Übungen zu Classical Groups — Blatt 2

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Throughout K is a field and V is a finite-dimensional K-vector space.

Exercise 5: Show that $\mathrm{PSL}_4(2)$ and $\mathrm{PSL}_3(4)$ are not isomorphic.
[**Hint**: Consider suitable elements of order 2.]

Exercise 6: Let $P \leq \mathrm{GL}(V)$ be the stabiliser of the flag $0 = V_0 < V_1 < \ldots < V_r = V$ of the n-dimensional vector space V and let $n_i := \dim V_i/V_{i-1}$ for $1 \leq i \leq r$.

(a) If $K = \mathbb{F}_q$ show $|P| = q^{n(n-1)/2} \prod_{i=1}^{r} \prod_{j=1}^{n_i} (q^j - 1)$.

(b) Let U be the subgroup of P of elements acting trivially on each factor V_i/V_{i-1}. Show that U is a normal subgroup of P generated by transvections, and $P/U \cong \prod_{i=1}^{r} \mathrm{GL}(V_i/V_{i-1})$.

Exercise 7: Let $K \leq K_1$ be a field extension of degree n.

(a) Show that $\mathrm{GL}_1(K_1)$ embeds into $\mathrm{GL}_n(K)$ in a natural way.

(b) Let $G = \mathrm{Aut}(K_1|K)$. Then G acts naturally on $\mathrm{GL}_1(K_1)$ and $\mathrm{GL}_1(K_1).G$ embeds into $\mathrm{GL}_n(K)$ in a natural way.

[**Hint**: Choose a basis of K_1 as K-vector space.]

Exercise 8: Let V be a symplectic space with symplectic basis $\{u_1, v_1, \ldots, u_m, v_m\}$. Show that with respect to $\{u_1, \ldots, u_m, v_1, \ldots, v_m\}$ any $\begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix}$ with $A \in \mathrm{GL}_m(K)$ lies in $\mathrm{Sp}(V)$, and conclude $\mathrm{Sp}_{2m}(K)$ contains a subgroup $\mathrm{GL}_m(K)$.