $|\mathbf{C}_S(\gamma^{p^{a-1}})|$. An obvious modification of the argument yields that such a prime also exists if S is a Suzuki or Ree group.

Let $s \in [G^*, G^*]$ be an r-element. By the construction of r, it does not divide $|\mathbf{Z}(G)|$ and thus by the argument in the proof of Lemma 2.4, there is a unique semisimple character χ_s of G in the Lusztig series corresponding to s, and χ_s is trivial on $\mathbf{Z}(G)$. Then, considered as character of S, χ_s has values in $\mathbb{Q}_{r^b}$ for some $b \geq 0$. Assume χ_s is fixed by some power γ^m. Then the class of s is fixed by the dual automorphism $(\gamma^*)^m$ of G^* (see [41, Prop. 7.2]). Since p does not divide $|S|$ and hence neither $|G^*|$, the class length of s is prime to p, and so a suitable conjugate of s itself is fixed by $(\gamma^*)^m$. As r does not divide $|\mathbf{C}_G(\gamma^{p^{a-1}})| = |\mathbf{C}_{G^*}(\gamma^{*p^{a-1}})|$, this forces $(\gamma^*)^m = 1$. We conclude that χ_s lies in an orbit of length $|\gamma|$ under A, so the induced character χ^A is irreducible.

By the choice of s, the values of χ^A lie in a subfield of index $|\gamma| = p^a > 1$ of $\mathbb{Q}_{r^b}$, which is in the fixed field of Ω.

Now assume $p = 2$. By [42, Thm C], there is a rational-valued character in $\mathrm{Irr}(A)$ of even degree unless $S = \mathrm{PSL}_2(q)$ with $q = 3^f$ and $f \geq 3$ is odd. Consider the latter case. Since A/S is a 2-group, we have $A \in \{S, \widetilde{S}\}$, where $\widetilde{S} := \mathrm{PGL}_2(q)$. Let r be a prime divisor of $(q-1)/2$ and χ an irreducible Deligne–Lusztig character of $\mathrm{SL}_2(q)$ labelled by an element in $\mathrm{PGL}_2(q)$ of order r, so of even degree $q+1$. Then χ has values in $\mathbb{Q}_r$ by [10, Tab. 2.6] and it has $\mathbf{Z}(\mathrm{SL}_2(q))$ in its kernel as r is odd. The unique involution in $\mathrm{Gal}(\mathbb{Q}_r/\mathbb{Q})$ acts as complex conjugation, but χ is real by loc. cit. Since $\mathbb{Q}(\chi) \leq \mathbb{Q}_r$, then χ is $\Omega(|S|)$-invariant. Moreover, χ is in fact the restriction of an irreducible character in $\mathrm{Irr}(\widetilde{S})$ with the same field of values, so the proof is complete. □

The following result on quasi-simple groups will be needed in the proof of Theorem 3:

Theorem 2.9. *Let p be a prime number. Let S be a non-abelian simple group with abelian Sylow p-subgroups. If G is quasi-simple with $G/\mathbf{Z}(G) = S$ and $|\mathbf{Z}(G)| = p$, then there exists a faithful $\chi \in \mathrm{Irr}_{\Omega(|G|)}(B_0(G))$ (and hence of degree divisible by p).*

Proof. First assume $p = 2$. The non-abelian simple groups with abelian Sylow 2-subgroups are $\mathrm{PSL}_2(q)$ with $q \equiv 3, 5 \pmod 8$ or q even, the Ree groups, and J_1. Of these, only

$\mathrm{PSL}_2(q)$ with $q \equiv 3,5 \pmod 8$ have covering groups $G = \mathrm{SL}_2(q)$ as in the claim. Now $\mathrm{SL}_2(q)$ has a rational faithful irreducible Deligne–Lusztig character of degree $q-1$ (if $q \equiv 3 \pmod 8$) respectively $q+1$ (if $q \equiv 5 \pmod 8$) labelled by an element of order 4 in the dual group $\mathrm{PGL}_2(q)$, hence lying in the principal 2-block (e.g. by [9, Prop. 6]).

So now assume $p \geq 3$. The non-abelian simple groups with order of the Schur multiplier divisible by an odd prime p are $\mathrm{PSL}_n(\epsilon q)$ with $p|\gcd(n, q-\epsilon)$, and for $p = 3$ groups of type $E_6(\epsilon q)$, $\mathfrak{A}_6$, $\mathfrak{A}_7$, and some of the sporadic simple groups. The groups $E_6(\epsilon q)$ have a section isomorphic to $\mathrm{PSL}_5(q)$, and $\mathrm{PSL}_n(\epsilon q)$ has a section $\mathrm{PSL}_p(\epsilon q)$ when $p|n$. Now the normalisers of a suitable maximal torus of $\mathrm{SL}_p(\epsilon q)$ contains a natural non-abelian subgroup $C_{q-\epsilon}^{p-1}.C_p$ whose image in $\mathrm{PSL}_p(\epsilon q)$ is easily seen to be non-abelian unless $p = 3$ and $\epsilon q \equiv 4,7 \pmod 9$. For the sporadic groups, the ones with abelian Sylow 3-subgroup can be identified from the Atlas [4]. Thus, the groups with abelian Sylow p-subgroups to consider only occur for $p = 3$, namely $\mathrm{PSL}_3(q)$ with $q \equiv 4,7 \pmod 9$, $\mathrm{PSU}_3(q)$ with $q \equiv 2,5 \pmod 9$, $\mathfrak{A}_6$, $\mathfrak{A}_7$, M_{22} and ON. Here, the groups $\mathrm{PSL}_3(\epsilon q)$ have faithful characters of degree $(q^2-1)(q-\epsilon)$ with values in $\mathbb{Q}_3$, labelled by an element of order 3 in $\mathrm{PGL}_3(\epsilon q)$, so lying in the principal 3-block. Using [43], the groups $3.\mathfrak{A}_6$, $3.\mathfrak{A}_7$, $3.M_{22}$, and $3.ON$ are seen to have faithful characters of degrees 6, 15, 231, 495 respectively lying in the principal 3-block, with values in $\mathbb{Q}_3$. This completes the proof. □

2.2. **Some observations on extending unipotent characters.** We continue to consider connected reductive linear algebraic groups $\mathbf{G}$ with a Steinberg endomorphism $F : \mathbf{G} \to \mathbf{G}$. The next result will be useful when dealing with graph automorphisms.

Lemma 2.10 (Digne–Michel). *Let $\mathbf{G}$ be of type A_{n-1}, E_6, or D_n with F inducing an $\mathbb{F}_q$-structure, and $G = \mathbf{G}^F$. Let τ be a non-trivial graph automorphism of G. Then any τ-invariant unipotent character χ lying in the principal series of G extends to a rational-valued character $\tilde\chi$ of $G\langle\tau\rangle$, unless either $G = \mathrm{E}_6(q)$ and $\chi \in \{\phi_{64,4}, \phi_{64,13}\}$, or $G = \mathrm{A}_{n-1}(q)$ and χ is labelled by a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n with*

$$\sum_i \binom{\lambda_i}{2} - \sum_j \binom{\lambda_j'}{2} + \binom{n}{2} \equiv 1 \pmod 2,$$

where $\lambda' = (\lambda_1', \ldots, \lambda_s')$ is the partition conjugate to λ. In the latter cases, $\mathbb{Q}(\tilde\chi) = \mathbb{Q}(\sqrt{q})$.

Proof. This follows from [8, Thm 3] but can also be extracted from [6, Thm II.3.3]. Note that all unipotent characters under consideration are rational-valued, by [2, Thm 2.9]. □

Given a normal subgroup N of a group G and $\theta \in \mathrm{Irr}(N)$, we will write $I_G(\theta)$ or G_θ to denote the inertia group of θ in G. In the case of field automorphisms, the following is also useful:

Lemma 2.11. *Let $\mathbf{G}$ be simply connected with a Frobenius map F such that $S = G/\mathbf{Z}(G)$ is simple, where $G := \mathbf{G}^F$. Let $\chi \in \mathrm{Irr}(S)$ be the deflation of a principal series unipotent character. If $S \leq A \leq \mathrm{Aut}(S)$ is such that A is generated by inner-diagonal and field automorphisms, then χ has an extension $\hat\chi$ to $I_A(\chi)$ with $\mathbb{Q}(\hat\chi) \leq \mathbb{Q}(\chi)$.*

Proof. By [25, Thm 2.4], every unipotent character of S extends to its stabilizer in $\mathrm{Aut}(S)$. We may view A as a subgroup of $\hat G_{\mathrm{ad}} := \widetilde S \rtimes \langle F_0\rangle$, where $\widetilde S := \mathbf{G}_{\mathrm{ad}}^F$ is the group of inner-diagonal automorphisms, induced by the fixed points of the group of adjoint type, and F_0

is an appropriate field automorphism. (See [31, Thm 24.24] and the discussion above it.) Then we view χ as the restriction of a principal series unipotent character $\widetilde{\chi}$ of $\widetilde{S}$ with the same inertia group in $\mathrm{Aut}(S)$ as χ, using [25, Prop. 2.1]. The claim then follows directly from [39, Prop. 2.6] (and its generalization [21, Prop. 8.7]) applied to $\hat{G}_{\mathrm{ad}}$ and $\widetilde{\chi}$, as in the notation of [39, Prop. 2.6], $\widetilde{\chi}$ is in the Harish–Chandra series of the cuspidal character $\delta = 1$ of a Levi subgroup L, in this case a torus of $\mathbf{G}_{\mathrm{ad}}^F$, which extends trivially to $\hat{\delta} = 1$ of $\hat{L} = L\langle F_0\rangle$. Namely, we obtain an extension $\hat{\widetilde{\chi}}$ of $\widetilde{\chi}$ to $I_{\hat{G}_{\mathrm{ad}}}(\widetilde{\chi})$ with $\mathbb{Q}(\hat{\widetilde{\chi}})) \leq \mathbb{Q}(\chi) = \mathbb{Q}(\widetilde{\chi})$ and let $\hat{\chi}$ be its restriction to $I_A(\chi)$. □

Using the above lemma, the following is proved in Lemma 4.4 of [33]:

Lemma 2.12. *Let p be a prime and $\mathbf{G}$ be simply connected defined in characteristic distinct from p, with a Steinberg map F such that $S = G/\mathbf{Z}(G)$ is simple, where $G := \mathbf{G}^F$. Then there exists a non-trivial, rational-valued unipotent character $\chi \in \mathrm{Irr}_{p'}(B_0(S))$ such that if $S \leq A \leq \mathrm{Aut}(S)$ and A is generated by inner-diagonal and field automorphisms, then there is an extension $\hat{\chi}$ of χ to $I_A(\chi)$ that is rational-valued.*

We will also use the following in the case $p = 2$.

Proposition 2.13. *Let S be a simple group of Lie type defined in odd characteristic, different from types A_1, ${}^2\mathrm{A}_2$ and ${}^2\mathrm{G}_2$. Then the principal 2-block $B_0(S)$ contains a rational unipotent principal series character χ of even degree such that if $S \leq A \leq \mathrm{Aut}(S)$ and A is generated by inner-diagonal and field automorphisms, then χ has a rational extension $\hat{\chi}$ to $I_A(\chi)$.*

Proof. Suppose first that S is of classical type. Then all unipotent characters of S are rational (see [10, Cor. 4.4.24]) and lie in the principal 2-block (see [9, Prop. 6]). Since the Weyl group of S has an irreducible character of even degree unless we are in the excluded cases, the claim follows by the arguments in [23, 2.3]. Now assume S is of exceptional type. Then by [9, Thm A] all principal series unipotent characters lie in $B_0(S)$. Almost all of these are rational by [10, Cor. 4.5.6]. From the lists of character degrees printed, for example, in [3, §13] it is then immediate to identify rational unipotent characters in the principal series having even degree: the characters denoted $\phi_{2,1}, \phi_{2,1}, \phi'_{8,3}, \phi_{6,1}, \phi'_{2,4}, \phi_{168,6}, \phi_{8,1}$ for the types $G_2, {}^3D_4, F_4, E_6, {}^2E_6, E_7, E_8$ respectively.
The second claim now follows from Lemma 2.11. □

2.3. **Existence of certain character extensions.** We work towards a proof of Conjecture 1 and begin with some additional results about extensions.

When S is a simple group of Lie type, we write $\mathrm{Aut}(S) = \widetilde{S} \rtimes D$, where $\widetilde{S}$ is the group of inner-diagonal automorphisms and D is an appropriate group of graph-field automorphisms, see [11, Thms 2.5.12 and 2.5.14].

Theorem 2.14. *Let A be an almost simple group with socle S. Suppose that A/S is a p-group. Then there exists $1_S \neq \varphi \in \mathrm{Irr}_\Omega(S)$ that extends to an Ω-invariant character $\hat{\varphi}$ of $I_A(\varphi)$. Furthermore, if p divides $|S|$ we can take $\hat{\varphi}$ to lie in the principal p-block of $I_A(\varphi)$. In this case, $\hat{\varphi} \in \mathrm{Irr}(B_0(I_A(\varphi)))$ can be chosen to be rational-valued, except possibly when $S = \mathrm{PSL}_2(p^f)$, or when $p = 2$ and $S = \mathrm{PSL}_3(\pm 2^f)\}$ or $A = S = {}^2\mathrm{B}_2(q^2)$.*

Proof. We see from [18, Lemma 4.1] that there is a non-trivial rational-valued character $\varphi \in \mathrm{Irr}(S)$ that extends to a rational-valued character of $\mathrm{Aut}(S)$, so the first statement holds.

For the second statement, since A/S is a p-group and hence $B_0(I_A(\varphi))$ is the unique p-block of $I_A(\varphi)$ above $B_0(S)$, it suffices to show that a φ satisfying the first statement can be chosen in $B_0(S)$. For the third statement, we wish to ensure that further $\hat{\varphi}$ is rational-valued, when S is not one of the listed exceptions.

For the sporadic groups and the alternating group $\mathfrak{A}_6$, this may be checked in GAP [43]. Next, suppose that $S = \mathfrak{A}_n$ with $n \geq 5$, $n \neq 6$. Then $A \in \{\mathfrak{A}_n, \mathfrak{S}_n\}$. The character of $\mathfrak{S}_n$ corresponding to any non-self-conjugate partition of n with p-core (r), where $n = ap + r$ with $0 \leq r < p$, will lie in $B_0(\mathfrak{S}_n)$ and restrict irreducibly to $B_0(\mathfrak{A}_n)$, giving a rational character of S that extends to a rational character in $B_0(A)$. More specifically, we may take the partition $(n-1, 1)$ if $r = 0$; $(ap-1, 2)$ if $r = 1$; and $(r, 1^{ap})$ if $r > 1$.

Now let S be a simple group of Lie type. First, assume that p is not the defining characteristic for S. Suppose that $p \in \{2, 3\}$. Then the Steinberg character St_S lies in $B_0(S)$, using [9, Thm A] and [24, Thm 6.6], and extends to a rational-valued character of $\mathrm{Aut}(S)$ by [40]. If instead $p \geq 5$, note that A is generated by inner-diagonal and field automorphisms, since A/S is a p-group. Then the result follows from Lemma 2.12.

Now assume that S is a group of Lie type defined in characteristic p. We have $\mathrm{Irr}(B_0(S)) = \mathrm{Irr}(S) \setminus \{\mathrm{St}_S\}$ by [17], so it suffices to know that there is a character $\varphi \notin \{1_S, \mathrm{St}_S\}$ that extends to an Ω-invariant (in fact rational, except in the stated exceptions) character of $I_A(\varphi)$. Recall that $\mathrm{Aut}(S) = \widetilde{S} \rtimes D$. We have $p \nmid |\widetilde{S}/S|$ and all Sylow subgroups of D are abelian, and hence the Sylow p-subgroups of $\mathrm{Aut}(S)/S$ are abelian. In particular, A/S is abelian.

If $S = {}^2\mathrm{F}_4(q^2)$ or ${}^2\mathrm{B}_2(q^2)$, then $p = 2$ and $|\mathrm{Aut}(S)/S|$ is odd, so in this case $A = S$. When $S = {}^2\mathrm{B}_2(q^2)$ we take for φ one of the two cuspidal unipotent characters, with values in $\mathbb{Q}_4$, and when $S = {}^2\mathrm{F}_4(q^2)$ any non-trivial rational unipotent character apart from the Steinberg character. Observe that for ${}^2\mathrm{B}_2(q^2)$ there is no suitable rational character.

If $S = {}^2\mathrm{G}_2(q^2)$, then $p = 3$ and we may take φ to be the semisimple character of degree $q^4 - q^2 + 1$ labelled by the class of involutions in $G^* \cong S$. Then φ is rational-valued, and note that $3 \nmid \varphi(1)$. Since A/S is a 3-group, φ extends to a rational-valued character of A by [36, Cors 6.2 and 6.4]. We therefore assume that S is not a Suzuki or Ree group.

First suppose that $S = \mathrm{PSL}_n(\epsilon q)$ with $n \geq 4$, $q = p^f$, and $\epsilon \in \{\pm 1\}$. Write $D = \langle \tau, F_p \rangle$, where τ is a graph automorphism of order 2 and F_p is a generating field automorphism of order f if $\epsilon = 1$, respectively $2f$ if $\epsilon = -1$. Let λ denote the partition $(\frac{n-4}{2} + 2, 2, 1^{\frac{n-4}{2}})$ if $n \equiv 0 \pmod 4$, the partition $(\frac{n-1}{2}, 1^{\frac{n-1}{2}})$ if $n \equiv 1 \pmod 4$, and the partition $(n-2, 2)$ if $n \equiv 2, 3 \pmod 4$. Then λ does not satisfies the condition in Lemma 2.10. Hence the corresponding unipotent character $\chi_\lambda \in \mathrm{Irr}(\widetilde{S})$ extends to a rational-valued character of $\widetilde{S}\langle \tau \rangle$. As χ_λ lies in the principal series, it further extends to a rational-valued character of $\widetilde{S}\langle F_p \rangle$, by Lemma 2.11. Note that if $p \geq 3$ or f is odd, then $A \leq \widetilde{S}\langle F_p \rangle$ or $A \leq \widetilde{S}\langle \tau \rangle$, since A/S is a p-group. Then $\varphi := (\chi_\lambda)_S$ extends to a rational-valued character of $I_A(\varphi)$ in these cases.

So now assume $p = 2$ and f is even. Since $\mathrm{Aut}(S)/\widetilde{S}$ is abelian and any unipotent character extends to $\mathrm{Aut}(S)$ by [25, Thms 2.4, 2.5], we have any character of $\mathrm{Aut}(S)$

lying above χ_λ is an extension by Gallagher's theorem. From above, any extension of χ_λ to $\widetilde{S}\langle\tau\rangle$ is rational. Choosing $\hat{\varphi}$ an extension of χ_λ to $\mathrm{Aut}(S)$ to be such that $\hat{\varphi}|_{\widetilde{S}\langle F_p\rangle}$ is the rational extension of φ guaranteed by Lemma 2.11, we obtain $\hat{\varphi}|_A$ is rational-valued.

Next, consider the case $S = \mathrm{PSL}_3(\epsilon q)$ with $q = p^f$. We remark first that if $\epsilon = 1$ and either f is even or p is odd, the same arguments as above but with $\lambda = (2,1)$ yield a unipotent character $\varphi \notin \{1_S, \mathrm{St}_S\}$ that extends to a rational character of $I_A(\varphi)$.

Let $\widetilde{G} := \mathrm{GL}_3(\epsilon q)$ and let $G := \mathrm{SL}_3(\epsilon q)$. Assume p is odd. Let $s = \mathrm{diag}(-1,-1,1) \in G \cong [\widetilde{G}^*, \widetilde{G}^*]$. Then the semisimple character χ_s of $\widetilde{G}$ is trivial on $\mathbf{Z}(\widetilde{G})$, restricts irreducibly to G, and has degree prime to p. Further, χ_s is rational-valued. Then taking $\varphi \in \mathrm{Irr}(S)$ to be the deflation of $\chi_s|_G$ and applying [36, Cors 6.2 and 6.4], we see φ extends to a rational-valued character of $I_A(\varphi)$.

Now assume $p = 2$. Let $s = \mathrm{diag}(\zeta, \zeta^{-1}, 1) \in G \cong [\widetilde{G}^*, \widetilde{G}^*]$ with $3 < |\zeta|$ a prime power dividing $q - \epsilon$. Note that this is possible since $q \neq 2$ in the case $\epsilon = -1$ as S is assumed simple, and $q \neq 4$ in the case $\epsilon = 1$ since f is odd. Then the semisimple character $\chi_s \in \mathrm{Irr}(\widetilde{G})$ restricts irreducibly to G, is trivial on $\mathbf{Z}(\widetilde{G})$, and has odd degree. Further, $\zeta^\sigma \in \{\zeta, \zeta^{-1}\}$ for any Galois automorphism σ of order 2 since $(\mathbb{Z}/|\zeta|\mathbb{Z})^\times$ is cyclic. Then $\chi_s^\sigma = \chi_s$ for any $\sigma \in \Omega$. Let $\varphi \in \mathrm{Irr}(S)$ be the deflation of the restriction of χ_s to G. Then again by [36, Cors 6.2 and 6.4], we have an extension of φ to $I_A(\varphi)$ that is also Ω-invariant.

Now consider the case $S = \mathrm{PSL}_2(q)$ with $q = p^f$. If $q \leq 9$, then we can check directly in GAP or use the well-known isomorphisms with alternating groups to see the statement holds. So, assume $q \geq 11$. If p is odd, then by [10, Tab. 2.6] there are two characters of S of degree $\frac{q+\eta}{2}$, where $\eta \in \{\pm 1\}$ with $q \equiv \eta \pmod 4$, which are Ω-stable and have p'-degree, so again possess an Ω-invariant extension to $I_A(\varphi)$ by [36, Cors 6.2 and 6.4]. We may therefore assume that $p = 2$. Let $\widetilde{G} := \mathrm{GL}_2(q)$ and note that $S = \mathrm{SL}_2(q)$. Let $s \in \widetilde{G}^*$ be an element with eigenvalues $\{\zeta, \zeta^{-1}\}$, where $|\zeta|$ is a prime dividing $q+1$. If $\sigma \in \Omega$ has order 2, then as above, $\zeta^\sigma \in \{\zeta, \zeta^{-1}\}$, so the corresponding semisimple character $\chi_s \in \mathrm{Irr}(\widetilde{G})$ is σ-invariant, whence $\chi_s \in \mathrm{Irr}_\Omega(\widetilde{G})$. Further, χ_s has odd degree and restricts irreducibly to S. Then letting φ denote this restricted character, we have $\varphi \in \mathrm{Irr}_{2',\Omega}(B_0(S))$ since $\varphi \neq \mathrm{St}_S$. Again applying [36, Cors 6.2 and 6.4], φ has an Ω-invariant extension to $I_A(\varphi)$.

We may now assume that $S = G/\mathbf{Z}(G)$ where G is not of type A, ^{2}A, or Suzuki or Ree type. The principal series unipotent characters are rational-valued except for a small number of exceptions for $S = \mathrm{E}_7(q)$ and $\mathrm{E}_8(q)$, by [2, Thm 2.9]. Further, in our remaining cases, there is always a principal series unipotent character $\varphi \notin \{1_S, \mathrm{St}_S\}$ (and hence in $B_0(S)$) that is rational-valued, distinct from the exceptions for $\mathrm{E}_6(q)$ in Lemma 2.10, and of degree divisible by p. (The principal series unipotent characters are described in [3, Sec. 13.8, 13.9].) In the case of $\mathrm{B}_2(2^n)$ with $n \geq 2$ and $\mathrm{F}_4(2^n)$ with $n \geq 1$, we may take φ to be the character indexed by the symbol $\binom{1,2}{0}$, respectively the character $\phi'_{8,3}$ in the notation of [3, Sec. 13.9], which is stable under field automorphisms but moved by the exceptional graph automorphism by [25, Thm. 2.5]. That is, in the latter cases, $I_A(\varphi)$ is generated by S and field automorphisms. Then using Lemmas 2.11 and 2.10, we can argue analogously to before to see the statement holds. □

For p a prime, as customary we denote by $\mathbf{O}^{p'}(G)$ the smallest normal subgroup of G with factor group of p'-order, that is, the normal subgroup generated by all p-elements. The following will be useful toward proving Conjecture 1 for certain almost simple groups (Theorem 2.17 below).

Lemma 2.15. *Let p be a prime and let A be an almost simple group such that $\mathbf{O}^{p'}(A) = A$ with socle S a simple group of Lie type. Assume $p \geq 5$ if $S = \mathrm{D}_4(q)$. Assume that one of the following holds:*

(1) *p is odd and $\mathrm{Irr}_\Omega(B_0(\widetilde{S}))$ contains a p-rational character of degree divisible by p that restricts irreducibly to S;*
(2) *p is odd and $\mathrm{Irr}_\Omega(B_0(A \cap \widetilde{S}))$ contains a p-rational character not invariant under A;*
(3) *$\mathrm{Irr}_\Omega(B_0(A \cap \widetilde{S}))$ contains a character of p'-degree not invariant under A.*

Then there is a character in $\mathrm{Irr}_\Omega(B_0(A))$ of degree divisible by p.

Proof. Since $\mathbf{O}^{p'}(A) = A$ and either D is abelian or $S = \mathrm{D}_4(q)$ and D is $\mathfrak{S}_3 \rtimes C$ with C cyclic, we have $A/(A \cap \widetilde{S}) \cong A\widetilde{S}/\widetilde{S}$ is a p-group. Then $B_0(A\widetilde{S})$ is the unique p-block covering $B_0(\widetilde{S})$ and $B_0(A)$ is the unique block covering $B_0(A \cap \widetilde{S})$.
(1) If $\mathrm{Irr}_\Omega(B_0(\widetilde{S}))$ contains a p-rational character of degree divisible by p that restricts irreducibly to S, then let φ be its restriction to $A \cap \widetilde{S}$. Then $\varphi \in \mathrm{Irr}_\Omega(B_0(A \cap \widetilde{S}))$ is p-rational and has degree divisible by p. By [19, Thm 6.30], if p is odd then φ extends to a unique p-rational character $\hat{\varphi}$ of $I_A(\varphi)$, which is therefore in $\mathrm{Irr}_\Omega(B_0(I_A(\varphi)))$. Then the induced character $\hat{\varphi}^A$ has degree divisible by p and lies in $\mathrm{Irr}_\Omega(B_0(A))$.
(2) Now assume $\mathrm{Irr}_\Omega(B_0(A \cap \widetilde{S}))$ contains a p-rational character φ not invariant under A. Then $[A : I_{A\cap\widetilde{S}}(\varphi)]$ is a power of p. As before, if p is odd then φ has a unique p-rational extension $\hat{\varphi}$ to $I_A(\varphi)$, which must lie in $B_0(I_A(\varphi))$ and be Ω-invariant. Then $\hat{\varphi}^A$ has degree divisible by p and lies in $\mathrm{Irr}_\Omega(B_0(A))$.
(3) If instead $\varphi \in \mathrm{Irr}_{p',\Omega}(B_0(A \cap \widetilde{S}))$ is not invariant under A, we may argue similarly, using [36, Cors 6.2 and 6.4] in place of [19, Thm 6.30]. □

Remark 2.16. Note that if $S = \mathrm{D}_4(q)$ and $A/(A \cap \widetilde{S})$ is a p-group, then the conclusion of Lemma 2.15(1, 2) still holds when $p = 3$ and that of Lemma 2.15(3) when $p \in \{2, 3\}$.

2.4. **Extension of Brauer's height zero conjecture for almost simple groups.** The following is Conjecture 1 for certain almost simple groups. It will be used in the proof of Theorem 3.

Theorem 2.17. *Let A be an almost simple group such that $\mathbf{O}^{p'}(A) = A$. Then A has abelian Sylow p-subgroups if and only if all the characters in $\mathrm{Irr}_\Omega(B_0(A))$ have p'-degree.*

Proof. The "only if" direction follows from that direction of the ordinary height zero conjecture, proved in [22]. For the converse, assume that A has non-abelian Sylow p-subgroups. We claim that $\mathrm{Irr}_\Omega(B_0(A))$ contains a character with degree divisible by p. Let S be the simple socle of A. We deal with the various possibilities for S in the subsequent Propositions 2.18–2.21. □

Proposition 2.18. *The conclusion of Theorem 2.17 holds for S not of Lie type.*

Proof. For $S = \mathfrak{A}_n$ with $n \geq 5$ the claim for $p = 2$ is shown in the proof of [29, Prop. 2.5]. For $p > 2$ the principal p-block of $\mathfrak{S}_n$ contains a character χ of degree divisible by p (if $\mathfrak{S}_n$ has non-abelian Sylow p-subgroups) by Brauer's height zero conjecture [30], and this is rational. Furthermore, the constituents of its restriction to $\mathfrak{A}_n$ are Ω-invariant and still of degree divisible by p. If S is sporadic or the Tits simple group, it can be checked with [43] that the only cases when $|A|$ is divisible by p^3 but A does not possess a rational character in the principal p-block of degree divisible by p is when $A = S = J_1$ with $p = 2$, or $S = ON$ with $p = 3$. But in either case, Sylow p-subgroups of A are abelian. □

For the groups of Lie type, as before, we write $\mathrm{Aut}(S) = \widetilde{S} \rtimes D$. Since $\mathbf{O}^{p'}(A) = A$, it follows that any field automorphism in A has p-power order, and further any graph-field automorphism in A has p-power order except possibly if $S = \mathrm{D}_4(q)$.

Proposition 2.19. *The conclusion of Theorem 2.17 holds for S of Lie type in characteristic p.*

Proof. Assume S is simple of Lie type defined in characteristic p. Recall again that $\mathrm{Irr}(B_0(S)) = \mathrm{Irr}(S) \setminus \{\mathrm{St}_S\}$. If $S = {}^2\mathrm{F}_4(q^2)$ or ${}^2\mathrm{B}_2(q^2)$ with $q^2 = 2^{2m+1}$ and $p = 2$, then the condition $\mathbf{O}^{p'}(A) = A$ means that $A = S$. In the case of $S = {}^2\mathrm{F}_4(q^2)$, there is a unique unipotent character of degree $q^2(q^4 - q^2 + 1)(q^8 - q^4 + 1)$, which is therefore rational-valued and has even degree. When $S = {}^2\mathrm{B}_2(q^2)$, there are two cuspidal unipotent characters of even degree, which take values in $\mathbb{Q}(\sqrt{-1})$, and therefore are Ω-stable since $p = 2$.

Next suppose $S = {}^2\mathrm{G}_2(q^2)$ with $q = 3^{2m+1}$ and $p = 3$. Then $\mathrm{Out}(S)$ is cyclic of size $2m+1$, comprised of field automorphisms. So $\mathbf{O}^{p'}(A) = A$ means A/S must be a 3-group and $B_0(A)$ is the unique block lying above $B_0(S)$. In this case, the unique character χ of degree $q^2(q^4 - q^2 + 1)$ has degree divisible by p and must be rational and invariant under A. Then we are done by Lemma 2.15(1).

Next, let $S = \mathrm{PSL}_2(q)$ with $q = p^f$. Here Sylow p-subgroups of A are abelian unless A induces field automorphisms of order p, thus $p|f$. Let r be a prime divisor of $(p^{2f} - 1)/(p^{2f/p} - 1)$, $s \in \mathrm{SL}_2(q)$ an element of order r and $\chi \in \mathrm{Irr}(\mathrm{PGL}_2(q))$ the corresponding (semisimple) Deligne–Lusztig character. By construction, χ is of p'-degree, but not invariant under any non-trivial subgroup of A (as s is not in any proper subfield subgroup). Thus $\chi_{A \cap \widetilde{S}}$ induces to an irreducible character $\hat{\varphi}$ of A, of degree divisible by p. If $r \not\equiv 1 \pmod{p}$ then χ and hence $\hat{\chi}$ are Ω-invariant. If $p|(r-1)$, then the field automorphism in A of order p fuses χ with its images under Ω, and again χ^A is Ω-invariant.

Next, consider the case $S = \mathrm{PSL}_3(\epsilon q)$ with $q = p^f$. Here $\widetilde{S} = \mathrm{PGL}_3(\epsilon q)$. If p is odd, the unique unipotent character in $\mathrm{Irr}(\widetilde{S}) \setminus \{1_{\widetilde{S}}, \mathrm{St}_{\widetilde{S}}\}$ has degree divisible by p, lies in $B_0(\widetilde{S})$, and is rational-valued, so we are done by Lemma 2.15. Assume $p = 2$ so $q = 2^f$. Then the unipotent character of $\widetilde{S} := \mathrm{PGL}_3(q)$ of degree $q(q+1)$ restricts irreducibly to $S = \mathrm{PSL}_3(q)$, lies in $B_0(\widetilde{S})$, has even degree, and extends to a character of $A\widetilde{S}$ whose values lie in $\mathbb{Q}(\sqrt{q}) \leq \mathbb{Q}_8$, by Lemmas 2.10 and 2.11, and so is Ω-invariant. This character restricts to an even-degree character in $\mathrm{Irr}_\Omega(B_0(A))$.

Now, let $S = \mathrm{PSU}_3(q)$ with $q = 2^f$ and $p = 2$. Observe that $A/(A \cap \widetilde{S})$ is cyclic. If $A \leq \widetilde{S}$ then $A = S$ and the unipotent character φ of degree $q(q-1)$ is as claimed. If $A \not\leq \widetilde{S}$ then A induces a graph-field automorphism. First, suppose that $A/(A \cap \widetilde{S})$ has order 2, so is generated by F_2^f. By [8, Thm 1], the cuspidal unipotent character φ extends to a character $\hat{\varphi}$ of A with field of values $\mathbb{Q}(\sqrt{-q})$. Note that this is again Ω-invariant for $p=2$. Then we may assume that $|A/(A\cap\widetilde{S})|\geq 4$, and hence that A contains $\gamma := F_2^{f/2}$. Consider $s\in \mathrm{SU}_3(q)=[\mathrm{GU}_3(q),\mathrm{GU}_3(q)]$ with eigenvalues $\{\zeta,\zeta^{-1},1\}$ where $|\zeta|\neq 3$ is a prime-power divisor of $q+1$. (Recall that $q\neq 2$ since S is simple.) Then the semisimple character χ_s of $\mathrm{GU}_3(q)$ has odd degree, is 2-rational since $|s|$ is odd, and is trivial on the centre. Further, each element of Ω maps ζ to $\zeta^{\pm 1}$, so χ_s is Ω-invariant. Then we view $\chi_s\in \mathrm{Irr}_\Omega(B_0(\widetilde{S}))$. Now, since $|\zeta|\mid(q+1)$, we have $\zeta^{2^{f/2}}\notin\{\zeta,\zeta^{-1}\}$. In particular, χ_s is not γ-invariant. Further, $\chi_s^\gamma\neq\chi_s\alpha$ for any $\alpha\in\mathrm{Irr}(\widetilde{S}/S)$ and $\chi_s\neq\chi_s\alpha$ for any $1\neq\alpha\in\mathrm{Irr}(\widetilde{S}/S)$, as otherwise $s^\gamma z$, respectively sz, would be conjugate to s for some $z\in\mathbf{Z}(\mathrm{GU}_3(q))$ by [10, Prop. 2.5.20] and [41, Prop. 7.2], contradicting our assumptions on $|\zeta|$. Thus χ_s restricts to an irreducible, 2-rational, odd-degree character of $\mathrm{Irr}_\Omega(B_0(S))$ that is not A-invariant, and we are done with this case by Lemma 2.15(3).

Now suppose $S=\mathrm{D}_4(q)$ with $p\leq 3$ and $q=p^f$. Let χ be one of the unipotent characters of $\widetilde{S}$ listed in [10, Thm 4.5.11(b)]: this character lies in the principal series, has degree divisible by p, and $I_{\mathrm{Aut}(S)}(\chi)$ contains all field automorphisms, but does not contain the triality graph automorphism. Then $I_{\mathrm{Aut}(S)}(\chi)/\widetilde{S}$ is a subgroup of $\langle\tau\rangle\times\langle F_0\rangle=C_2\times C_f$, where τ is a graph automorphism of order 2 and F_0 is a field automorphism of order f. Note that our assumption $\mathbf{O}^{p'}(A)=A$ means that $A/(A\cap\widetilde{S})$ is a subgroup of $\langle\tau\rangle\times\langle F_0^m\rangle$, where $m=f_{p'}$. By Lemmas 2.10 and 2.11, χ extends to a rational character χ_1 of $\widetilde{S}\langle\tau\rangle$ and to a rational character χ_2 of $\widetilde{S}\langle F_0^m\rangle$. Since both possible choices of χ_1 must therefore be rational, we may assume $\chi_1\in\mathrm{Irr}(B_0(\widetilde{S}\langle\tau\rangle))$. Since F_0^m has p-power order, we also have $\chi_2\in\mathrm{Irr}(B_0(\widetilde{S}\langle F_0^m\rangle))$. Then taking the unique common extension of χ_1 and χ_2, we see χ extends to a rational character of $B_0(I_{A\widetilde{S}}(\chi))$. Then the corresponding unipotent character $\varphi:=\chi|_S$ of S extends to a rational character $\hat{\varphi}$ of $B_0(I_A(\varphi))$. The induced character $\hat{\varphi}^A$ then also has degree divisible by p and is rational-valued. Further, by [34, Cor. 6.2 and Thm 6.7], we have $\hat{\varphi}^A\in\mathrm{Irr}(B_0(A))$.

We may now assume that S is not a Suzuki or Ree group and not of types A_1, A_2, ${}^2\mathrm{A}_2$, nor D_4 when $p\leq 3$. In particular, $A/(A\cap\widetilde{S})$ is a p-group. Then the unipotent character $\varphi\in\mathrm{Irr}_\Omega(B_0(S))$ exhibited in the proof of Theorem 2.14 restricts irreducibly from a member of $\mathrm{Irr}_\Omega(B_0(\widetilde{S}))$ and has degree divisible by p. Further, note $I_A(\varphi)$ contains $A\cap\widetilde{S}$. Then the extension $\hat{\varphi}\in\mathrm{Irr}_\Omega(B_0(I_A(\varphi)))$ from Theorem 2.14 has degree divisible by p and the induced character $\hat{\varphi}^A$ lies in $\mathrm{Irr}_\Omega(B_0(A))$ and has degree divisible by p, as required. □

Proposition 2.20. *The conclusion of Theorem 2.17 holds for S of Lie type in characteristic not p when Sylow p-subgroups of S are abelian or when $p=3$ and $S=\mathrm{PSL}_3(\epsilon q)$ with $q\equiv\epsilon \pmod 3$.*

Proof. In these cases, we will see that the characters with positive height in $B_0(A)$ constructed in [27, Props 3.10 and 3.11] will satisfy our conditions.

Let $\mathbf{G}$ be of simply connected type such that $S=G/\mathbf{Z}(G)$ with $G=\mathbf{G}^F$. We may also identify $\widetilde{S}$ with $\mathbf{G}_{\mathrm{ad}}^F$, with $\mathbf{G}_{\mathrm{ad}}$ the corresponding group of adjoint type.

First, assume that $p \geq 5$. In this case, since Sylow p-subgroups of S are abelian, we have $|\mathbf{Z}(G)|$ is not divisible by p (see e.g. [26, Sec. 2.1]). In [27, Prop. 3.11], a character of $\mathrm{Irr}(B_0(A))$ is constructed with height 1. This is done by constructing a semisimple character $\chi_s \in \mathrm{Irr}_{p'}(B_0(G))$ trivial on $\mathbf{Z}(G)$ and indexed by a p-element $s \in G^*$ that is not invariant under A. Since $|s|$ is a power of p and hence relatively prime to $|\mathbf{Z}(G)|$, as in the proof of Lemma 2.4 we have χ_s is the unique semisimple character in its Lusztig series and, as an element of $\mathrm{Irr}(S)$, is the restriction of a semisimple character $\widetilde{\chi}_{\widetilde{s}}$ of $\widetilde{S}$ with $|\widetilde{s}|$ also of p-power order. Then we have $\widetilde{\chi}_{\widetilde{s}}^{\sigma} = \widetilde{\chi}_{\widetilde{s}}$ for any $\sigma \in \Omega$, since such σ stabilize pth roots of unity. Then we are done by Lemma 2.15(3).

If $p = 2$, then $S = \mathrm{PSL}_2(q)$ with $q \equiv \epsilon 3 \pmod 8$ for $\epsilon \in \{\pm 1\}$, whose group of field automorphisms has odd order. Then $A = \widetilde{S} = \mathrm{PGL}_2(q)$, and again the character of A discussed in [27, Prop. 3.11], labelled by an element of order 4, with even degree $q + \epsilon$, lies in $\mathrm{Irr}_\Omega(B_0(A))$.

Now let $p = 3$; then our assumption S has abelian Sylow 3-subgroups forces $S = \mathrm{PSL}_2(q)$ or $\mathrm{PSL}_3(\epsilon q)$ with $\epsilon \in \{\pm 1\}$.

Assume first that $S = \mathrm{PSL}_3(\epsilon q)$ with $q \equiv -\epsilon \pmod 3$ or $S = \mathrm{PSL}_2(q)$. Here S has cyclic Sylow 3-subgroups. Since by assumption A contains field automorphisms of 3-power order, we have $q = q_0^3$ for q_0 some power of the defining characteristic, and we may consider the field automorphism F_{q_0} to lie in A. Let $\zeta \in \mathbb{F}_{q^2}^\times$ be a 3-element of order dividing $(q^2-1)_3$ but not $(q_0^2-1)_3$. Let $\widetilde{\chi}$ be the semisimple character of $\widetilde{S}$ corresponding to a semisimple element $s \in \mathrm{SL}_3(\epsilon q)$, resp. $\mathrm{SL}_2(q)$ with eigenvalues $\{\zeta, \zeta^{-1}, 1\}$, resp. $\{\zeta, \zeta^{-1}\}$. Then $\widetilde{\chi}|_S$ is irreducible and $\widetilde{\chi}$ is not fixed by F_{q_0}, and hence is not stable under A. However, this character is stable under Ω, since any $\sigma \in \Omega$ fixes 3-power roots of unity. Since $\widetilde{\chi}$ lies in $B_0(\widetilde{S})$ by [16, Cor. 3.4], we are done in this case by Lemma 2.15(3).

Now let $S = \mathrm{PSL}_3(\epsilon q)$ with $q \equiv \epsilon \pmod 3$. Here again we use the same characters in [27, Props 3.10, 3.11]. In this case, A/S is a 3-group. First, assume A contains non-field automorphisms. Note that the three characters of $3'$-degree $\frac{1}{3}(q+\epsilon)(q^2+\epsilon q+1)$ of S take values in $\mathbb{Q}_3$ and hence are Ω-invariant. Further, it is shown in [27, Prop. 3.10] that $[A : I_A(\chi)] = 3$ if χ is one of these characters. Then by the same arguments in Lemma 2.15(3), there is a character of degree divisible by 3 in $\mathrm{Irr}_\Omega(B_0(A))$.

So now assume that A contains only field automorphisms. Again the proof of [27, Prop. 3.10] yields a semisimple character $\chi := \chi_s \in \mathrm{Irr}(B_0(S))$, for s a 3-element, that has degree divisible by 3, is invariant under A, and restricts irreducibly from a semisimple character $\widetilde{\chi}$ of $\widetilde{S}$. Then χ and $\widetilde{\chi}$ are Ω-invariant as before. By [38, Prop. 6.7] and its proof, $\widetilde{\chi}$ extends to an Ω-invariant character of $\widetilde{S}A$, since A is generated by S and field automorphisms. (Indeed, note that letting $A = S\langle F'\rangle$ for a field automorphism F' of 3-power order, the second paragraph of the proof of [38, Thm. 7.4] yields that we may apply the proof of [38, Prop. 6.7] to each $\sigma \in \Omega$. While our Ω is not in the group $\mathcal{H}$ there, since $\widetilde{\chi}$ is indexed by a semisimple 3-element, and therefore stable under Ω, we may still apply the arguments there with $x_\sigma = \sigma\widetilde{t}^{-1}$ in the notation of loc. cit.) Then χ extends to an Ω-invariant character $\hat{\chi}$ of A. Since A/S is a 3-group, $B_0(A)$ is the unique 3-block above $B_0(S)$, and therefore $\hat{\chi}$ is a character with degree divisible by 3 in $\mathrm{Irr}_\Omega(B_0(A))$, as desired. □

Proposition 2.21. *The conclusion of Theorem 2.17 holds for S of Lie type in characteristic not p when Sylow p-subgroups of S are non-abelian.*

Proof. Let S be of Lie type in characteristic $r \neq p$. Notice that when $p \geq 5$, or when $p = 3$ and S is not of type D_4, we will be done by Lemma 2.15(1) if $B_0(\widetilde{S})$ contains a rational unipotent character of degree divisible by p, since unipotent characters restrict irreducibly to S.

If $p \geq 5$, then by the first paragraph of the proof of [23, Thm 2.18] there exists a unipotent character $\chi \in \mathrm{Irr}(B_0(\widetilde{S}))$ of degree divisible by p. By [23, Lemma 2.8] this continues to hold when $p = 3$ except when $S = \mathrm{PSL}_3(\epsilon q)$ with $q \equiv \epsilon \pmod 3$. However, we may assume S is not the latter case, by Proposition 2.20. If S is of classical type, χ is rational, see [10, Cor. 4.4.24]. If S is of exceptional type, by our assumption on p we either have $p = 3$, or $p = 5$ and S is of type E, or $p = 7$ and $S = \mathrm{E}_8(q)$. Assume $p = 3$. Here,

$$\rho_2', \phi_{9,2}, \phi_{81,6}, \phi_{9,6}', \phi_{27,2}, \phi_{567,6}$$

are rational unipotent characters in $B_0(S)$ of degree divisible by 3 for

$$S = {}^2F_4(q^2), F_4(q), E_6(q), {}^2E_6(q), E_7(q), E_8(q)$$

respectively, by [9] and [10, Cor. 4.5.6], and

$$G_2[1], \phi_{2,1}, {}^3D_4[1], \phi_{2,2}$$

are such rational unipotent characters for $G_2(q)$ with $q \equiv 1, -1 \pmod 3$, ${}^3D_4(q)$ with $q \equiv 1, -1 \pmod 3$ respectively. Similarly, for the primes $p = 5, 7$ an inspection of the tables in [3, 13.9] and [9] shows the existence of a rational unipotent character as desired.

If $S = \mathrm{D}_4(q)$ and $p = 3$, the same unipotent character χ of $\widetilde{S}$ considered above in the case of defining characteristic has degree divisible by 3 in this case. Further, choosing χ more specifically with symbol $\binom{2}{2}$, we see this character lies in $B_0(\widetilde{S})$, since the 1-core and 1-cocore of this symbol are both trivial in this case. Then the same considerations as above yield a rational character in $\mathrm{Irr}(B_0(A))$ of degree divisible by p above $\varphi := \chi|_S$. This completes the proof when $p \geq 3$.

Finally, assume $p = 2$. The Ree groups ${}^2\mathrm{G}_2(q^2)$ have abelian Sylow 2-subgroups so do not occur here. Consider the case $S = \mathrm{G}_2(3^n)$ with $n \geq 1$, so that $\mathrm{Aut}(S)$ contains an exceptional graph automorphism τ, and suppose that A contains τF_1 for some (possibly trivial) field automorphism F_1. Then the character $\chi = \phi_{1,3}'$ in the notation of [3, Sec. 13.9] has odd degree, is rational-valued by [10, Cor. 4.5.6], lies in $B_0(S)$ by [9, Thm A], and is fixed by the field automorphisms but not by τ by [25, Thm 2.5]. Then $A \neq I_A(\chi)$, and by Lemma 2.15(3), our statement holds in this case. Then we may assume that A/S is comprised of field automorphisms in the case that $S = \mathrm{G}_2(3^n)$.

Now, assume for the moment that $S \neq \mathrm{PSL}_2(q), \mathrm{PSL}_3(\epsilon q)$. Let χ be a rational-valued, principal series unipotent character in $B_0(S)$ with even degree guaranteed by Proposition 2.13. In the case that $S = \mathrm{PSL}_n(\epsilon q)$ with $n \geq 4$, let λ be the partition $(n-2, 2)$ if $n \equiv 0, 3 \pmod 4$ and $\lambda = (n-2, 1^2)$ if $n \equiv 1, 2 \pmod 4$. Then taking χ more specifically to be the character indexed by λ, we have χ is such a character using the degree formula in [10, Prop. 4.3.2], but also does not satisfy the condition in Lemma 2.10. Then in all relevant cases, χ extends to a rational-valued character of $\widetilde{S}\langle\tau\rangle$ if τ is a non-trivial graph automorphism stabilizing χ, by Lemma 2.10.

If $A/(A\cap\widetilde{S})$ is abelian, note that it must be a 2-group, using our assumption that $\mathbf{O}^{2'}(A)=A$. Then arguing as before, with Lemmas 2.10 and 2.11 and unique common extensions, we obtain an extension $\hat{\varphi}$ of $\varphi:=\chi|_S$ in $\mathrm{Irr}_\Omega(B_0(I_A(\varphi)))$. Again applying [34, Cor. 6.2 and Thm 6.7], we have $\hat{\varphi}^A\in\mathrm{Irr}_\Omega(B_0(A))$ with even degree.

Next, we assume that $A/(A\cap\widetilde{S})$ is non-abelian, so $S=\mathrm{D}_4(q)$. We choose χ in this case to be the unique (unipotent) character of degree $q(q^2+1)^2$, so that χ extends to a rational character of $\mathrm{Aut}(S)$ by the second paragraph of [42, Sec. 4.5], To find a character in $B_0(A)$, note that $A/(A\cap\widetilde{S})\leq X:=\mathfrak{S}_3\times\langle F_0^m\rangle$, where F_0 is a field automorphism of order f and $f_{2'}=m$, for $q=r^f$. Since every character of X is Ω-invariant (as the characters of $\mathfrak{S}_3$ are rational and Ω stabilizes 2-power roots of unity), it follows by Gallagher's theorem that every character of A lying above $\chi':=\chi_{A\cap\widetilde{S}}$ is Ω-invariant. In particular, we know that there is a character above χ' in $B_0(A)$, which therefore has even degree and lies in $\mathrm{Irr}_\Omega(B_0(A))$.

It remains to discuss the groups $\mathrm{PSL}_2(q)$ and $\mathrm{PSL}_3(\epsilon q)$. First, let $S=\mathrm{PSL}_2(q)$. Since Sylow 2-subgroups of S are non-abelian by assumption, we have $q\equiv\pm1\pmod 8$. Here $A/(A\cap\widetilde{S})$ is generated by field automorphisms of 2-power order. Let $s\in\mathrm{SL}_2(q)$ be a 2-element of maximal 2-power order. Then the corresponding Deligne–Lusztig character χ of $\widetilde{S}=\mathrm{PGL}_2(q)$ of degree $q\pm1$ has values in $\mathbb{Q}_{2^f}$ for some $f\geq1$, is not invariant under any field automorphism of 2-power order, and it restricts irreducibly to S. Thus its irreducible induction to A is as required.

Finally, let $S=\mathrm{PSL}_3(\epsilon q)$ with $q\not\equiv\epsilon\pmod 3$ odd, and continue to assume $p=2$. In the first paragraph of [29, Prop. 2.12], a character $\psi\in\mathrm{Irr}_\Omega(B_0(S))$ is constructed such that $I_{\mathrm{Aut}(S)}(\psi)=\widetilde{S}$ and $\psi=\chi_S$ for a semisimple $\chi\in\mathrm{Irr}_\Omega(B_0(\widetilde{S}))$ with even degree corresponding to a suitable 2-element. (We note that, although only a specific $\sigma\in\Omega$ is considered there, χ is Ω-stable for the same reason — that any such Galois automorphism fixes 2-power roots of unity). Then as in loc. cit., the induced character $(\chi_{\widetilde{S}\cap A})^A$ satisfies our statement. □

3. The Galois Itô–Michler theorem

In this section we prove Theorem 4. We will use the Alperin–Dade character correspondence.

Lemma 3.1. *Suppose that N is a normal subgroup of G, with G/N a p'-group. Let $P\in\mathrm{Syl}_p(G)$ and assume that $G=N\mathbf{C}_G(P)$. Then restriction of characters defines a natural bijection between the irreducible characters of the principal p-blocks of G and N. In particular restriction defines a bijection*

$$\mathrm{res}:\mathrm{Irr}_\Omega(B_0(G))\to\mathrm{Irr}_\Omega(B_0(N)).$$

Proof. The case where G/N is solvable was proved in [1] and the general case in [5]. We prove the last assertion. It is clear that if $\chi\in\mathrm{Irr}_\Omega(B_0(G))$, then $\chi_N\in\mathrm{Irr}_\Omega(B_0(N))$. Conversely, let $\theta\in\mathrm{Irr}_\Omega(B_0(N))$ and let $\chi\in\mathrm{Irr}(B_0(G))$ such that $\chi_N=\theta$. Let $\tau\in\Omega$. Then, since τ acts on $\mathrm{Irr}(B_0(G))$, we have $\chi^\tau\in\mathrm{Irr}(B_0(G))$ and $(\chi^\tau)_N=\theta^\tau=\theta=\chi_N$. We conclude that $\chi=\chi^\tau$. Hence $\chi\in\mathrm{Irr}_\Omega(B_0(G))$, as wanted. □

If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, we denote by $\mathrm{Irr}(G|\theta)$ the set of irreducible characters of G lying over θ, and by $\mathrm{cd}(G|\theta)$ the set of degrees of the characters in $\mathrm{Irr}(G|\theta)$.

Lemma 3.2. *Suppose that N is a normal subgroup of G, with G/N a p'-group. Let $\Omega = \Omega(|G|)$. If $\theta \in \mathrm{Irr}_\Omega(N)$ then there exists $\chi \in \mathrm{Irr}_\Omega(G)$ over θ. Furthermore, if $\theta \in \mathrm{Irr}_\Omega(B_0(N))$ then there exists $\chi \in \mathrm{Irr}_\Omega(B_0(G))$ over θ.*

Proof. We argue by induction on $|G : N|$. Suppose that $G_\theta < G$. By the inductive hypothesis there exists $\psi \in \mathrm{Irr}_\Omega(G_\theta|\theta)$. By Clifford's correspondence, $\chi = \psi^G$ is irreducible and by the formula for the induced character $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\psi)$ so χ is Ω-invariant. Moreover, if $\psi \in \mathrm{Irr}(B_0(G_\theta))$, then $\chi \in \mathrm{Irr}(B_0(G))$ by [34, Cor. 6.2 and Thm 6.7]. Hence, we may assume that θ is G-invariant in both statements.

For $d \geq 1$ set $\mathrm{Irr}_d(G|\theta) := \{\chi \in \mathrm{Irr}(G|\theta) \mid \chi(1) = d\}$. Since θ is Ω-invariant, Ω acts on $\mathrm{Irr}_d(G|\theta)$ for every d. Since

$$|G : N| = \sum_{\chi \in \mathrm{Irr}(G|\theta)} \left(\frac{\chi(1)}{\theta(1)}\right)^2 = \sum_d |\mathrm{Irr}_d(G|\theta)| \frac{d^2}{\theta(1)^2}$$

is a p'-number, we conclude that there exists $d \in \mathrm{cd}(G|\theta)$ such that $|\mathrm{Irr}_d(G|\theta)|$ is not divisible by p. Since $|\Omega|$ is a power of p, it follows that there exists $\chi \in \mathrm{Irr}_d(G|\theta)$ that is Ω-invariant.

Now assume that $\theta \in \mathrm{Irr}_\Omega(B_0(N))$. If G/N is not simple, let $M \trianglelefteq G$ with $N < M < G$. By induction there is $\psi \in \mathrm{Irr}_\Omega(B_0(M))$ lying over θ. Again by induction, there is $\chi \in \mathrm{Irr}_\Omega(B_0(G))$ lying over ψ (and hence over θ). Hence we may assume that G/N is simple.

Now, let $P \in \mathrm{Syl}_p(G)$ and notice that $P \leq N$. By the Frattini argument, $G = N\mathbf{N}_G(P)$ and hence $M := N\mathbf{C}_G(P) \trianglelefteq G$. Since G/N is simple we have that $M = N$ or $M = G$. If $M = G$, by Lemma 3.1 there exists $\chi \in \mathrm{Irr}_\Omega(B_0(G))$ such that $\chi_N = \theta$, and we are done. Hence we may assume that $M = N$. In this case, $\mathbf{C}_G(P) \leq N$ and by [27, Lemma 4.2] we have that $B_0(G)$ is the only block covering $B_0(N)$. By the first part of the proof, there is $\chi \in \mathrm{Irr}_\Omega(G|\theta)$, hence $\chi \in \mathrm{Irr}_\Omega(B_0(G)|\theta)$ and we are done. □

The following is the p-group case of Theorem 4.

Lemma 3.3. *Let p be a prime number and let P be a p-group. Then*

$$\mathrm{Irr}(P) = \mathrm{Irr}_\Omega(P).$$

Proof. By the definition of Ω, every $\tau \in \Omega$ fixes the p-power roots of unity, so the result is clear. □

We need the following technical lemma.

Lemma 3.4. *Let p and q be distinct primes. Suppose that $G = VP$, where $P > 1$ is an abelian p-group, V is an elementary abelian q-subgroup of G, minimal normal in G, and $\mathbf{C}_P(V) = 1$. Suppose that p divides $q - 1$. Then there exists $\chi \in \mathrm{Irr}(G)$ of degree divisible by p such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_q$ and p divides $|\mathbb{Q}_q : \mathbb{Q}(\chi)|$.*

Proof. Since V is a faithful irreducible P-module, and $\mathbf{C}_V(x) \trianglelefteq G$ for every $x \in P$, we have that $\mathbf{C}_V(x) = 1$ for every $1 \neq x \in P$. Thus G is a Frobenius group with abelian complement P. Let $h \in P$ be an element of order p and put $H = \langle h \rangle$.

Now, write $V = V_1 \oplus \cdots \oplus V_t$ as a direct sum of irreducible H-modules. Since p divides $q-1$, the field $\mathbb{F}_q$ contains primitive pth roots of unity, so all irreducible H-modules have dimension 1. Since $\mathbf{C}_P(V) = 1$, there exists i and $\lambda \in \mathrm{Irr}(V_i)$ that is not H-invariant by [32, Prop. 12.1]. Write $V = U \oplus W$, so that $U = V_i$ and $W \trianglelefteq VH$. We view λ as a character of V/W (hence of V) that induces irreducibly to $\mu = \lambda^{VH} \in \mathrm{Irr}(VH)$. Let $U = \langle v \rangle$, so that $\lambda(v) = \varepsilon$ for some primitive qth root of unity ε (note that since λ is not H-invariant $\lambda(v) \neq 1$). Since H does not act trivially on U, $v^{h^{-1}} = v^i$ for some i that has order p modulo q. Let $\tau \in \mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q})$ be the Galois automorphism such that $\tau(\varepsilon) = \varepsilon^i$, so that $\lambda^h(v) = \lambda(v)^\tau$. Note that τ has order p since $\tau^p(\varepsilon) = \varepsilon^{i^p} = \varepsilon$. Note that

$$\mu_V = \sum_{j=0}^{p-1} \lambda^{g^j}$$

so that

$$\mu(v) = \varepsilon + \varepsilon^i + \cdots + \varepsilon^{i^{p-1}} = \varepsilon + \varepsilon^\tau + \cdots + \varepsilon^{\tau^{p-1}} = \mu^\tau(v).$$

Thus μ is τ-invariant and hence $\mathbb{Q}(\mu)$ is τ-invariant, that is, $\tau \in \mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}(\mu))$. Since τ has order p, the fundamental theorem of Galois theory implies that $|\mathbb{Q}_q : \mathbb{Q}(\mu)|$ is divisible by p.

Since G is a Frobenius group, $\chi = \lambda^G \in \mathrm{Irr}(G)$ (so p divides $\chi(1)$). Note that $\mathbb{Q}(\chi) = \mathbb{Q}(\chi_V)$ (because χ vanishes off V). By Clifford's theorem, χ_{VH} is a sum of conjugates of μ. Therefore,

$$\mathbb{Q}(\chi) = \mathbb{Q}(\chi_{VH}) \subseteq \mathbb{Q}(\mu).$$

It follows that $|\mathbb{Q}_q : \mathbb{Q}(\chi)|$ is divisible by p, as wanted. □

Given a group G, we write $\mathbf{O}_p(G)$ to denote the largest normal p-subgroup of G. Analogously $\mathbf{O}_{p'}(G)$ is the largest normal p'-subgroup of G. The following is Theorem 4:

Theorem 3.5. *Suppose that G does not have composition factors isomorphic to S with (S,p) one of the pairs listed in Theorem 2.8. Then all characters in $\mathrm{Irr}_\Omega(G)$ have p'-degree if and only if G has a normal abelian Sylow p-subgroup.*

Proof. The "if" direction is clear, so we prove the "only if" direction. Let P be a Sylow p-subgroup of G.

Step 1. If P is normal in G, then P is abelian.

Let $\theta \in \mathrm{Irr}(P)$. By Lemma 3.3 we have that $\theta \in \mathrm{Irr}_\Omega(P)$. By Lemma 3.2, there exists $\chi \in \mathrm{Irr}_\Omega(G)$ over θ. By hypothesis, χ has p'-degree. Since $\chi(1)$ is a multiple of $\theta(1)$, this implies that $\theta(1)$ is p' and hence θ is linear. Thus P is abelian.

In the following we want to prove that P is normal in G. Let G be a minimal counterexample.

Step 2. If $1 < N \trianglelefteq G$, G/N has normal and abelian Sylow p-subgroups. In particular, there is a unique minimal normal subgroup of G and $\mathbf{O}_p(G) = 1$.

Since $\mathrm{Irr}_\Omega(G/N) \subseteq \mathrm{Irr}_\Omega(G)$, the hypothesis is inherited by G/N. By the minimality of G as a counterexample, G/N has a normal and abelian Sylow p-subgroup. Now, if M, N are distinct minimal normal subgroups, G is isomorphic to a subgroup of $G/N \times G/M$, and hence G has a normal and abelian Sylow subgroup, as wanted. Finally, if $1 < N = \mathbf{O}_p(G)$, P/N is normal in G, so P is normal in G, and we are done.

Step 3: We have that $G = \mathbf{O}^{p'}(G)$. In particular, if N is the unique minimal normal subgroup of G, then $G = PN$.

Let $L = \mathbf{O}^{p'}(G)$. Suppose that $L < G$. Let $\theta \in \mathrm{Irr}_\Omega(L)$. By Lemma 3.2, there exists $\chi \in \mathrm{Irr}_\Omega(G)$ over θ. By hypothesis, $\chi(1)$ is p'. We conclude that all characters in $\mathrm{Irr}_\Omega(L)$ have p'-degree. By induction, L has a normal Sylow p-subgroup and hence the same holds for G. Thus, we may assume that $G = \mathbf{O}^{p'}(G)$. By Step 2, PN/N is normal in G/N, so PN is normal in G, which forces $G = PN$.

Step 4: Let N be the unique minimal normal subgroup of G. Then N is a direct product of non-abelian simple groups.

Suppose that N is elementary abelian. By Step 2, N is a q-group for some prime $q \neq p$ and by Step 3, $G = PN$. Notice that $P \cong G/N$ is abelian by Step 2. Then $\mathbf{C}_P(N)$ is normal in G, so $\mathbf{C}_P(N) = 1$ again by Step 2. Note that $\Omega(|G|)$ is cyclic. If p does not divide $q-1$ then Ω is trivial and we finish the proof using the Itô–Michler theorem. Hence, we may assume that p divides $q-1$, so that Ω is cyclic of order p. By Lemma 3.4, there exists $\chi \in \mathrm{Irr}(G)$ of degree divisible by p and such that p divides $|\mathbb{Q}_q : \mathbb{Q}(\chi)|$. Then there is $\tau \in \mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}(\chi))$ of order p. It follows that Ω is generated by an extension of τ, whence $\chi \in \mathrm{Irr}_\Omega(G)$ has degree divisible by p, a contradiction.

Step 5: Completion of the proof.

Let N be the unique minimal normal subgroup of G. By Step 4, we know that N is a direct product of isomorphic non-abelian simple groups. Write $N = S_1 \times \cdots \times S_t$, where $S_i \cong S$ is non-abelian simple. By Theorem 2.8, we may assume that $t > 1$. Write $H = \bigcap_{i=1}^t \mathbf{N}_G(S_i)$ so that G/H is isomorphic to a transitive permutation group on t letters and $N \leq H \leq \mathrm{Aut}(S) \times \cdots \times \mathrm{Aut}(S)$. By Step 2 and Step 3, G/H is an abelian p-group. Therefore, all point stabilizers are trivial.

By Theorem 2.14, there exists $1_S \neq \varphi \in \mathrm{Irr}(S)$ that extends to an Ω-invariant character of its inertia group in $\mathrm{Aut}(S)_p$, where $\mathrm{Aut}(S)_p/S$ is a Sylow p-subgroup of $\mathrm{Aut}(S)/S$. Since $G_\nu \leq H$, $\nu = \varphi \times 1_S \times \cdots \times 1_S$ extends to an Ω-invariant character $\tilde{\nu}$ of $G_\nu < G$ (using also Step 3). Hence, since $G/H \neq 1$, $\tilde{\nu}^G \in \mathrm{Irr}(G)$ has degree divisible by p and is Ω-invariant. This contradicts the hypothesis. □

The following is the p-solvable case of Theorem 3.

Corollary 3.6. *Let p be a prime. Let G be p-solvable and let $P \in \mathrm{Syl}_p(G)$. Then all characters in $\mathrm{Irr}_\Omega(B_0(G))$ have height zero if and only if P is abelian.*

Proof. Since G is p-solvable, $\mathrm{Irr}(B_0(G)) = \mathrm{Irr}(G/\mathbf{O}_{p'}(G))$. Now, the result follows from Theorem 3.5 and Hall–Higman's Lemma 1.2.3. □

Remark 3.7. We remark that if p is odd, our arguments can be adapted to show that we can replace Ω by $\mathcal{P} = \{\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}) \mid o(\sigma) = p\}$ in Theorem 3.5 and Corollary 3.6 (using that the field of values of an irreducible character of an odd order p-group is a full cyclotomic field). This is not possible, however, in Theorem 3. Fix $p = 3$ and let $G = X \star Y$ be the central product of a cyclic group X of order 9 and the 3-fold cover Y of the alternating group on 6 letters. Note that G does not have abelian Sylow 3-subgroups. We claim that every character in $\mathrm{Irr}_{\mathcal{P}}(B_0(G))$ has $3'$-degree. We can check in [4] that every character of degree divisible by 3 of $B_0(Y)$ has field of values of 2-power degree. Hence, all of these characters are invariant under a Sylow 3-subgroup of the Galois group. Hence, $B_0(G) = B_0(X) \star B_0(Y)$ (see [29, Lemma 4.1]) does not have $\mathcal{P}$-invariant characters of degree divisible by 3, as claimed.

We conclude this section with a theorem that generalizes results of Dolfi–Navarro–Tiep [7, Thm A], who consider the case that σ is complex conjugation, as well as the $p = 2$ case of Grittini [14, Thm A], which considers the same statement for p-solvable groups (there for an arbitrary prime p). This is also part of [15, Thm A].

Theorem 3.8. *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ have order 2 and let G be a finite group of order dividing n. If every $\chi \in \mathrm{Irr}(G)$ fixed by σ has odd degree, then G has a normal Sylow 2-subgroup.*

Proof. We proceed by induction on $|G|$. Using [14, Prop. 2.3] and following Steps 1–2 of the proof of [14, Thm A], we see that we may assume $G = \mathbf{O}^{2'}(G) = NP$, where N is the unique minimal normal subgroup of G and P is a Sylow 2-subgroup of G.

Suppose first that N is abelian. Then the result follows from [14, Thm A].

Hence, we assume that $N = S^{x_1} \times \cdots \times S^{x_t}$ is the product of conjugates of a non-abelian simple subgroup S and let $H := \mathbf{N}_G(S)/\mathbf{C}_G(S)$. Now, by [42, Thm C] and Theorem 2.8, there is a σ-invariant character $\chi \in \mathrm{Irr}(H)$ of even degree, and such that there is some non-trivial $\theta \in \mathrm{Irr}(S)$ lying under χ. From here, we argue exactly as in the proof of [7, Thm A]. Namely, taking $\eta := \theta \times 1_{S^{x_2}} \times \cdots \times 1_{S^{x_t}} \in \mathrm{Irr}(N)$ we have $I_G(\eta) \leq H$. Since χ (now viewed as the inflation to $\mathbf{N}_G(S)$) lies over η, we obtain that the induced character χ^G is a σ-invariant, irreducible character of even degree, a contradiction. □

4. The Galois height zero conjecture

We assume that the reader is familiar with the basic properties of the generalized Fitting subgroup $\mathbf{F}^*(G)$ (see e.g. [20, Sec. 9A]). We write $\mathbf{E}(G)$ to denote the layer of G and $\mathbf{F}(G)$ to denote the Fitting subgroup of G.

Now, we prove Theorem 2 and Theorem 3. We handle both of them simultaneously.

Theorem 4.1. *Let G be a finite group, let p be a prime number and let $P \in \mathrm{Syl}_p(G)$. Suppose that G does not have composition factors isomorphic to S with (S, p) one of the pairs listed in Theorem 2.8. Then all characters in $\mathrm{Irr}_\Omega(B_0(G))$ have height zero if and only if P is abelian.*

Proof. The "if" part follows from the "if" part of Brauer's height zero conjecture, proved in [22]. We prove the "only if" part. Let P be a Sylow p-subgroup of G. We want to see that P is abelian. Let G be a minimal counterexample.

Step 1: We have that $\mathbf{O}_{p'}(G) = 1$. Furthermore, for every $1 < N \trianglelefteq G$, G/N has abelian Sylow p-subgroups and G has a unique minimal normal subgroup.

Put $M = \mathbf{O}_{p'}(G)$ and assume that $M > 1$. Since $\mathrm{Irr}_\Omega(B_0(G/M)) \subseteq \mathrm{Irr}_\Omega(B_0(G))$, all the characters in $\mathrm{Irr}_\Omega(B_0(G/M))$ have p'-degree. By the minimality of G as a counterexample, G/M and hence G, have abelian Sylow p-subgroups. The second part is analogous. For the last part, suppose that N, M are minimal normal subgroups of G. Then G is isomorphic to a subgroup of $G/N \times G/M$ and the conclusion holds.

Step 2: We have that $G = \mathbf{O}^{p'}(G)$.

Write $M = \mathbf{O}^{p'}(G)$ and assume that $M < G$. Let $\psi \in \mathrm{Irr}_\Omega(B_0(M))$. By Lemma 3.2, there exists $\chi \in \mathrm{Irr}_\Omega(B_0(G))$ over ψ of p'-degree. Hence ψ has p'-degree. By the minimality hypothesis M has abelian Sylow p-subgroups and hence the same holds for G.

In the following, let N be the unique minimal normal subgroup of G. Set $K/N = \mathbf{O}_{p'}(G/N)$, so that by Step 1, Step 2 and [27, Thm 4.1], $G/K = X/K \times Y/K$, where X/K is an abelian p-group and Y/K is a direct product of non-abelian simple groups of order divisible by p with abelian Sylow p-subgroups.

We first consider the case when N is an elementary abelian p-group. If $Y = K$, G is p-solvable, so it has just one p-block by Step 1. In this case the result holds by Corollary 3.6. So we may assume that $Y > K$.

Step 3: Let $M = \mathbf{C}_G(N)$. Then G/M is an abelian p-group.

Let Q be a Sylow p-subgroup of M, so that $\mathbf{C}_G(Q) \subseteq \mathbf{C}_G(N) = M$ (note that since N is a normal p-subgroup of M, $N \subseteq Q$). By [27, Lemma 4.2], $\mathrm{Irr}(G/M) \subseteq \mathrm{Irr}(B_0(G))$, so $\mathrm{Irr}_\Omega(G/M) \subseteq \mathrm{Irr}_\Omega(B_0(G))$. By hypothesis, this implies that all characters in $\mathrm{Irr}_\Omega(G/M)$ have p'-degree, then by Theorem 3.5 and Step 2 we obtain that $G/M = PM/M$ is abelian.

Step 4: We have that $K > N$.

Suppose that $K = N$, so $G/N = X/N \times Y/N = X/N \times S_1/N \times \cdots \times S_t/N$, where S_i/N is non-abelian simple of order divisible by p for every $i = 1, \ldots, t$. Now, $X \geq N$ is a p-group, so $1 < \mathbf{Z}(X) \lhd G$ and then $N \subseteq \mathbf{Z}(X)$. Hence $X \subseteq \mathbf{C}_G(N) = M$. Since $G/X \cong Y/N$ does not have normal subgroups of p-power index, we necessarily have that $M = G$. Then $N \subseteq \mathbf{Z}(G)$ and $|N| = p$. Now G is the central product of X with $S_1, \ldots, S_t$, where $Y/N \cong S_1/N \times \cdots \times S_t/N$. Since N is the unique minimal normal subgroup of G, $N \subseteq S_i'$. Hence all S_i are perfect, so S_i is quasi-simple with centre N, for every i. Let $1_N \neq \lambda \in \mathrm{Irr}(N)$. By Theorem 2.9 there exists $\psi_i \in \mathrm{Irr}_{\Omega(|S_i|)}(B_0(S_i)) = \mathrm{Irr}_\Omega(B_0(S_i))$ lying over λ. (If necessary, replacing ψ_i by a Galois conjugate.) Now, let $\xi \in \mathrm{Irr}(X|\lambda)$, then $\xi \in \mathrm{Irr}_\Omega(B_0(X)|\lambda)$. By [29, Lemma 4.1] the central product of characters

$$\chi = \xi \star \psi_1 \star \psi_2 \star \cdots \star \psi_t$$

lies in the principal block of G. Hence $\chi \in \mathrm{Irr}_\Omega(B_0(G))$ has degree divisible by p, which contradicts the assumption $K = N$.

Step 5: We have that $\mathbf{F}(G) = \mathbf{F}^*(G)$.

In this step we use arguments from the proof of Theorem 4.6 of [27]. By Step 1 and the assumption that N is abelian, $F = \mathbf{F}(G) = \mathbf{O}_p(G) > 1$. Suppose that $E = \mathbf{E}(G) > 1$ and let $Z = \mathbf{Z}(E)$. Since N is the unique minimal normal subgroup of G, $N \subseteq Z$ (notice that $Z > 1$ since otherwise $\mathbf{F}^*(G) = \mathbf{F}(G) \times E$ in contradiction to Step 1). We claim that $E/Z = S_1/Z \times \cdots \times S_n/Z$, where $S_i \trianglelefteq G$ for every i. Let W/Z be a non-abelian chief factor of G/Z contained in E/Z. By the Schur–Zassenhaus theorem and Step 1, we know that $|W/Z|$ is divisible by p. Now, by [27, Thm 4.1] applied to G/Z, we have that

W/Z is simple and the claim follows. Write $S = S_1$, so that S' is a quasi-simple normal subgroup of G. Using again that N is the unique minimal normal subgroup of G, we have that $N \subseteq \mathbf{Z}(S) \cap S' \subseteq \mathbf{Z}(S')$. Looking at the Schur multipliers of the simple groups [4], if $p \geq 5$, we deduce that $\mathbf{Z}(S')$ has cyclic Sylow p-subgroups. Arguing as in Step 4 of the proof of Theorem 4.6 of [27] we have that $\mathbf{Z}(S')$ has cyclic Sylow p-subgroups for $p = 2, 3$ as well.

In all cases, we conclude that N is cyclic and hence $|N| = p$. Now, the order of $G/\mathbf{C}_G(N)$ divides $p-1$. By Step 2, $G = \mathbf{C}_G(N)$, so N is central in G. Thus K is the direct product of N and a p-complement H. Since H is normal in G, we have a contradiction with Step 1. Hence $E = 1$ and $\mathbf{F}(G) = \mathbf{F}^*(G)$ as wanted.

Step 6: We have that $N = \mathbf{F}(G) = \mathbf{F}^*(G)$.

Let H be a p-complement of K, so $K = HN$ and $H \cap N = 1$. Then by the Frattini argument and the Schur–Zassenhaus theorem, we have $G = N\mathbf{N}_G(H)$. Write $L = \mathbf{N}_G(H)$. Now, since N is abelian and normal in G, $\mathbf{N}_N(H)$ is normal in $G = N\mathbf{N}_G(H) = NL$, and hence $\mathbf{N}_N(H) = 1$ or $\mathbf{N}_N(H) = N$. If $\mathbf{N}_N(H) = N$, then $H \lhd G$, and we get a contradiction since $\mathbf{O}_{p'}(G) = 1$ by Step 1. Thus, $L \cap N = \mathbf{N}_N(H) = 1$ and L is a complement of N in G.

Let $F = \mathbf{F}(G) = \mathbf{O}_p(G)$. We will show that $F = N$. Notice that since $\mathbf{Z}(F) > 1$, we have $N \subseteq \mathbf{Z}(F)$. Let $F_1 = F \cap L$. Then $F_1 \lhd L$ and since $G = NL$, we have $F_1 \lhd G$. Since N is the unique minimal normal subgroup, this forces $F_1 = 1$, so $F = N$ as wanted. Since $\mathbf{F}(G) = \mathbf{F}^*(G)$, the claim follows, and the proof of the Step is completed.

Step 7: Completion of the proof when N is elementary abelian.

Step 6 implies that $\mathbf{C}_G(N) = N$. By Step 3, G/N is a p-group, but then G is a p-group and we are done since $\mathrm{Irr}_\Omega(B_0(G)) = \mathrm{Irr}(G)$ in this case (see Lemma 3.3).

Hence, from now on we may assume that N is a direct product of non-abelian simple groups. Write $N = S_1 \times \cdots \times S_t$, where $S_i \cong S$ is non-abelian simple. By Theorem 2.17, we may assume $t > 1$. Write $H = \bigcap_{i=1}^t \mathbf{N}_G(S)$ so that G/H is isomorphic to a transitive permutation group on t letters and $N \leq H \leq \mathrm{Aut}(S) \times \cdots \times \mathrm{Aut}(S)$.

Step 8: We have that $\mathbf{C}_G(P) \subseteq H$. In particular, G/H is a non-trivial p-group and G/N has a normal p-complement K/N.

Let $R \in \mathrm{Syl}_p(S)$, so that $Q = R \times \cdots \times R \subseteq P$ is a Sylow p-subgroup of N. Since $\mathbf{O}_{p'}(G) = 1$, $R > 1$. Let $g \in G - H$. Since g permutes the copies of S, we may assume without loss of generality that g does not centralize $(x, 1, \ldots, 1)$, where $1 \neq x \in R$. The first part follows.

Now, using [27, Lemma 4.2] we have $\mathrm{Irr}_\Omega(G/H) \subseteq \mathrm{Irr}_\Omega(B_0(G))$. By hypothesis, p does not divide the degree of any character in $\mathrm{Irr}_\Omega(G/H)$. It follows from Theorem 3.5 that G/H has a normal abelian Sylow p-subgroup. Since $\mathbf{O}^{p'}(G) = G$, we conclude that G/H is a (non-trivial) p-group, as desired.

Finally, we prove the third claim. Since H/N is isomorphic to a subgroup of $\mathrm{Out}(S)^t$, it follows from Schreier's conjecture that H/N is solvable. Since G/H is also solvable, we conclude that G/N is solvable. By Step 1, G/N has abelian Sylow p-subgroups.

Therefore, by Hall–Higman's Lemma 1.2.3 applied to G/K and the fact that $\mathbf{O}^{p'}(G) = G$, we conclude that G/N has a normal p-complement K/N.

Step 9: We have that $G = NP$.

In this step we follow the arguments from [29]. Let $Q = P \cap N \in \mathrm{Syl}_p(N)$. By the Frattini argument, $G = N\mathbf{N}_G(Q)$. Therefore, $M = N\mathbf{C}_G(Q) \trianglelefteq G$. By [27, Lemma 4.2], all irreducible characters of G/M belong to $B_0(G)$. By Theorem 4, G/M has a normal Sylow p-subgroup. Since $\mathbf{O}^{p'}(G) = G$, we conclude that G/M is a p-group. Hence $K \subseteq M = N\mathbf{C}_G(Q)$. Therefore, $K = N\mathbf{C}_K(Q)$.

Let $\alpha \in \mathrm{Irr}_\Omega(B_0(NP))$. We want to show that α has p'-degree. Let $\theta \in \mathrm{Irr}(N)$ under α, so θ belongs to the principal p-block of N. By Alperin's isomorphic blocks (Lemma 3.1), there exists a unique extension η of θ in $B_0(K)$. Let $I = G_\eta$, so that $J = I \cap PN = (PN)_\theta$ (using uniqueness in Alperin's theorem). Let $\mu \in \mathrm{Irr}(J)$ be the Clifford correspondent of α over θ. By the Isaacs restriction correspondence ([36, Lemma 6.8]), let $\rho \in \mathrm{Irr}(I|\eta)$ be such that $\rho_J = \mu$. By the Clifford correspondence, $\chi = \rho^G \in \mathrm{Irr}(G)$ lies in the principal p-block of G (because η does and G/K is a p-group). Now, let $\sigma \in \Omega$. Since α is σ-invariant, we have $\theta^\sigma = \theta^{g_\sigma}$ for some $g_\sigma \in P$ by Clifford's theorem. By the uniqueness in the Alperin–Dade correspondence we have that, $\eta^{g_\sigma} = \eta^\sigma$. By uniqueness in the Clifford correspondence and the Isaacs correspondence we obtain that $\mu^\sigma = \mu^{g_\sigma}$, $\rho^\sigma = \rho^{g_\sigma}$ and $\chi^\sigma = \chi$. Since this occurs for every $\sigma \in \Omega$, we conclude that $\chi \in \mathrm{Irr}_\Omega(B_0(G))$. By hypothesis, χ has p'-degree. Thus $I = G$, $\chi = \rho$ and $\chi_{PN} = \alpha$ has p'-degree as wanted. By the minimality of G as a counterexample, we may assume that $G = NP$, as claimed.

Step 10: Completion of the proof.

By Theorem 2.14, there exists $1_S \neq \varphi \in \mathrm{Irr}_\Omega(B_0(S))$ that extends to an Ω-invariant character of the principal p-block of its inertia group in $\mathrm{Aut}(S)_p$, where $\mathrm{Aut}(S)_p/S$ is a Sylow p-subgroup of $\mathrm{Aut}(S)/S$. Therefore, $\nu = \varphi \times 1_S \times \cdots \times 1_S \in \mathrm{Irr}(B_0(N))$ extends to an Ω-invariant character $\tilde{\nu}$ of the principal p-block of $G_\nu \subseteq H$ (using again that G/H is an abelian transitive permutation group). Hence, $\tilde{\nu}^G \in \mathrm{Irr}(B_0(G))$ has degree divisible by p and is Ω-invariant. This contradicts the hypothesis. □

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