Übungen zu Classical Groups — Blatt 3

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Throughout K is a field and V is a finite-dimensional K-vector space.

Exercise 9: Let B be a symplectic form on V with Gram matrix $\widehat{B}$ with respect to a basis. Then $g \in \mathrm{GL}(V)$ lies in $\mathrm{Sp}(V)$ if and only if its representing matrix $\hat{g}$ satisfies $\hat{g}^t \widehat{B} \hat{g} = \widehat{B}$.

Exercise 10: (a) Let V be a symplectic space. Show that all maximal totally isotropic subspaces of V are $\mathrm{Sp}(V)$-conjugate and have dimension $\dim V/2$.

(b) Let V be a symplectic space of dimension $2m$ over $\mathbb{F}_q$. Show that the number of totally isotropic subspaces of dimension $k \leq m$ is $\prod_{i=0}^{k-1} \frac{q^{2m-2i}-1}{q^{i+1}-1}$.

Exercise 11: Let V be a symplectic space with alternating form B. The *conformal symplectic group* is

$$\mathrm{CSp}(V) := \{g \in \mathrm{GL}(V) \mid \exists\, c(g) \in K :\ B(g.v, g.w) = c(g)B(v, w)\ \forall v, w \in V\}.$$

Show that $\mathrm{CSp}(V) \to K^\times$, $g \mapsto c(g)$, is a surjective homomorphism. Determine $Z(\mathrm{CSp}(V))$. If $K = \mathbb{F}_q$, determine $|\mathrm{CSp}(V)|$.

[**Hint**: For $c \in K^\times$ find $g \in \mathrm{CSp}(V)$ acting by suitable scalars on the elements of a symplectic basis of V, such that $c(g) = c$.]

Exercise 12: Let V be a $2m$-dimensional symplectic space, W a maximal totally isotropic subspace and $P \leq \mathrm{Sp}(V)$ the stabiliser of a flag $0 = V_0 < V_1 < \cdots < V_r = W$ of totally isotropic subspaces of V with $m_i := \dim V_i/V_{i-1}$ for $1 \leq i \leq r$.

(a) Show that $|P| = q^{m^2}\Big(\prod_{i=1}^{r-1}\prod_{j=1}^{m_i}(q^j - 1)\Big)\prod_{j=1}^{m_r}(q^{2j} - 1)$ if $K = \mathbb{F}_q$.

(b) Let U be the subgroup of P of elements acting trivially on each factor V_i/V_{i-1}. Show that U is a normal subgroup of P generated by transvections, and

$$P/U \cong \mathrm{Sp}(V_r/V_{r-1}) \times \prod_{i=1}^{r-1} \mathrm{GL}(V_i/V_{i-1}).$$

[**Hint**: P also stabilises $W = V_r^\perp < V_{r-1}^\perp \ldots < 0^\perp = V$.]