FUSION SYSTEMS AND THE MARTINO-PRIDDY CONJECTURE

Seminar for the Group and Representation Theory group at the AGAG-RPTU
Sommersemester 2025

Functorial methods, in particular methods of homological and homotopical algebra, have often found some resistance when applied to finite group theory. A reason behind this behavior may simply lie in the intrinsic suboptimal properties of the category of groups (and of its subcategory of finite groups) with respect to such tools: for example, applications have earlier and more successfully reached the theory of representations of groups helped by the fact the category of R-modules is abelian for any (also non-commutative) unitary ring R, including in particular the case of a group algebra $R = KG$ of some group G with respect to a field K, a property which is not shared by the entire category of groups.
In the realm of finite group theory, what is sometimes called *local group theory* (i.e. the study of normalizers of non-trivial p-subgroups of a group) is arguably the most important collection of investigative tools that has been developed. Aim of this seminar consists in providing the background necessary for a comprehensive understanding of the key points that led to the modern proof of the Martino-Priddy conjecture, which constitutes the most striking and high-reaching connection between the p-local structure of finite groups and the homotopy theory of topological spaces.
It should be noted that a first proof of the conjecture was completed in 2006 (see [11] and [12]), but this heavily relied on the classification of the finite simple groups. A successive collective effort led to the modern proof, which is based on a very different argument and is deeply rooted in the theory of fusion systems. Through the work of Broto, Levi and Oliver [3], [4], [13], of Chermak [5] and of Glauberman and Lynd [7], a proof of a more general statement of the conjecture, independent on the classification of the finite simple groups, was eventually achieved.

IMPORTANT DATES

28^{th} April, 2025	Talk 0: *Brief introduction to the Martino-Priddy conjecture*, E. Salati
5^{th} May, 2025	Talk 1: *Homotopy theory*, M. Albert
12^{th} May, 2025	Talk 2: *Relationship between homotopy theory and group theory*, D. Rossi
19^{th} May, 2025	Talk 3: *Fusion in groups*, G. Malle
26^{th} May, 2025	Talk 4: *Realizability*, A. Bartelt
23^{rd} June, 2025	Talk 5: *Insights on the proof of the Martino-Priddy conjecture I*, E. Salati (?)
30^{th} June, 2025	Talk 6: *Insights on the proof of the Martino-Priddy conjecture II*, E. Salati

0. Brief introduction to the Martino-Priddy conjecture

15-20 minutes talk for introducing the objects involved in the Martino-Priddy conjecture and its importance in connecting the theory of finite groups with homotopy theory.

1. Homotopy theory

Throughout we will refer to [8] if not differently specified.

1.1: **Homotopy and the homotopical category of topological spaces.**
Definitions of homotopic maps and homotopy equivalent spaces on page 3.
Definition of CW-complexes on page 5; more properties in the Appendix.

1.2: **Homotopy groups, weak homotopy equivalence and Whitehead's Theorem.**
Pages 25-26 for the definition of the fundamental group. Pages 340-341 for higher homotopy groups.
Pages 352-353 for weak homotopy equivalences and CW-approximation; Theorem 4.5 on page 246 for Whitehead's Theorem.
Alternatively, [6, Theorem 3.39], where (i) is Whitehead's Theorem and (ii) is CW-approximation.

1.3: **Simplicial sets and their use in homotopy theory.**
Definition and examples: [6], pages 56-58; naive construction of a sphere as a simplicial set $\Delta^2/\partial\Delta^2$ and/or construction of the simplicial set for the projective real plane.

2. Relationship between homotopy theory and group theory

2.1: **The nerve and geometric realization functors.**
[6, Definition 3.8] for the nerve of a small category; page 63 and [6, Definition 3.17] for the geometric realization of $|\Delta^n|$ and of a generic simplicial set.
[6, Proposition 3.21 and Corollary 3.22] as interesting consequences.

2.2: **The classifying space of a group.**
[6, Definition 3.25 and 3.27] and [6, Proposition 3.37].
IMPORTANT: Groups as unique algebraic models of Eilenberg-Mac Lane spaces (in the more restrictive sense of CW-complexes) of the form $K(G,1)$, up to homotopy.

2.3: **The Bousfield-Kan p-completion functor.**
[1, III:1.4]: provide with properties of p-completions, until Proposition 1.11.
The construction of the space may be found in [6, Section 3.4] or in the original version in [2], but it is out of the scope of this seminar.

2.4: **The Martino-Priddy conjecture (original statement)**
This can be found in [11], page 2, (stated for odd primes, but the conjecture was formulated, and holds, for all primes).

3. Fusion in groups

If not differently specified, we refer to [6].

3.1: **Control of fusion in groups.**
- **Burnside's Theorem.** Definition 1.2 and Theorems 1.1, 1.4 and 1.5.
- **Fusion categories in groups.** Definition 1.7 and Proposition 1.9, Corollary 1.10.
- **Frobenius normal p-complement Theorem.** Theorems 1.12, 1.13.

3.2: **Fusion systems.** Definition 1.34, 1.36 and 1.37 (saturation; compare with Definition 4.11. On page 98 down to Proposition 4.10 one may find an exhaustive explanation of the saturation axioms and their meaning for fusion systems). Proposition 1.38 and Theorem 1.39.

3.3: **Statement of the Martino-Priddy conjecture with fusion systems.**
Theorem 3.1.
IMPORTANT: the geometric realization of the fusion system has the wrong homotopy type!

4. Realizability

4.1: **The universal fusion system over a p-group and subsystems.**
[6, Definitions 4.2 and 4.3].

4.2: **Normal and centric subgroups; constrained fusion systems.**
[1, I.4.1 and I.4.3] for normal subgroups and for $O_p(\mathcal{F})$; [1, I.4.8] for constrained fusion systems and models; [1, I.Theorem 4.9] for the Model Theorem.

4.3: **The Leary-Stancu groups.**
[9, Theorem 2](Follows from properties of HNN extensions)

4.4: **The Park groups.**
[14, Theorem 1]; here one can provide good insight of the construction and of the structure of the realizing group.

5. Insights on the proof of the Martino-Priddy conjecture I

5.1: **Transporter categories for groups and the centric linking system.**
[1, III.3.1] and [1, III.Lemma 2.3] as motivation for the centric linking system. [1, III.Theorem 1.2].

5.2: **A generalization of Martino-Priddy.**
The statement of the Martino-Priddy conjecture generalized to all saturated fusion systems.

5.3: **Abstract centric linking systems: existence and uniqueness problems.**
[1, III.4.1] (not necessarily in full detail!)
IMPORTANT: Uniqueness for centric linking systems coming from finite groups; existence and uniqueness for centric linking systems over arbitrary fusion systems.

5.4: **(Right) derived functors and higher limits.**
Injective resolutions and the derived functors of a left-exact functor.
Limits in categories and the lim functor.

6. Insights on the proof of the Martino-Priddy conjecture II

6.1: **The orbit category of a fusion system, the functor $\mathcal{Z}$ and the obstruction groups.**
[1, III.5] until Section 5.1 for the definition of and discussion over the orbit category.
[1, III.5.5], the very first lines, for the definition of the functor $\mathcal{Z}$ and Proposition 5.11 for the obstruction groups.

6.2: **The vanishing result and the fundamental group-theoretic used knowledge.**

- **Intervals and the reduction results about vanishing of groups**: [13, Definition 1.4 and 1.5, Lemma 1.6 and 1.7].
- **Reduction to quadratic offenders:** define first quadratic best offenders as in [13, Definition 2.1], then provide the restricted vanishing result [13, Proposition 3.2] and the necessary FF-module theorem: see [13, Theorem 4.2], compare with [10, Theorem 1 and 2].
- **The final vanishing result:** see [13, Theorem 3.4], explain the inductive argument.

References

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